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LOCAL EXISTENCE THEOREM FOR FIRST ORDER SEMILINEAR HYPERBOLIC SYSTEMS IN SEVERAL SPACE DIMENSIONS

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Abstract

This paper studies first order semilinear hyperbolic systems in n ($n \geq 2$) space dimensions. Under the hypothesis that the system satisfies so called 'null condition', the local well-posedness for its Cauchy problem with initial data in $H^{\frac{n-1}{2}}$ is proved.

Keywords Semilinear hyperbolic systems, Local well posedness, Cauchy problem
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§1. Introduction

As is well known, the proof of the local existence theorem for nonlinear hyperbolic systems in R^{n+1}, with $n \geq 2$, rests entirely on energy estimates and Sobolev inequalities. This requires that the initial conditions have to belong to the Sobolev spaces $H^s(R^n)$ for relatively large s. Thus, for the following Cauchy problem of the first order semilinear hyperbolic systems

$$u_t + \sum_{i=1}^{n} A_i u_{x_i} = Q(u), \quad u = (u^1, \cdots, u^m), \tag{1.1}$$

$$t = 0: \quad u = u_0(x), \quad x \in R^n, \tag{1.2}$$

the minimum amount of regularity needed is

$$u_0 \in H^s(R^n) \quad \text{for} \quad s > \frac{n}{2}. \tag{1.3}$$

However, this condition is not always optimal. In this paper, we shall prove the local well-posedness of the Cauchy problem (1.1), (1.2) for $s = \frac{n-1}{2}$ in (1.3) under the hypothesis that the system of equations satisfies the so called 'null condition'.

Here we assume that A_i are constant matrices and $Q(u) = (Q^1(u), \cdots, Q^m(u))$ with each $Q^i(u)$ quadratic in u, that is,

$$Q^i(u) = \sum_{jk} \Gamma^i_{jk} u^j u^k \tag{1.4}$$

in which Γ^i_{jk} are constants.

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Our work is motivated by the recent work of Klainerman and Machedon[3] on the Cauchy problem for the systems of semilinear wave equations:

$$\Box\phi = Q(\phi, D\phi), \tag{1.5}$$

$$t = 0: \quad \phi = f, \phi_t = g, \tag{1.6}$$

where $\Box = \partial_t^2 - \triangle$ is the wave operator, $D = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$ and $\phi = (\phi^1, \cdots, \phi^n)$. They showed in three space dimensions that if Q is quadratic in $D\phi$ and satisfies the 'null condition', then (1.1)(1.2) is locally well-posed for

$$f \in H^{\frac{n+1}{2}}, \quad g \in H^{\frac{n-1}{2}}. \tag{1.3}$$

Later, Beals and Bezard[1] proved that the same conclusion holds in n $(n \geq 4)$ space dimensions even without the 'null condition'. However, In three space dimensions, Lindblad[4] has found examples of nonlinear scalar wave equations of type (1.5) which do not satisfy the null condition such that the initial value problem is ill posed for data $f \in H^2, g \in H^1$.

In order to state precisely our above mentioned results, we need to make some assumptions.

[H1]: $A(\xi) = \sum\limits_{i=1}^{n} A_i\xi_i$ is diagonalizable, more precisely, there exist invertible matrix $R(\xi) = (R_{ij}(\xi))_{m\times m}$ and diagonal matrix

$$\Lambda(\xi) = \mathrm{diag}[\lambda_1(\xi), \cdots \lambda_m(\xi)] \tag{1.7}$$

such that

$$R^{-1}(\xi)A(\xi)R(\xi) = \Lambda(\xi) \tag{1.8}$$

where $R_{ij}(\xi)(i, j = 1, \cdots, m)$ and $\lambda_i(\xi)(i = 1, \cdots, m)$ are homogeneous functions of ξ of degree 0 and 1 repectively and

$$\lambda_i(\xi), R_{ij}(\xi) \in C^\infty(S^{n-1}) \tag{1.9}$$

with S^{n-1} denoting the unit sphere in R^n.

The fact that

$$A(\xi) = -A(-\xi) \tag{1.10,)}$$

implies that $\lambda(\xi)$ is the eigenvalue of $A(\xi)$ if and only if $-\lambda(-\xi)$ is the eigenvalue of $A(\xi)$. Thus, we make the following assumptions.

