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ON A PROBLEM OF SUMS OF MIXED POWERS (II)***

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Abstract

Let $R_{b,c}(n)$ denote the number of representations of n as the sum of one square, four cubes, one b-th power and one c-th power of natural numbers. It is shown that if $b = 4$, $4 \le c \le 35$, or $b = 5, 5 \le c \le 13$, or $b = 6, 6 \le c \le 9$, or $b = c = 7$, then $R_{b,c}(n) \gg n^{5/6+1/b+1/c}$ for all sufficiently large n.

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§1. Introduction

Let $R_{b,c}(n)$ denote the number of representations of n as the sum of one square, four cubes, one b-th power and one c-th power of natural numbers. C. Hooley[3] first got an asymptotic formula for $R_{3,5}(n)$. From Brüdern's work[2], one can easily get the asymptotic formulae for $R_{3,c}(n)$ (see [5]). By improving technique, Lu[5] obtained the asymptotic formulae for $R_{4,c}(n)$, where $4 \le c \le 6$.

For the larger c's, however, it is somewhat difficult to get the asymptotic formulae of $R_{4,c}(n)$. But to obtain lower estimates of the expected order of magnitude for $R_{4,c}(n)$, furthermore, for $R_{b,c}(n)$, is of interest to us. The following results are due to Lu[5,Theorem2]: If $b = 4, 7 \le c \le 17$, or $b = 5, 5 \le c \le 9$, or $b = 6, 6 \le c \le 7$, then

$$R_{b,c}(n) \gg n^{\frac{5}{6}+\frac{1}{b}+\frac{1}{c}} \tag{1.1}$$

for all sufficiently large n.

By combining the pruning technique with Wooley's work[10], we obtain a larger rectangle of (b, c) in which (1.1) is true. Exactly, we have

Theorem. *If $b = 4, 4 \le c \le 35$, or $b = 5, 5 \le c \le 13$, or $b = 6, 6 \le c \le 9$, or $b = c = 7$, then for all sufficiently large natural number n,* (1.1) *is true.*

The manner used in the proof of each case is similar, and specially, cases $b = 4$ and $b = 5$ require furtheremore a result of Davenport's method. For the reason, the present paper gives the proof of (1.1) for the case $b = 5, 10 \le c \le 13$ only.

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§2. Notation and Auxiliary Results

Throughout η is a sufficiently small but fixed positive number, and ε is a sufficiently small positive number, not necessarily the same in different places. Let

$$X = n^{\frac{1}{105}}, \quad R = n^{\eta},$$
$$W = n^{\delta}, \quad \delta = 10^{-10}, \tag{2.1}$$

$$\mathcal{A}(n^{\frac{1}{3}}, R) = \{x : x \le n^{\frac{1}{3}}, p \text{ is prime}, \ p|x \text{ implies } p \le R\}, \tag{2.2}$$

$$g_3(\alpha) = \sum_{x \in \mathcal{A}(n^{\frac{1}{3}}, R)} e(\alpha x^3), \tag{2.3}$$

$$f_k(\alpha, m) = \sum_{\substack{x \le n^{\frac{1}{k}} \\ (x,m)=1}} e(\alpha x^k), f_k(\alpha) = f_k(\alpha, 1), \tag{2.4}$$

$$f_k(m^k\alpha) = \sum_{x \le n^{\frac{1}{k}}/m} e(\alpha m^k x^k). \tag{2.5}$$

For $S = n^{\rho}(0 < \rho \le \frac{1}{2})$ and $1 \le a \le q \le S$ with $(a, q) = 1$, let $\mathfrak{M}(q, a)$ denote the set of real numbers α with

$$|q\alpha - a| \le S/n,$$

$\mathfrak{M}$ denote their union. One can observe that $\mathfrak{M}(q, a)$ are pairwise disjoint and contained in $(S/n, 1 + S/n]$.

