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THE EXISTENCE AND UNIQUENESS AND STABILITY OF ALMOST PERIODIC SOLUTIONS FOR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAYS**

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Abstract

This paper deals with the problems on the existence and uniqueness and stability of almost periodic solutions for functional differential equations with infinite delays. The author obtains some sufficient conditions which ganrantee the existence and uniqueness and stability of almost periodic solutions with module containment. The results extend all the results of the paper [1] and solve the two open problems proposed in [1] under much weaker conditions than that proposed in [1].

Keywords Existence, Uniqueness, Stability, Almost periodic solution, Functional differential equation, Infinite delay
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§1. Introduction

In the paper [1], G. Seifert used Medvedev's method[2] for some almost periodic (a. p. for short) systems with infinite time delays to obtain an existence theorem for a.p. solutions, but he did not obtain the uniqueness and stability for such a.p. solutions. Therefore he proposed two open problems on uniqueness and stability of such a.p. solutions for further study in the finality of [1]. In this paper, we study the problems on the existence and uniqueness and stability of a.p. solutions for such systems. We obtain some sufficient conditions which garantee the existence and uniqueness and stability of a.p. solutions with module containment. Our results extend all the results of [1], and solve the two open problems proposed in [1] under much weaker conditions than that proposed in [1].

§2. Notation, Definition, and Main Results

Let R^n denote the set of real n-vectors, and $|x|$ any convenient norm for $x \in R^n$; also let $R = R^1$.

By CB we denote the set of R^n-valued functions continuous and bounded on $(-\infty, 0]$; for each $\phi \in CB$ we define $\|\phi\| = \sup\{|\phi(s)| : s \leq 0\}$. Thus $\{CB, \|\|\}$ is a real Banach space.

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If $x(t)$ is an R^n-valued function on $(-\infty, b), b \le \infty$, we define for each $t \in (-\infty, b)$, $x_t(s) = x(t+s),\ s \le 0$. Clearly if $x(t)$ is continuous and bounded on each interval $(-\infty, b_1], b_1 < b$, then $x_t \in CB$ for $t \in (-\infty, b)$.

The R^n-valued function $F(t, x, \phi)$ on $R \times R^n \times CB$ is said to satisfy condition:

(A_1) if it is a.p. in t uniformly for (x, ϕ) in closed bounded subsets of $R^n \times CB$, i.e., if $S \subset R^n \times CB$ is closed and bounded, then $\{F(t, x, \phi) : (x, \phi) \in S\}$ is a uniformly a.p. family in the sense of [3, p.17];

(A_2) if there exists an $M > 0$ such that $|F(t, 0, 0)| \le M$ for $t \in R$;

(A_3) if for $x(t)$ uniformly continuous and bounded on $R, F(t, x(t), x_t)$ is uniformly continuous on R;

(A_4) if there exist positive numbers p, h, and r such that $ph < 1,\ p \ge M/r$ where M is as in (A_2), such that

$$|x(t) - y(t) + h(F(t, x(t), x_t) - F(t, y(t), y_t))| \le (1 - ph)\|x_t - y_t\| \tag{2.1}$$

for $t \in R$ and any functions $x(t), y(t)$ uniformly continuous and such that $|x(t)| \le r, |y(t)| \le r$ on R;

(A_5) if for each $r > 0$ there exists an $M_1(r) > 0$ such that $|F(t, x, \phi)| \le M_1(r)$ for $|x| \le r, \|\phi\| \le r, t \in R$;

(A_6) if in the condition (A_4), (2.1) is also valid for any functions $x(t), y(t)$ continuous on R and there is $p > M/r$.

Remark 2.1. It follows easily that if F is uniformly continuous on $R \times R^n \times CB$, it satisfies (A_3). In this case , however, the function $F(t, x(t), x_t)$ is not necessarily continuous in t for $x(t)$ only continuous and bounded on R. This follows because x_t need not be continuous in t for such $x(t)$.