[H2] either

$$\lambda_i(\xi) \equiv \lambda_j(\xi) \tag{1.11}$$

or

$$\lambda_i(\xi) \equiv -\lambda_j(-\xi) \tag{1.12}$$

or the following conditions are satisfied:

$$\lambda_i(\xi) - \lambda_j(\xi) \neq 0, \quad \forall\xi \neq 0 \tag{1.13}$$

and

$$\lambda_i(\xi) + \lambda_j(-\xi) \neq 0, \quad \forall\xi \neq 0 \tag{1.14}$$

and

$$\zeta \cdot (\nabla\lambda_i(\xi) - \nabla\lambda_j(\eta)) \neq 0 \tag{1.15}$$

for all $\zeta, \eta, \xi \neq 0$ such that

$$\zeta \cdot \xi = 0, \quad \zeta \cdot \eta \neq 0 \tag{1.16}$$

in which $\xi \cdot \eta$ denotes the iner product of ξ and η.

We remark that (1.13)-(1.16) is satisfied, for instance, by $\lambda_i(\xi) = \pm|\xi|, \lambda_j(\xi) = 2|\xi|$.

If $\lambda_i(\xi)$ is a homogeneous function of degree 1, then

$$\sum_i \xi_i \partial_i \lambda(\xi) \equiv \lambda(\xi), \tag{1.17}$$

so

$$\sum_{i,j} \xi_i \partial_i \partial_j \lambda(\xi) \equiv 0 \quad (j = 1, \cdots, n). \tag{1.18}$$

Thus, the matrix

$$(\partial_i \partial_j \lambda(\xi))_{n \times n}$$

has a zero eigenvalue. So we assume

[H3]

$$\operatorname{rank}(\lambda_{\xi_i \xi_j}) = n - 1 \tag{1.19}$$

and moreover the $n-1$ nonzero eigenvalues are all positive (or all negative) for all $\lambda = \lambda_i \quad (i = 1, \cdots, m)$.

We define the following concept.

Definition 1.1. *The system* (1.1) *satisfies 'null condition' if any plane wave solution* $u = u(\tau t + \sum_{i=1}^{n} \xi_i x_i)$ $(u(0) = 0$ *and* τ, ξ_i *are constants such that* $\tau^2 + |\xi|^2 \neq 0)$ *to the linearized system*

$$u_t + \sum_{i=1}^{n} A_i u_{x_i} = 0$$

is always a solution to the original system (1.1).

We then assume

[H4] (1.1) satisfies 'null condition'.

The null condition will be analysed in Section 2.

Our main result is that (1.1)(1.2) is locally well-posed for

$$u_0 \in H^{\frac{n-1}{2}},$$

provided that [H1]-[H4] are satisfied in $n = 2, 3$ space dimensions or [H1]-[H3] are satisfied in n $(n \geq 4)$ space dimensions.

We remark that if only [H1]-[H3] are satisfied in $n = 2$ or 3 space dimensions, then (1.1)-(1.2) is locally well-posed for

$$u_0 \in H^s, \quad s > \frac{3}{4} \quad \text{if} \quad n = 2, \tag{1.20}$$

$$u_0 \in H^s, \quad s > 1 \quad \text{if} \quad n = 3. \tag{1.21}$$

This will be proved in Appendix (see [6]).

As in [3], the proof of the above metioned results were based on establishing new space-time estimates for the null bilinear forms $Q^i(u)$. Let us state these estimates as follows

Theorem 1.1. *Under assumptions* [H1]-[H4], *suppose that* v, w *satisfy respectively*

$$v_t + \sum_{i=1}^{n} A_i v_{x_i} = F, \tag{1.22}$$

$$t = 0: \quad v = v_0(x), \quad x \in R^n \tag{1.23}$$

and

$$w_t + \sum_{i=1}^{n} A_i w_{x_i} = G, \tag{1.24}$$

$$t = 0: \quad w = w_0(x), \quad x \in R^n \tag{1.25}$$

in which $n = 2, 3$. *Then*

$$\begin{aligned} \|Q^i(v,w)\|_{H^{\frac{n-1}{2}}(R^{n+1})} \le C\Big(\|v_0\|_{H^{\frac{n-1}{2}}} + \int_0^{+\infty} \|F(t,\cdot)\|_{H^{\frac{n-1}{2}}}\,dt\Big) \\ \cdot \Big(\|w_0\|_{H^{\frac{n-1}{2}}} + \int_0^{+\infty} \|G(t,\cdot)\|_{H^{\frac{n-1}{2}}}\,dt\Big), \end{aligned} \tag{1.26}$$

where

$$Q^i(v,w) = \sum_{j,k=1}^{m} \Gamma^i_{jk} v^j w^k. \tag{1.27}$$

Theorem 1.2. *Under assumptions* [H1]-[H3], *suppose that* v, w *satisfy respectively* (1.22), (1.23) *and* (1.24), (1.25), *in which* $n \ge 4$. *Then* (1.26) *holds.*