For $(a, q) = 1$, put

$$S_k(q, a) = \sum_{m=1}^{q} e(am^k/q), \tag{2.6}$$

$$V_k(\alpha; q, a) = q^{-1} S_k(q, a) \int_0^{n^{\frac{1}{k}}} e\Big(\Big(\alpha - \frac{a}{q}\Big)t^k\Big) dt \tag{2.7}$$

and define $V_k(\alpha)$ and $\Delta_k(\alpha)$ on $\mathfrak{M}$ by

$$V_k(\alpha) = V_k(\alpha, q, a), \quad \alpha \in \mathfrak{M}(q, a), \tag{2.8}$$

$$\Delta_k(\alpha) = f_k(\alpha) - V_k(\alpha). \tag{2.9}$$

Theorem 2 of Vaughan[7] showed that

$$\Delta_k(\alpha) \ll q^{\frac{1}{2}+\varepsilon}(1 + n|\alpha - a/q|)^{\frac{1}{2}}. \tag{2.10}$$

Lemma 2.1.[10]

$$\int_0^1 |f_3(\alpha)|^2 |g_3(\alpha)|^4 d\alpha \ll n^{\frac{13}{12}+\varepsilon}.$$

Lemma 2.2. *Suppose that* $t > \max(4, k+1)$. *Then*

$$\int_{\mathfrak{M}} |V_k(\alpha)|^t d\alpha \ll n^{\frac{t}{k}-1}.$$

Lemma 2.3. *Suppose that* $X \le M \le XW$. *Then*

$$\int_0^1 \sum_{M < p \le 2M} |f_3(\alpha; p)|^2 |f_5(p^5\alpha)|^4 d\alpha \ll n^{\frac{11}{15}+\varepsilon} M^{-1}.$$

The proofs of Lemmas 2.2 and 2.3 are similar to that of Lemma 2.3 of Lu[5] and that of Theorem 2 of Lu[4], respectively.

Next, let $\mathfrak{M}(K)$ denote the union of intervals $[\frac{a}{q}-\frac{K}{qn},\frac{a}{q}+\frac{K}{qn}]$ with

$$1\le a\le q\le K \quad\text{and}\quad (a,q)=1.$$

Lemma 2.4. *Suppose that*

$$h_k(\alpha)=\sum_{x\in\mathfrak{B}_k}e(\alpha x^k),$$

where $\mathfrak{B}_k\subset[1,n^{\frac1k}]$ *is a set of integers. If* K *is sufficiently large,* $\log n\ll\log K$*, we have*

$$\int_{\mathfrak{M}(K)}|h_k(\alpha)|^2d\alpha\ll n^{\frac1k-1}(n^{\frac1k}+K)K(\log K)^{k+2}.$$

Proof. Let $\eta(h)$ denote the number of solutions of

$$h=x^k-y^k$$

subject to $x,y\in\mathfrak{B}_k$. Obviously, we have

$$\begin{aligned}\int_{\mathfrak{M}(K)}|h_k(\alpha)|^2d\alpha&\ll\sum_{q\le K}\sum_{a=1}^{q}\sum_{|h|\le n}\eta(h)e\Big(\frac{ah}{q}\Big)\int_{-\frac{K}{qn}}^{\frac{K}{qn}}e(h\beta)d\beta\\&\ll n^{-1}K\sum_{q\le K}q^{-1}\sum_{|h|\le n}\eta(h)\sum_{a=1}^{q}e(ah/q)\\&=n^{-1}K\sum_{q\le K}\sum_{\substack{|h|\le n\\q|h}}\eta(h).\end{aligned}\tag{2.11}$$

If $(q,l)=1$, we know that the number of solutions of

$$x^k\equiv l(\bmod q)$$

is $O(k^{\omega(q)})$, where $\omega(q)$ is the number of distinct prime factors of q. Thus, we have

$$\begin{aligned}\sum_{q\le K}\sum_{\substack{|h|\le n\\q|h}}\eta(h)&\ll n^{\frac1k}K+\sum_{q\le K}\sum_{m\le n^{\frac1k}}\sum_{\substack{x\le n^{\frac1k}/m\\(x,y)=1,x\ne y\\m^k(x^k-y^k)\equiv0(\bmod q)}}\sum_{y\le n^{\frac1k}/m}1\\&\ll n^{\frac1k}K+\sum_{q}\sum_{m}\sum_{\substack{x\\x^k\equiv y^k\left(\bmod\frac{q}{(m^k,q)}\right)}}\sum_{\substack{y\\\left(y,\frac{q}{(q,m^k)}\right)=1}}1\\&\ll n^{\frac1k}K+\sum_{q}\sum_{m}\frac{n^{\frac1k}}{m}\cdot k^{\omega(\frac{q}{(m^k,q)})}\Big(1+\frac{n^{\frac1k}(q,m^k)}{qm}\Big)\\&=n^{\frac1k}K+\sum,\text{say.}\end{aligned}$$