Remark 2.2. Obviously, the conditions (A_2)-(A_5) are much weaker than the conditions (H_2)-(H_4) in [1], and we do not need the condition (H_5) in [1].

Remark 2.3. The condition (A_6) contains the condition (A_4).

We now consider the functional differential equations with infinite time delays

$$x'(t) = F(t, x(t), x_t). \tag{2.2}$$

Definition 2.1. *A bounded solution $x(t, t_0, \phi_1)$, which satisfies $x_{t_0} = \phi_1$ with $\phi_1 \in CB$, of* (2.2) *for $t \ge t_0$ is uniformly stable if, for each $\varepsilon > 0$ and each $t_0 \ge 0$, there exists a positive number $\delta = \delta(\varepsilon)$ (independent of t_0) such that $|x(t, t_0, \phi_1) - y(t, t_0, \phi_2)| < \varepsilon$, whenever $\|\phi_1 - \phi_2\| < \delta$ and $t \ge t_0$, where $y(t, t_0, \phi_2)$, which satisfies $y_{t_0} = \phi_2$ with $\phi_2 \in CB$, is any solution of* (2.2) *for $t \ge t_0$.*

Theorem 2.1. *Let F have* (A_2)-(A_5). *Then* (2.2) *has only one solution $\bar{x}(t)$ with $|\bar{x}(t)| \le r$ for $t \in R$.*

Theorem 2.2. *Let F have* (A_2), (A_3), (A_5) *and* (A_6). *Then* (2.2) *has only one bounded solution $\bar{x}(t)$ which is uniformly stable and satisfies $|\bar{x}(t)| \le r$ for $t \in R$.*

Theorem 2.3. *If F is periodic in t with period T independent of (x, ϕ) and if F satisfies* (A_2)-(A_4), *then* (2.2) *has only one T-periodic solution $\bar{x}(t)$ with $|\bar{x}(t)| \le r$ for $t \in R$.*

Theorem 2.4. *If F is periodic in t with period T independent of (x, ϕ) and if F has* (A_2), (A_3), (A_5) *and* (A_6), *then* (2.2) *has only one T-periodic solution $\bar{x}(t)$ which is uniformly stable and satisfies $|\bar{x}(t)| \le r$ for $t \in R$.*

Theorem 2.5. *Let F have* (A_1)-(A_4). *Then* (2.2) *has unique a.p. solution* $\bar{x}(t)$ *with* $|\bar{x}(t)| \le r$ *for* $t \in R$ *and* $\mod(\bar{x}) \subset \mod(F)$.

Theorem 2.6. *If F satisfies* (A_1)-(A_3) *and* (A_5), (A_6), *then* (2.2) *has unique uniformly stable a.p. solution* $\bar{x}(t)$ *with* $|\bar{x}(t)| \le r$ *for* $t \in R$ *and* $\mod(\bar{x}) \subset \mod(F)$.

Remark 2.4. Obviously, the results of Theorems 2.1, 2.3, 2.5 are much better than all the results of §2 in [1], and the conditions of our theorems are much weaker than that of the theorems in §2 of [1].

Remark 2.5. Theorems 2.5 and 2.6 solve the two open problems[1] under much weaker conditions than that proposed in [1].

Remark 2.6. From Definition 2.1, it is easy to see that the condition (A_6) in Theorems 2.2, 2.4, 2.6 is reasonable.

§3. Proofs of Theorems

Proof of Theorem 2.1. First, we write (2.2) in the following form:

$$x'(t) = F(t, x(t), x_t) = -\frac{1}{h}x(t) + g(t, x(t), x_t) + F(t, 0, 0), \tag{3.1}$$

where $g(t, x(t), x_t) = \frac{1}{h}[x(t) + h(F(t, x(t), x_t) - F(t, 0, 0))]$. Then there is $g(t, 0, 0) = 0$ and it follows from (A_3) and (A_4) that $g(t, x(t), x_t)$ satisfies the following condition:

(A_7) it is uniformly continuous on R and satisfies

$$|g(t, x(t), x_t) - g(t, y(t), y_t)| \le \frac{(1-ph)}{h}\|x_t - y_t\| \tag{3.2}$$

for $t \in R$ and any functions $x(t), y(t)$ uniformly continuous on R such that $|x(t)| \le r, |y(t)| \le r$.