§2. Null Condition

For the simplicity of the exposition we will assume that the system is strictly hyperbolic. If the system is nonstrictly hyperbolic with constant multiplicity, the similar analysis applies.

By strict hyperbolicity, the plane wave solution of the linearized system

$$u_t + \sum_{i=1}^{n} A_i u_{x_i} = 0$$

is of the form

$$u^i = R_{ij}(\xi) w_j\Big(-\lambda_j(\xi)t + \sum_{i=1}^{n} \xi_i x_i\Big), \quad i = 1, \cdots, m$$

for some fixed j. Therefore, system (1.1) satisfies null condition if and only if

$$\sum_{j,k} \Gamma^i_{jk} R_{jl}(\xi) R_{kl}(\xi) \equiv 0, \quad \forall \xi, \tag{2.1}$$

for all $j, l = 1, \cdots, m$. Noticing (1.10), it follows from strict hyperbolicity that there exists a permutation σ of the set $\{1, \cdots, m\}$ such that

$$\lambda_i(\xi) = -\lambda_{\sigma(i)}(-\xi), \tag{2.2}$$

$$R_{ij}(\xi) = R_{i,\sigma(j)}(-\xi). \tag{2.3}$$

Therefore, we conclude

$$\lambda_i(\xi) + \lambda_j(-\xi) \ge c_0|\xi|, \quad \forall \xi \ne 0, \quad j \ne \sigma(i), \tag{2.4}$$

$$\sum_{j,k} \Gamma^i_{jk} R_{jl}(\xi) R_{k\sigma(l)}(-\xi) \equiv 0, \quad \forall \xi, \tag{2.5}$$

for all $i, l = 1, \cdots, m$. We remark that (2.4) is related to the condition on the wave front set of two distributions in order to define a product. (See [2, p.297, Theorem 8.5.3]).

§3. Preliminaries

Lemma 3.1. *Let $\lambda(\xi)$ be a smooth homogeneous function of degree 1 that verifies [H3]. Then we have*

$$|\eta_i \partial_i \lambda(\xi) - \eta_i \partial_i \lambda(\eta)| \ge c_0 \Big|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}\Big|^2 |\eta|, \tag{3.1}$$

where c_0 is a positive constant.

Proof. Without loss of generality, we assume that the nonzero eigenvalues of $(\partial_i \partial_j \lambda(\xi))$ are all positive. Let $\vartheta = \frac{\eta}{|\eta|}$, $\omega = \frac{\xi}{|\xi|}$. When $\vartheta = -\omega$, we have

$$\vartheta_i \partial_i \lambda(\xi) - \vartheta_i \partial_i \lambda(\eta) = \lambda(\vartheta) + \lambda(-\vartheta).$$

It follows from Jensen's inequality that

$$\lambda(\vartheta) + \lambda(-\vartheta) > 2\Big(\lambda\Big(\frac{\vartheta - \vartheta}{2}\Big)\Big) = 0.$$

Thus, by continuity, there exist an ε sufficiently small such that when

$$\vartheta \cdot \omega + 1 \le \varepsilon, \tag{3.2}$$

we have

$$\vartheta_i \partial_i \lambda(\xi) - \vartheta_i \partial_i \lambda(\eta) \ne 0; \tag{3.3}$$

when

$$\vartheta \cdot \omega + 1 \ge \varepsilon, \tag{3.4}$$

we argue as follows:

$$\begin{aligned}
\sum_{i=1}^n \vartheta_i \partial_i \lambda(\xi) - \vartheta_i \partial_i \lambda(\eta) &= \sum_{i=1}^n (\vartheta_i - \omega_i) \partial_i \lambda(\xi) + \frac{\lambda(\xi)}{|\xi|} - \frac{\lambda(\eta)}{|\eta|} \\
&= \sum_{i=1}^n (\vartheta_i - \omega_i) \partial_i \lambda(\omega) + \lambda(\omega) - \lambda(\vartheta) \\
&= \sum_{i=1}^n (\vartheta_i - \omega_i) \Big(\partial_i \lambda(\omega) - \int_0^1 \partial_i \lambda(s\omega + (1-s)\vartheta) ds\Big) \\
&= \sum_{i,j=1}^n (\vartheta_i - \omega_i) \int_0^1 \int_0^1 (1-s) \partial_i \partial_j \lambda((\tau + (1-\tau)s)\omega \\
&\quad + (1-\tau)(1-s)\vartheta) ds d\tau (\vartheta_j - \omega_j).
\end{aligned} \tag{3.5}$$

Noticing (3.4), we have

$$\begin{aligned}
|(1-a)\omega + a\vartheta|^2 &= (1-a)^2 + a^2 + 2a(1-a)\omega \cdot \vartheta \\
&= (1-2a)^2 + 2a(1-a)(1 + \omega \cdot \vartheta) \\
&\ge (1-2a)^2 + 2a(1-a)\varepsilon > 0,
\end{aligned} \tag{3.6}$$

where $a = (1-\tau)(1-s)$. So if

$$|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| \geq \epsilon \tag{3.7}$$

for some positive constant ϵ, then (3.1) holds with some constant $c_0 = c_0(\epsilon)$.

If

$$|\frac{\xi}{|\xi|} - \frac{\eta}{|\eta|}| = \delta \leq \epsilon \tag{3.8}$$

and ϵ is sufficiently small, then

$$\sum_{i=1}^{n} \vartheta_i \partial_i \lambda(\xi) - \vartheta_i \partial_i \lambda(\eta) \geq c\{|\vartheta - \omega|^2 - ((\vartheta - \omega)r)^2\} \tag{3.9}$$

where r is an eigenvector of

$$\int_0^1 \int_0^1 (1-s)\partial_i \partial_j \lambda((\tau + (1-\tau)s)\omega - (1-\tau)(1-s)\vartheta) ds d\tau$$

corresponding to the smallest eigenvalue. Noticing (3.8) and the fact that

$$\int_0^1 \int_0^1 (1-s)\partial_i \partial_j \lambda((\tau + (1-\tau)s)\omega - (1-\tau)(1-s)\vartheta) ds d\tau = \partial_i \partial_j \lambda(\omega) + O(\delta),$$

we get

$$r = \omega + O(\delta).$$

Therefore

$$(\vartheta - \omega) r = (\vartheta - \omega)\omega + O(\delta^2), \quad \frac{1}{2}|\vartheta + \omega|^2 + O(\delta^2) = O(\delta^2).$$

Thus (3.1) follows from (3.5).

Replacing η by $-\eta$, we get

Lemma 3.2. *Let $\lambda(\xi)$ be a smooth homogeneous function of degree 1 that verifies* [H3]. *Then we have*

$$|\eta_i \partial_i \lambda(\xi) - \eta_i \partial_i \lambda(-\eta)| \geq c_0 \Big| \frac{\xi}{|\xi|} - \frac{\eta}{|\eta|} \Big|^2 |\eta|, \tag{3.10}$$

where c_0 is a positive constant.

Lemma 3.3. *Let $n \geq 2$ and $|\overline{\omega}| = 1$. Then*

$$\int_{S^{n-1}} \frac{d\omega}{|\overline{\omega} + \omega|^l} < +\infty, \tag{3.11}$$

where $l < n-1$.

Proof. By rotational invariance, we have

$$\int_{S^{n-1}} \frac{d\omega}{|\overline{\omega} + \omega|^l} = \int_{S^{n-1}} \frac{d\omega}{|e_1 + \omega|^l} = \int_{S^{n-1}} \frac{d\omega}{(2 + 2\omega_1)^{\frac{l}{2}}},$$

where $e_1 = (1, 0, \cdots, 0)$. Thus, it is enough to prove

$$\int_{\omega_1 < -\frac{1}{2}} \frac{d\omega}{(1 + \omega_1)^{\frac{l}{2}}} < +\infty. \tag{3.12}$$

Noticing

$$\int_{\omega_1 < -\frac{1}{2}} \frac{d\omega}{(1 + \omega_1)^{\frac{l}{2}}} = C \int_0^{\frac{\pi}{6}} \frac{\sin^{n-2}\theta}{(1 + \cos\theta)^{\frac{l}{2}}} d\theta = C \int_0^{\frac{\pi}{6}} \frac{\sin^{n-2}\theta}{\sin^l \frac{\theta}{2}} d\theta < +\infty,$$

we have (3.11).