Observing that Selberg's work[6] implies that

$$\sum_{n\le x}k^{\omega(n)}\ll x(\log x)^{k-1},$$

where the constant implied by the symbol $\ll$ depends on k only, we have

$$\begin{aligned}\sum &\ll \sum_{q\le K}\sum_{m\le n^{1/k}}\frac{n^{\frac{1}{k}}}{m}k^{\omega(q)}+\sum_{rs\le K}\sum_{m\le n^{1/k}}\sum_{s|m^k}\frac{n^{\frac{2}{k}}k^{\omega(r)}}{rm^2}\\ &\ll n^{\frac{1}{k}}K(\log K)^k+n^{\frac{2}{k}}\sum_{r\le K}\sum_{m\le n^{\frac{1}{k}}}\frac{k^{\omega(r)}d(m^k)}{rm^2}\\ &\ll n^{\frac{1}{k}}K(\log K)^k+n^{\frac{2}{k}}(\log K)^k.\end{aligned}\tag{2.12}$$

From (2.11) and (2.12), we complete the proof.

§3. The Proof of Case $b=5, 10\le c\le 13$.

By an elementary discussion, it is easy to see that

$$R_{5,c}(n)\gg\int_{\mathcal{R}}f_2(\alpha)f_3(\alpha)g_3^3(\alpha)\Big(\sum_{X\le p\le XW}f_5(p^5\alpha)\Big)f_c(\alpha)e(-n\alpha)d\alpha,\tag{3.1}$$

where $\mathcal{R}$ is any interval with length 1.

Let

$$Q_1=n^{\frac{11}{20}-\frac{1}{c}+3\delta},\quad Q_2=n^{\frac{59}{168}+7\delta},\quad Q_3=n^{\frac{197}{560}-\frac{3}{2c}+5\delta},\quad Q_4=(\log n)^{\delta}.\tag{3.2}$$

For $1\le a\le q\le Q_j$ with $(a,q)=1$ denote by $\mathfrak{M}_j(q,a)$ the set of real numbers α with $|q\alpha-a|<Q_j/n$, and denote by $\mathfrak{M}_j$ their union. Note that $\mathfrak{M}_j(q,a)$ are pairwise disjoint and $\mathfrak{M}_j\subset\mathfrak{M}_{j-1}(2\le j\le 4), \mathfrak{M}_1\subset\mathfrak{U}=(Q_1/n,1+Q_1/n]$. We choose $\mathcal{R}=\mathfrak{U}$ in (3.1) and let $\mathfrak{m}=\mathfrak{U}\backslash\mathfrak{M}_1$, and write $\sigma=\frac{31}{30}+\frac{1}{c}$ henceforth.

Lemma 3.1. *We have*

$$J_1=\int_{\mathfrak{m}}f_2(\alpha)f_3(\alpha)g_3^3(\alpha)(\sum_p f_5(p^5\alpha))f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma-\delta},\tag{3.3}$$

$$J_2=\int_{\mathfrak{M}_1}\Delta_2(\alpha)f_3(\alpha)g_3^3(\alpha)(\sum_p f_5(p^5\alpha))f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma-\delta}.\tag{3.4}$$

Proof. By Weyl's inequality, Lemma 2.1 and (3.1), we have

$$\begin{aligned}J_1&\ll\Big(\frac{n}{Q_1}\Big)^{\frac{1}{2}+\varepsilon}\Big(\int_0^1|f_3^2(\alpha)g_3^4(\alpha)|d\alpha\Big)^{\frac{1}{2}}\Big(\int_0^1|f_3(\alpha)f_5(\alpha)f_c(\alpha)|^2d\alpha\Big)^{\frac{1}{2}}\\ &\ll\Big(\frac{n}{Q_1}\Big)^{\frac{1}{2}+\varepsilon}\cdot n^{\frac{13}{24}+\varepsilon}\cdot n^{\frac{1}{6}+\frac{1}{10}+\frac{1}{2c}+\varepsilon}\\ &\ll n^{\sigma-\delta}.\end{aligned}$$

By (2.9) and in the same manner as for J_1, (3.4) can be deduced.