(1) We prove that (2.2) has a solution $\bar{x}(t)$ with $|\bar{x}(t)| \le M/p$ for $t \in R$.

We first construct the function sequences as follows:

$$x^0(t) = \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)F(s, 0, 0)ds \quad (t \in R), \tag{3.3}$$

$$x^{m+1}(t) = x^0(t) + \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)g(s, x^m(s), x_s^m)ds \quad (t \in R), \tag{3.4}$$

when $m = 0, 1, 2, 3, \cdots$.

(i) We prove that $x^m(t)(m = 0, 1, 2, 3, \cdots)$ are uniformly continuous and $|x^m(t)| < M/p$ for $t \in R$.

From (A_2) and (A_3), there is

$$\begin{aligned} |x^0(t)| &\le \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)|F(s, 0, 0)|ds \\ &\le M\int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)ds \\ &= Mh < M/p \quad (t \in R). \end{aligned} \tag{3.5}$$

And it follows from (A_3) and (3.3) that $x^0(t)$ is uniformly continuous on R. Using (A_7) and

(3.4), (3.5), we have

$$\begin{aligned}|x^1(t)| &\le |x^0(t)| + \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)|g(s, x^0(s), x_s^0)|ds \\ &\le Mh + \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\frac{(1-ph)}{h}\|x_s^0\|ds \\ &\le Mh + (1-ph)M\int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)ds \\ &= Mh[1+(1-ph)] \quad (t \in R).\end{aligned} \tag{3.6}$$

Since $0 < 1 - ph < 1$, there is

$$\frac{1}{ph} = \frac{1}{1-(1-ph)} = \sum_{j=0}^{\infty}(1-ph)^j. \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$|x^1(t)| \le Mh[1+(1-ph)] < M/p \quad (t \in R). \tag{3.8}$$

From (A_7), (3.4)-(3.6) and the uniform continuity of $x^0(t)$ on R, we see easily that $x^1(t)$ also is uniformly continuous on R.

In general, for any natural number k, we can assume inductively that

$$|x^K(t)| \le Mh\sum_{j=0}^{K}(1-ph)^j < M/p \quad (t \in R), \tag{3.9}$$

and $x^K(t)$ is uniformly continuous on R.

Then from (3.2), (3.4), (3.5) and (3.9), there is

$$\begin{aligned}|x^{K+1}(t)| &\le |x^0(t)| + \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)|g(s, x^K(s), x_s^K)|ds \\ &\le Mh + \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\frac{(1-ph)}{h}\|x_s^K\|ds \\ &\le Mh + (1-ph)M\sum_{j=0}^{K}(1-ph)^j\int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)ds \\ &= Mh\sum_{j=0}^{K+1}(1-ph)^j < M/p \quad (t \in R).\end{aligned} \tag{3.10}$$

Because of (A_7), (3.4) and the uniform continuity of $x^0(t)$ and $x^K(t)$ on $R, x^{K+1}(t)$ is uniformly continuous on R.

Therefore by induction, for any natural number m, there is

$$|x^m(t)| \le Mh\sum_{j=0}^{m}(1-ph)^j < M/p \quad (t \in R), \tag{3.11}$$

and $x^m(t)$ is uniformly continuous on R.

(ii) We prove that $\{x^m(t)\}$ is uniformly convergent on R and its limit function $\bar{x}(t)$ is uniformly continuous and satisfies $|\bar{x}(t)| \le M/p$ for $t \in R$.