§4. Estimate of the 'Null Forms'

Now we shall prove Theorem 1.1 and Theorem 1.2.

Let

$$\hat{W}(t,\xi)=R^{-1}(\xi)\hat{u}(t,\xi). \tag{4.1}$$

Then

$$\hat{u}(t,\xi)=R(\xi)\hat{W}(t,\xi). \tag{4.2}$$

So

$$\begin{aligned} Q(u)(t,x)&=\sum_{jk}\Gamma_{jk}u^j(t,x)u^k(t,x)=\sum_{jk}\iint e^{ix(\xi+\eta)}\Gamma_{jk}\hat{u}^j(t,\xi)\hat{u}^k(t,\eta)d\xi d\eta\\ &=\sum_{jk}q_{jk}(w_j,w_k), \end{aligned}\tag{4.3}$$

where q_{jk} is of the form

$$q(f,g)=\iint e^{ix(\xi+\eta)}q(\xi,\eta)\hat{f}(\xi)\hat{g}(\eta)d\xi d\eta \tag{4.4}$$

with its symbol

$$q_{jk}(\xi,\eta)=\sum_{lm}\Gamma_{lm}r_{lj}(\xi)r_{mk}(\eta). \tag{4.5}$$

So the null condition implies

$$q_{jj}(\xi,\xi)\equiv 0,\quad \forall\xi\neq 0, \tag{4.6}$$

and

$$q_{j\sigma(j)}(\xi,-\xi)\equiv 0,\quad \forall\xi\neq 0. \tag{4.7}$$

Therefore, Theorem 1.1 and Theorem 1.2 follow from the following three theorems.

Theorem 4.1. *Let F_λ be the Fourier multiplier defined by*

$$\hat{F_\lambda}f(t,\xi)=e^{i\lambda(\xi)t}\hat{f}(\xi), \tag{4.8}$$

where λ verifies [H3]. *Suppose that $q(\cdot,\cdot)$ is a bilinear form defined by* (4.4), *and moreover*

$$q(\xi,\xi)\equiv 0\quad if\quad n=2,3. \tag{4.9}$$

Then

$$\|q(F_\lambda f,F_\lambda g)\|_{H^{\frac{n-1}{2}}(R^{n+1})}\le C|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}. \tag{4.10}$$

Proof. It follows from duality that we only have to prove

$$\begin{aligned}&\iint|\xi|^{\frac{n-1}{2}}q(\xi-\eta,\eta)h(\lambda(\xi-\eta)+\lambda(\eta),\xi)f(\xi-\eta)g(\eta)d\xi d\eta\\ &\le C|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}|h|_{L^2(R^{n+1})},\end{aligned}\tag{4.11}$$

where $|\xi-\eta|\ge\frac{|\xi|}{2}$ in the region of integration. Noticing

$$\begin{aligned}&\iint|\xi|^{\frac{n-1}{2}}h(\lambda(\xi-\eta)+\lambda(\eta),\xi)q(\xi-\eta,\eta)f(\xi-\eta)g(\eta)d\xi d\eta\\ &\le|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}\\ &\quad\cdot\left(\int|\xi|^{n-1}|\xi-\eta|^{-n+1}|\eta|^{-n+1}q^2(\xi-\eta,\eta)h^2(\lambda(\xi-\eta)+\lambda(\eta),\xi)d\eta d\xi\right)^{\frac{1}{2}}\end{aligned}$$

(4.10) follows from Lemma 3.1 and Lemma 3.3 by making a transformation from $\lambda(\xi-\eta)+\lambda(\eta)$ to $|\eta|$ for fixed ξ and $\frac{\eta}{|\eta|}$.

Theorem 4.2. *Let* F_λ *be the Fourier multiplier as in Theorem* 4.1 *where* λ *verifies* [H3]. *Suppose that* $q(\cdot,\cdot)$ *is a bilinear form defined by* (4.4), *and moreover*

$$a(\xi,-\xi)\equiv 0 \quad if \quad n=2,3. \tag{4.12}$$

Then

$$\|q(F_\lambda f, F_{\lambda^*} g)\|_{H^{\frac{n-1}{2}}(R^{n+1})} \le C|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}, \tag{4.13}$$

where $\lambda^*(\xi)=-\lambda(-\xi)$.