Lemma 3.2.

$$J_3=\int_{\mathfrak{M}_1\backslash\mathfrak{M}_2}V_2(\alpha)f_3(\alpha)g_3^3(\alpha)\Big(\sum_p f_5(p^5\alpha)\Big)f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma-\delta},\tag{3.5}$$

$$J_4=\int_{\mathfrak{M}_2\backslash\mathfrak{M}_3}V_2(\alpha)f_3(\alpha)g_3^3(\alpha)\Big(\sum_p f_5(p^5\alpha)\Big)f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma-\delta},\tag{3.6}$$

$$J_5=\int_{\mathfrak{M}_3}V_2(\alpha)\Delta_3(\alpha)g_3^3(\alpha)\Big(\sum_p f_5(p^5\alpha)\Big)f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma-\delta}.\tag{3.7}$$

Proof. By Lemmas 2.1 and 2.3, (3.1) and Lemma 2.2 of Lu[5], we have

$$\begin{aligned} J_3 &\ll (XW)^{\frac14}\Big(\int_{\mathfrak{M}_1\setminus\mathfrak{M}_2}|V_2(\alpha)|^4\Big|\sum_p f_5(p^5\alpha)\Big|^2|f_c(\alpha)|^4d\alpha\Big)^{\frac14}\Big(\int_0^1|f_3(\alpha)|^2|g_3(\alpha)|^4d\alpha\Big)^{\frac12}\\ &\quad\times\Big(\int_0^1|g_3(\alpha)|^6d\alpha\Big)^{\frac18}\Big(\int_0^1 XW\sum_p|g_3(\alpha)|^2|f_5(p^5\alpha)|^4d\alpha\Big)^{\frac18}\\ &\ll (XW)^{\frac14}(n(n^{\frac15+\frac2c}+Q_2^{-1}n^{\frac25+\frac4c}))^{\frac14}\cdot n^{\frac{13}{24}+\varepsilon}\cdot n^{\frac{13}{96}+\varepsilon}\cdot n^{\frac{11}{120}+\varepsilon}\cdot W^{\frac14}\\ &\ll n^{\sigma-\delta} \end{aligned}$$

and by Lemma 2.1 of Lu[5] and Lemma 2 of Brüdern[1], we have

$$\begin{aligned} J_4 &\ll \Big(\Big(\frac{n}{Q_3}\Big)^{\frac13}+Q_2^{\frac12+\varepsilon}\Big)\Big(\int_{\mathfrak{M}_2\setminus\mathfrak{M}_3}|V(\alpha)|^2\Big|\sum_p f_5(p^5\alpha)\Big|^2|f_c(\alpha)|^2d\alpha\Big)^{\frac12}\Big(\int_0^1|g_3(\alpha)|^6d\alpha\Big)^{\frac12}\\ &\ll \Big(\Big(\frac{n}{Q_3}\Big)^{\frac13}+Q_2^{\frac12+\varepsilon}\Big)(n^{\frac15+\frac1c}Q_2+n^{\frac25+\frac2c})^{\frac12}n^{\frac{13}{24}+\varepsilon}\\ &\ll n^{\sigma-\delta}. \end{aligned}$$

From (2.9), (3.1) and Hua's inequality, we have

$$\begin{aligned} J_5 &\ll n^{\frac1c}\cdot Q_3^{\frac12+\varepsilon}\Big(\int_{\mathfrak{M}_3}|V_2(\alpha)|^4d\alpha\Big)^{\frac14}\Big(\int_0^1|g_3(\alpha)|^6d\alpha\Big)^{\frac12}\Big(\int_0^1|f_5(\alpha)|^4d\alpha\Big)^{\frac14}\\ &\ll n^{\frac1c}\cdot Q_3^{\frac12+\varepsilon}\cdot n^{\frac14+\varepsilon}\cdot n^{\frac{13}{24}+\varepsilon}\cdot n^{\frac1{10}+\varepsilon}\\ &\ll n^{\sigma-\delta}. \end{aligned}$$

Lemma 3.3. *We have*

$$J_6=\int_{\mathfrak{M}_3\setminus\mathfrak{M}_4}V_2(\alpha)V_3(\alpha)g_3^3(\alpha)\Big(\sum_p f_5(p^5\alpha)\Big)f_c(\alpha)e(-n\alpha)d\alpha\ll n^{\sigma}(\log n)^{-\frac{\delta}{49}}. \tag{3.8}$$