Let

$$L_{m+1} = \sup\{|x^{m+1}(t) - x^m(t)| : t \in R\} \quad (m = 0, 1, 2, 3, \cdots). \tag{3.12}$$

Then from (3.2), (3.4), (3.12), (A_7), and (i) we have

$$\begin{aligned}|x^{m+1}(t)-x^m(t)| &\le \int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)\\ &\quad\cdot\|g(s,x^m(s),x_s^m)-g(s,x^{m-1}(s),x_s^{m-1})|ds\\ &\le \frac{(1-ph)}{h}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)\|x_s^m-x_s^{m-1}\|ds\\ &\le \frac{(1-ph)L_m}{h}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)ds\\ &=(1-ph)L_m\quad(t\in R).\end{aligned}\tag{3.13}$$

Hence there is

$$L_{m+1}\le(1-ph)L_m\quad(m=0,1,2,3,\cdots).$$

Since $0<1-ph<1$, $\{x^m(t)\}$ is uniformly convergent on R. Let $x^m(t)\longrightarrow\bar{x}(t)$ uniformly in $t\in R$. Because $x^m(t)(m=0,1,2,3,\cdots)$ are uniformly continuous on R and $|x^m(t)|\le M/p$ for $t\in R$, $\bar{x}(t)$ is uniformly continuous on R and $|\bar{x}(t)|\le M/p$ for $t\in R$.

(iii) We prove that

$$\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,x^m(s),x_s^m)ds\longrightarrow\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,\bar{x}(s),\bar{x}_s)ds$$

uniformly in $t\in R$ when $m\longrightarrow\infty$.

Because $x^m(t)\longrightarrow\bar{x}(t)$ uniformly on R when $m\longrightarrow\infty$, for each $\varepsilon>0$, there is a natural number $N=N(\varepsilon)$ sufficiently large such that

$$|x^m(t)-\bar{x}(t)|<\varepsilon\quad(t\in R),$$

when $m\ge N$. Then when $m\ge N$, using (A_7), we have

$$\begin{aligned}&\Big|\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,x^m(s),x_s^m)ds-\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,\bar{x}(s),\bar{x}_s)ds\Big|\\ &\le\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)|g(s,x^m(s),x_s^m)-g(s,\bar{x}(s),\bar{x}_s)|ds\\ &\le\frac{(1-ph)}{h}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)\|x_s^m-\bar{x}_s\|ds\\ &\le\frac{(1-ph)\varepsilon}{h}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)ds\\ &=(1-ph)\varepsilon<\varepsilon\end{aligned}$$

for all $t\in R$, i.e.,

$$\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,x^m(s),x_s^m)ds\longrightarrow\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,\bar{x}(s),\bar{x}_s)ds$$

uniformly for $t\in R$.

Now, taking its limit from the both sides of (3.4), we obtain

$$\bar{x}(t)=x^0(t)+\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)g(s,\bar{x}(s),\bar{x}_s)ds\quad(t\in R).\tag{3.14}$$

Then, from the right side of (3.14), it is easy to see that $\bar{x}(t)$ is continuously differentiable on R. Immediately, differentiating the both sides of (3.14), we have that $\bar{x}'(t)=F(t,\bar{x}(t),\bar{x}_t)$

for $t \in R$. Therefore $\bar{x}(t)$ is a bounded solution of (2.2) such that $|\bar{x}(t)| \le M/p$ for $t \in R$.

(2) We prove that $\bar{x}(t)$ is a unique bounded solution of (2.2) such that $|x(t)| \le r$ for $t \in R$.

In fact, if this conclusion is not valid, then there is another bounded solution $y(t)$ of (2.2) such that $|y(t)| \le r$ for $t \in R$ and $y(t) \not\equiv \bar{x}(t)$. It follows from ($A_5$) that $y(t)$ is uniformly continuous on R. Since

$$\bar{x}'(t) - y'(t) = -\frac{1}{h}[\bar{x}(t) - y(t)] + [g(t, \bar{x}(t), \bar{x}_t) - g(t, y(t), y_t)],$$

there is

$$\begin{aligned}\bar{x}(t) - y(t) = \exp\Big(-\frac{1}{h}(t-t_0)\Big)[\bar{x}(t_0) - y(t_0)] + \int_{t_0}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\\ \cdot [g(s, \bar{x}(s), \bar{x}_s) - g(s, y(s), y_s)]ds \quad (t \ge t_0).\end{aligned} \tag{3.15}$$