Theorem 4.3. *Let* F_λ *and* F_μ *be the Fourier multiplier as in Theorem* 4.1 *where* λ,μ *satisfies* (1.13)-(1.16). *Suppose that* $q(\cdot,\cdot)$ *is a bilinear form defined by* (4.4). *Then*

$$\|q(F_\lambda f, F_\mu g)\|_{H^{\frac{n-1}{2}}(R^{n+1})} \le C|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}. \tag{4.14}$$

Proof. It follows from duality that we only have to prove

$$\begin{aligned}&\iint |\xi|^{\frac{n-1}{2}} q(\xi-\eta,\eta)h(\lambda(\xi-\eta)+\mu(\eta),\xi)f(\xi-\eta)g(\eta)d\xi d\eta\\ &\le C|f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}|h|_{L^2(R^{n+1})}.\end{aligned} \tag{4.15}$$

Without loss of generality, we may assume $|\xi-\eta|\ge\frac{|\xi|}{2}$ in the region of integration. Noticing

$$\begin{aligned}&\iint |\xi|^{\frac{n-1}{2}} q(\xi-\eta,\eta)h(\lambda(\xi-\eta)+\mu(\eta),\xi)f(\xi-\eta)g(\eta)d\xi d\eta\\ &\le |f|_{H^{\frac{n-1}{2}}(R^n)}|g|_{H^{\frac{n-1}{2}}(R^n)}\\ &\quad\cdot\left(\int |\xi|^{n-1}|\xi-\eta|^{-n+1}|\eta|^{-n+1}q^2(\xi-\eta,\eta)h^2(\lambda(\xi-\eta)+\mu(\eta),\xi)d\eta d\xi\right)^{\frac12},\end{aligned}$$

we only have to prove

$$\iint |\eta|^{-n+1}q^2(\xi-\eta,\eta)h^2(\lambda(\xi-\eta)+\mu(\eta),\xi)d\eta d\xi \le C\|h\|^2_{L^2}. \tag{4.16}$$

We divide the region of integration into three parts

(I) $\frac{\xi-\eta}{|\xi-\eta|}\frac{\eta}{|\eta|}\ge 1-\varepsilon$,

(II) $\frac{\xi-\eta}{|\xi-\eta|}\frac{\eta}{|\eta|}\le -1+\varepsilon$,

(III) $-1+\varepsilon\le\frac{\xi-\eta}{|\xi-\eta|}\frac{\eta}{|\eta|}\le 1-\varepsilon$, where ε is to be determined later.

In the first case, if

$$\frac{\zeta}{|\zeta|}=\frac{\eta}{|\eta|},$$

then

$$\sum_{i=1}^n -\eta_i\partial_i\lambda(\zeta)+\eta_i\partial_i\mu(\eta)=|\eta|(\mu(\frac{\eta}{|\eta|})-\lambda(\frac{\eta}{|\eta|}))\ne 0.$$

Thus, it follows that

$$|\eta|^{-1}\sum_{i=1}^n -\eta_i\partial_i\lambda(\zeta)+\eta_i\partial_i\mu(\eta)\ne 0$$

for

$$\frac{\zeta}{|\zeta|}\frac{\eta}{|\eta|}\ge 1-\varepsilon$$

with ε sufficiently small. So for fixed ξ and $\frac{\eta}{|\eta|}$ the transformation from $\lambda(\xi-\eta)+\mu(\eta)$ to $|\eta|$ is nonsingular. Thus, (4.16) follows.

The second case can be dealt with in a similar way.