Proof. By (3.1), Lemmas 2.2 and 2.4 and Hua's inequality, we have

$$\begin{aligned} J_6 &\ll \sum_{\frac{\log Q_4}{\log 2}<t\le[\frac{\log Q_3}{log 2}]+1}\int_{\mathfrak{M}(2^t)\setminus\mathfrak{M}(2^{t-1})}|V_2(\alpha)V_3(\alpha)g_3^3(\alpha)||f_c(\alpha)||\sum_p|f_5(p^5\alpha)|d\alpha\\ &\ll n^{\frac1c}\cdot\sum_t\max_{\alpha\in\mathfrak{M}(2^t)\setminus\mathfrak{M}(2^{t-1})}|V_2(\alpha)||V_3(\alpha)|^{\frac1{16}}\Big(\int_{\mathfrak{M}(2^t)}|V_3(\alpha)|^{\frac{30}{7}}d\alpha\Big)^{\frac7{32}}\\ &\quad\times\Big(\int_0^1|g_3(\alpha)|^8d\alpha\Big)^{\frac14}\Big(\int_{\mathfrak{M}(2^t)}|g_3(\alpha)|^2d\alpha\Big)^{\frac12}\Big(\int_0^1|f_5(\alpha)|^{32}d\alpha\Big)^{\frac1{32}}\\ &\ll n^{\sigma}\sum_t(2^t)^{-\frac12}\cdot(2^t)^{-\frac1{48}}\cdot t^5\cdot\Big((2^t)^{\frac12}+\frac{2^t}{n^{\frac13}}\Big)\\ &\ll n^{\sigma}\sum_t t^5(2^t)^{-\frac1{48}}\\ &\ll n^{\sigma}(\log n)^{-\frac{\delta}{49}}, \end{aligned}$$

which completes the proof.

Let

$$W^*(\alpha)=\Big(\sum_{X<p\le XW}p^{-1}\Big)W_3^3(\alpha)V_5(\alpha)V_c(\alpha),$$

where

$$W_3(\alpha)=W_3(\alpha;q,a)=q^{-1}S_3(q,a)\omega_3(\alpha-a/q),\qquad \alpha\in\mathfrak{M}_4(q,a),$$
$$\omega_3(\beta)=\sum_{Q_4^3<m\le n}\frac{1}{3}m^{-\frac{2}{3}}\rho\Big(\frac{\log m}{3\log Q_4}\Big)e(m\beta),$$

where $\rho(t)$ is Dickman's function.

From Vaughan's discussion[9], we know that

$$g_3(\alpha)=W_3(\alpha;q,a)+O\Big(\frac{n^{\frac{1}{3}}q}{\log n}(1+n|\alpha-\frac{a}{q}|)\Big),\quad \alpha\in\mathfrak{M}_4(q,a),$$

which implies that

$$g_3^3(\alpha)-W_3^3(\alpha)\ll n(\log n)^{\sigma-1},\qquad \alpha\in\mathfrak{M}_4. \tag{3.9}$$

From (2.10), we have

$$f_c(\alpha)-V_c(\alpha)\ll(\log n)^{\sigma} \tag{3.10}$$

when $\alpha\in\mathfrak{M}_4$. And by §6 of Vaughan[8], if $\alpha\in\mathfrak{M}_4(q,a)$, we have

$$\begin{aligned} f_5(p^5\alpha)&=(qp)^{-1}\sum_{x=1}^{q}e\Big(\frac{ap^5x^5}{q}\Big)v_4\Big(\alpha-\frac{a}{q}\Big)+\Delta_5(\alpha;p)\\ &=p^{-1}V_5(\alpha)+O((\log n)^{\delta}).\end{aligned} \tag{3.11}$$

From (3.1), (3.10) and (3.11), it can be deduced that

$$\Big(\sum_p f_5(p^5\alpha)\Big)g_3^3(\alpha)f_c(\alpha)-W^*(\alpha)\ll\Big(\sum_p\frac{1}{p}\Big)n^{\frac{6}{5}+\frac{1}{c}}(\log n)^{\delta-1}\ll n^{\frac{6}{5}+\frac{1}{c}}(\log n)^{\delta-1}. \tag{3.12}$$

Hence, we can easily obtain

Lemma 3.4.

$$J_7=\int_{\mathfrak{M}_4}V_2(\alpha)V_3(\alpha)(g_3^3(\alpha)\Big(\sum_p f_5(p^5\alpha)\Big)f_c(\alpha)-W^*(\alpha))e(-n\alpha)d\alpha\ll n^{\sigma}(\log n)^{-\frac{1}{2}}. \tag{3.13}$$

Now, from Lemmas 3.1 to 3.4, and (3.1), in order to prove the theorem for the case $b=5, 10\le c\le 13$, it suffices to show that

$$\int_{\mathfrak{M}_4}V_2(\alpha)V_3(\alpha)W^*(\alpha)e(-n\alpha)d\alpha\gg n^{\sigma}. \tag{3.14}$$

Similar to that of Lu[5], the proof of (3.14) can be easily obtained.

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