Using (A_7), we have

$$\begin{aligned}|\bar{x}(t) - y(t)| \le \exp\Big(-\frac{1}{h}(t-t_0)\Big)|\bar{x}(t_0) - y(t_0)|\\ + \frac{(1-ph)}{h}\int_{t_0}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\|\bar{x}_s - y_s\|ds \quad (t \ge t_0).\end{aligned} \tag{3.16}$$

Let $c_1 = \sup\{|\bar{x}(t) - y(t)| : t \in R\}$. Then there are $c_1 > 0$ and $t_1 \in R$ such that

$$|\bar{x}(t_1) - y(t_1)| \ge c_1(1 - ph/4). \tag{3.17}$$

Taking $t_0 = t_1 - T$, where $T > 0$ is sufficiently large such that

$$\exp(-T/h) \le ph/2, \tag{3.18}$$

from (3.16)-(3.18), we obtain

$$\begin{aligned}c_1(1 - ph/4) &\le |\bar{x}(t_1) - y(t_1)| \le \exp(-T/h)c_1 + (1-ph)c_1\\ &\le phc_1/2 + (1-ph)c_1 = (1 - ph/2)c_1,\end{aligned}$$

which is a contradiction because of $0 < 1 - ph < 1$ and $c_1 > 0$. Therefore $\bar{x}(t)$ is a unique bounded solution of (2.2) such that $|\bar{x}(t)| \le r$ for $t \in R$. It completes the proof.

Proof of Theorem 2.2. From Remark 2.3 and Theorem 2.1, we can see that it suffices to prove that $\bar{x}(t)$ is uniformly stable. In fact, for each $\varepsilon > 0$ (especially $\varepsilon < (r - M/p)$) and each $t_0 \ge 0$, taking a positive number $\delta = \delta(\varepsilon) = \min\{(r - M/p)/2, ph\varepsilon/2\}$ (obviously, $\delta(\varepsilon)$ is independent of t_0), we see that, when $\|\bar{x}_{t_0} - \phi\| < \delta$, there is

$$|\bar{x}(t) - y(t)| < \varepsilon \quad (t \ge t_0), \tag{3.19}$$

where $y(t) = y(t, t_0, \phi)$, which satisfies $y_{t_0} = \phi$ with $\phi \in CB$, is any solution of (2.2) for $t \ge t_0$. Otherwise, if (3.19) is not valid, there is $t_1 > t_0$ such that

$$|\bar{x}(t_1) - y(t_1)| = \varepsilon \tag{3.20}$$

and

$$|\bar{x}(t) - y(t)| < \varepsilon \quad (t_0 < t < t_1). \tag{3.21}$$

Therefore there is

$$\|\bar{x}_t - y_t\| \le \varepsilon \quad (t \le t_1). \tag{3.22}$$

Because

$$\begin{aligned}\bar{x}'(t)-y'(t)&=F(t,\bar{x}(t),\bar{x}_t)-F(t,y(t),y_t)\\&=-\frac{1}{h}[\bar{x}(t)-y(t)]+[g_1(t,\bar{x}(t),\bar{x}_t)-g_1(t,y(t),y_t)]\quad(t\ge t_0),\end{aligned}\tag{3.23}$$

where $g_1(t,x(t),x_t)=\frac{1}{h}[x(t)+hF(t,x(t),x_t)]$, and from ($A_6$), we have

$$|g_1(t,x(t),x_t)-g_1(t,y(t),y_t)|\le\frac{(1-ph)}{h}\|x_t-y_t\|\tag{3.24}$$

for any functions $x(t),y(t)$ continuous on R such that $|x(t)|\le r$ and $|y(t)|\le r$. It follows from (3.23) that

$$\begin{aligned}\bar{x}(t_1)-y(t_1)=\exp\Big(-\frac{1}{h}(t_1-t_0)\Big)[\bar{x}(t_0)-y(t_0)]+\int_{t_0}^{t_1}\exp\Big(-\frac{1}{h}(t_1-s)\Big)\\\cdot[g_1(s,\bar{x}(s),\bar{x}_s)-g_1(s,y(s),y_s)]ds.\end{aligned}\tag{3.25}$$