Now, let us look at the third case. By the compactness of $S^{n-1}\times S^{n-1}$, we only have to concentrate in a small neighbourhood. So we assume

$$\Big|\frac{\xi-\eta}{|\xi-\eta|}-\zeta_0\Big|\le\delta,\quad \Big|\frac{\eta}{|\eta|}-\eta_0\Big|\le\delta \tag{4.17}$$

for some $\zeta_0,\eta_0\in S^{n-1}$ such that $1-|\zeta_0\cdot\eta_0|\ge\varepsilon$. By (1.15), (1.16), there exists $\zeta_1\in S^{n-1}$ such that $\zeta_1\cdot\zeta_0=0,\quad \zeta_1\cdot\eta_0\ne0$ and $\zeta_1\cdot(\nabla\lambda(\zeta_0)-\nabla\mu(\eta_0))\ne0$. It follows by continuity that for δ sufficiently small $\zeta_1\cdot\eta\ge c|\eta|$ and $\zeta_1\cdot(\nabla\lambda(\xi-\eta)-\nabla\mu(\eta))\ne0$. So the transformation from $\lambda(\xi-\eta)+\mu(\eta)$ to $\zeta_1\cdot\eta$ for fixed ξ is nonsingular. (4.16) follows from making this transformation.

§5. Main Results

The main results of this paper are

Theorem 5.1. *Under assumptions* [H1]-[H4], *consider the Cauchy problem* (1.1), (1.2) *in* $n=2,3$ *space dimensions. There exists a positive number* T *depending only on* $\|u_0\|_{H^{\frac{n-1}{2}}}$ *and a unique solution* u *of* (1.1), (1.2) *defined in* $[0,T]\times R^n$ *verifying the following conditions*

$$\|Q^i(u)\|_{H^{\frac{n-1}{2}}([0,T]\times R^n)}<+\infty, \tag{5.1}$$

$$\sup_{[0,T]}\|u(t,\cdot)\|_{H^{\frac{n-1}{2}}(R^n)}<+\infty. \tag{5.2}$$

Theorem 5.2. *Under assumptions* [H1]-[H3], *the same conclusions of Theorem* 5.1 *hold true in* $n\ge4$ *space dimensions.*

Using Theorem 1.1 and Theorem 1.2, Theorem 5.1 and Theorem 5.2 can be proved in the same way as in [3].

§6. Appendix: Stichartz's Inequality

The following estimate is a generalization of an inequality due to Strichartz[7] (see also [5]).

Theorem 6.1. *Suppose that* [H3] *is satisfied. Let*

$$u=F_\lambda f. \tag{6.1}$$

Then

$$|u|_{L_t^rL_x^q}\le C|f|_{H^s(R^n)}, \tag{6.2}$$

where $2\le q<\infty$ *and* $r=\frac{4q}{(n-1)(q-2)}$ *and* $s=\frac{n+1}{(n-1)r}$

From Theorem 6.1 and Sobolev embedding theorem, we easily get

Corollary 6.1. *Under the assumptions* [H1] *and* [H3], *let* u *be the solution to the Cauchy problem*

$$u_t+\sum_{i=1}^{2}A_iu_{x_i}=F, \tag{6.3}$$

$$t=0:\ u=u_0. \tag{6.4}$$

Then it holds that

$$|u|_{L^4([0,T];L^\infty(R^2))} \le CT^d\Big(|u_0|_{H^s(R^2)} + \int_0^T \|F(\tau,\cdot)\|_{H^s(R^2)} d\tau\Big), \tag{6.5}$$

where $s > \frac{3}{4}$ *and* $d < s - \frac{3}{4}$

Corollary 6.2. *Under the assumptions* [H1] *and* [H3], *let* u *be the solution to the Cauchy problem*

$$u_t + \sum_{i=1}^{3} A_i u_{x_i} = F, \tag{6.6}$$

$$t = 0: \;\; u = u_0. \tag{6.7}$$

Then it holds that

$$|u|_{L^2([0,T];L^\infty(R^3))} \le CT^d\Big(|u_0|_{H^s(R^3)} + \int_0^T \|F(\tau,\cdot)\|_{H^s(R^3)} d\tau\Big), \tag{6.8}$$

where $s > 1$ *and* $d < s - 1$

Using Corollary 6.1 and Corollary 6.2, it is easy to prove the following (see [6])

Theorem 6.1. *Under assumptions* [H1]-[H3], *consider the Cauchy problem* (1.1), (1.2) *in* $n = 2, 3$ *space dimensions. There exists a positive number* T *depending only on* $\|u_0\|_{H^s}$ *and a unique solution* u *of* (1.1), (1.2) *defined in* $[0,T]\times R^n$ *verifying the following conditions*

$$\sup_{[0,T]} \|u(t,\cdot)\|_{H^s(R^n)} < +\infty,$$

where $s > \frac{3}{4}$ *if* $n = 2$, $s > 1$ *if* $n = 3$.

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