Using (3.20)-(3.22), (3.24) and (3.25), we can obtain

$$\begin{aligned}\varepsilon=|\bar{x}(t_1)-y(t_1)|&\le\exp\Big(-\frac{1}{h}(t_1-t_0)\Big)|\bar{x}(t_0)-y(t_0)|\\&\quad+\int_{t_0}^{t_1}\exp\Big(-\frac{1}{h}(t_1-s)\Big)|g_1(s,\bar{x}(s),\bar{x}_s)-g_1(s,y(s),y_s)|ds\\&\le\exp\Big(-\frac{1}{h}(t_1-t_0)\Big)\delta+\frac{(1-ph)}{h}\int_{t_0}^{t_1}\exp\Big(-\frac{1}{h}(t_1-s)\Big)\|\bar{x}_s-y_s\|ds\\&\le\delta+\frac{(1-ph)\varepsilon}{h}\int_{t_0}^{t_1}\exp\Big(-\frac{1}{h}(t_1-s)\Big)ds\\&\le\delta+(1-ph)\varepsilon\le ph\varepsilon/2+(1-ph)\varepsilon=(1-ph/2)\varepsilon<\varepsilon,\end{aligned}$$

which is a contradiction. Therefore $\bar{x}(t)$ is uniformly stable. The proof is complete.

Proof of Theorem 2.3. Because a continuous T-periodic solution of (2.2) is uniformly continuous on R, the condition (A_5) is not needed. From Theorem 2.1, obviously, it suffices to prove that $\bar{x}(t)$ is a T-periodic solution of (2.2). Using (3.14) and the conditions of this theorem, we obtain

$$\begin{aligned}\bar{x}(t+T)&=\int_{-\infty}^{t+T}\exp\Big(-\frac{1}{h}(t+T-s)\Big)[g(s,\bar{x}(s),\bar{x}_s)+F(s,0,0)]ds\\&\overset{s=u+T}{=\!=\!=}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-u)\Big)[g(u+T,\bar{x}(u+T),\bar{x}_{u+T})\\&\quad+F(u+T,0,0)]du\\&=\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-u)\Big)[g(u,\bar{x}(u+T),\bar{x}_{u+T})+F(u,0,0)]du\\&=\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)[g(s,\bar{x}(s+T),\bar{x}_{s+T})+F(s,0,0)]ds.\end{aligned}\tag{3.26}$$

Let $L=\sup\{|\bar{x}(t)-\bar{x}(t+T)|:t\in R\}$. Then from (3.14), (3.26) and (A_7), we have

$$\begin{aligned}|\bar{x}(t)-\bar{x}(t+T)|&\le\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)|g(s,\bar{x}(s),\bar{x}_s)-g(s,\bar{x}(s+T),\bar{x}_{s+T})|ds\\&\le\frac{(1-ph)L}{h}\int_{-\infty}^{t}\exp\Big(-\frac{1}{h}(t-s)\Big)ds=(1-ph)L,\end{aligned}$$

i.e., $L \le (1-ph)L$. It follows from $0 < 1-ph < 1$ that $L = 0$. Therefore $\bar{x}(t)$ is a T-periodic solution of (2.2). The proof is complete.

Proof of Theorem 2.4. The proof of this theorem follows immediately from Theorems 2.2 and 2.3.

Proof of Theorem 2.5. Because an a.p. solution of (2.2) is uniformly continuous on R, the condition (A_5) is not needed. From Theorem 2.1, obviously, it suffices to prove that $\bar{x}(t)$ is an a.p. solution of (2.2) which satisfies mod $(\bar{x}) \subset$ mod (F). Using (3.14), we have

$$\begin{aligned}\bar{x}(t+\tau) &= \int_{-\infty}^{t+\tau} \exp\Big(-\frac{1}{h}(t+\tau-s)\Big)[g(s,\bar{x}(s),\bar{x}_s)+F(s,0,0)]ds \\ &\underline{\underline{s=u+\tau}} \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-u)\Big)[g(u+\tau,\bar{x}(u+\tau),\bar{x}_{u+\tau})+F(u+\tau,0,0)]du \\ &= \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)[g(s+\tau,\bar{x}(s+\tau),\bar{x}_{s+\tau})+F(s+\tau,0,0)]ds\end{aligned} \tag{3.27}$$

for all $t, \tau \in R$. It follows from (3.14) and (3.27) that

$$\begin{aligned}\bar{x}(t)-\bar{x}(t+\tau) &= \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\frac{1}{h}[\bar{x}(s)-\bar{x}(s+\tau)+h(F(s,\bar{x}(s),\bar{x}_s) \\ &\quad - F(s+\tau,\bar{x}(s+\tau),\bar{x}_{s+\tau}))]ds \\ &= \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\{\frac{1}{h}[\bar{x}(s)-\bar{x}(s+\tau)+h(F(s,\bar{x}(s),\bar{x}_s) \\ &\quad - F(s,\bar{x}(s+\tau),\bar{x}_{s+\tau}))]+[F(s,\bar{x}(s+\tau),\bar{x}_{s+\tau}) \\ &\quad - F(s+\tau,\bar{x}(s+\tau),\bar{x}_{s+\tau})]\}ds\end{aligned} \tag{3.28}$$

for all $t, \tau \in R$. By (A_1), for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$ such that every interval of R of length $l(\varepsilon)$ contains a $\tau = \tau(\varepsilon)$ such that

$$|F(t,\bar{x}(t+\tau),\bar{x}_{t+\tau}) - F(t+\tau,\bar{x}(t+\tau),\bar{x}_{t+\tau})| < \varepsilon \tag{3.29}$$

for all $t \in R$. Let $L_0 = \sup\{|\bar{x}(t)-\bar{x}(t+\tau)| : t \in R, \tau = \tau(\varepsilon)\}$. Thus, for such a τ, it follows from (A_7) and (3.28), (3.29) that

$$\begin{aligned}|\bar{x}(t)-\bar{x}(t+\tau)| &\le \int_{-\infty}^{t} \exp\Big(-\frac{1}{h}(t-s)\Big)\Big[\frac{(1-ph)}{h}\|\bar{x}_s-\bar{x}_{s+\tau}\|+\varepsilon\Big]ds \\ &\le (1-ph)L_0 + h\varepsilon,\end{aligned}$$

i.e., $L_0 \le (1-ph)L_0 + h\varepsilon$. Therefore there is $L_0 \le \varepsilon/p$. Thus τ is an ε/p-translation number for $\bar{x}(t)$, and since $\varepsilon > 0$ is arbitrary, $\bar{x}(t)$ is a.p. and the module of $\bar{x}(t)$ is contained in the module of F. The proof is complete.

Proof of Theorem 2.6. The proof of this theorem follows immediately from Theorems 2.2 and 2.5.

§4. An Application

The following equation can arise in a study of the dynamics of a single-species population model (cf. [4, p.123]):

$$N'(t) = N(t)(1-k(t)N(t)-l(t)\int_{-\infty}^{0} N(t+s)d\eta(s)). \tag{4.1}$$

Since in this equation $N(t)$ represents population density, we are only concerned with positive solutions, and can make the change of variable $x = \log N$ to get the equation

$$x'(t) = 1 - k(t)\exp x(t) - l(t)\int_{-\infty}^{0}(\exp x(t+s))d\eta(s). \tag{4.2}$$

This is in the form of the scalar case of (2.2) with

$$F(t,x,\phi) = 1 - k(t)\exp x(t) - l(t)\int_{-\infty}^{0}(\exp\phi(s))d\eta(s). \tag{4.3}$$

We assume that

(A_8) $\eta(s)$ is nondecreasing on $(-\infty,0]$ with $\int_{-\infty}^{0} d\eta(s) = B < \infty$.

(A_9) $k(t)$ and $l(t)$ are a.p. with $\inf\{k(t) : t \in R\} = k > 0$, and $l(t) \geq 0$ for $t \in R$.

(A_{10}) Define $d = \sup\{|1-k(t)| : t \in R\}, L = \sup\{l(t) : t \in R\}$. Then there is

$$d + LB < k \tag{4.4}$$

and there exists at least an $r > 0$ such that

$$d + LB < (ke^{-r} - LBe^{r})r. \tag{4.5}$$

Remark 4.1. Obviously, if d and LB are sufficiently small, then k must be near 1, and thus (A_{10}) must be valid.

The conditions (A_8)-(A_{10}) contain the conditions (A_1)-(A_3) and (A_5), (A_6) for (4.3). In fact, setting $M = d + LB$ and $M_1 = 1 + d_1e^r + LBe^r$, where $d_1 = \sup\{k(t) : t \in R\}$, we have

$$|F(t,0,0)| = |1 - k(t) - l(t)\int_{-\infty}^{0} d\eta(s)| \leq |1-k(t)| + l(t)\int_{-\infty}^{0} d\eta(s) \leq M$$

and $|F(t,x,\phi)| \leq M_1$ for $t \in R, |x| \leq r$ and $\|\phi\| \leq r$. Therefore it is easy to see that the conditions (A_1)-(A_3) and (A_5) are satisfied. Taking $p = (ke^{-r} - LBe^{r}) > 0$, where r is as in (A_{10}), and taking $h > 0$ such that $ph < 1$ and $h < d_1e^r/2$, from (4.5) we have $M < pr$. And it follows from the Mean Value Theorem that

$$\begin{aligned}
&|x(t) - y(t) + h(F(t,x(t),x_t) - F(t,y(t),y_t))| \\
&\leq |x(t) - y(t) - hk(t)(\exp x(t) - \exp y(t))| \\
&\quad + hl(t)\int_{-\infty}^{0} |\exp x(t+s) - \exp y(t+s)| d\eta(s) \\
&\leq |1 - hk(t)\exp\bar{x}(t)|\,|x(t)-y(t)| + hl(t)\int_{-\infty}^{0}(\exp\bar{x}(t+s)) \\
&\qquad \cdot |x(t+s) - y(t+s)| d\eta(s) \\
&\leq (1 - hke^{-r})\|x_t - y_t\| + hLe^r\|x_t - y_t\|\int_{-\infty}^{0} d\eta(s) \\
&= [1 - h(ke^{-r} - LBe^{r})]\|x_t - y_t\| \\
&= (1 - ph)\|x_t - y_t\|
\end{aligned} \tag{4.6}$$

for all $t \in R$, and any functions $x(t), y(t)$ with $|x(t)| \leq r$ and $|y(t)| \leq r$; here $\bar{x}(t) = y(t) + \theta(t)(x(t) - y(t))$ for some function $\theta(t), 0 < \theta(t) < 1$ (thus there is $|\bar{x}(t)| \leq r$), i.e., the condition (A_6) also is valid. Therefore from Theorem 2.6, we have

Theorem 4.1. *If the conditions* (A_8)-(A_{10}) *are satisfied, then* (4.1)*, and hence also* (4.2)*, has only one a.p. solution* $\bar{x}(t)$ *with* $\operatorname{mod}(\bar{x}) \subset \operatorname{mod}(l(t), k(t))$ *and* $|\bar{x}(t)| \leq r$ *for* $t \in R$

where r is as in (A_{10}). *Moreover, this a.p. solution* $\bar{x}(t)$ *is uniformly stable.*

Remark 4.2. Theorem 4.1 is much better than Theorem 3 of [1].

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