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CONDITIONED SUPERPROCESSES***

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Abstract

A class of superprocesses which dies out is investigated. Under the condition of nor–extinction, a new superprocess is constructed, its life time is infinite, and its distribution is determined by the moment function. Several limit theorems about this superprocess and its occupation time process are obtained.

Keywords Superprocess, Extinction time, Occupation time process
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§1. Introduction

Let E be a topological Lusin space and $M(E)$ denote the space of finite measures on E with the topology of weak convergence. $\mathcal{B}(E)$ is the space of bounded nonnegative Borel–measurable functions on E. Suppose that $\xi = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, \xi_t, P^x)$ is a Borel right Markov process with state space $(E, \mathcal{B}(E))$ and semigrop (S_t) such that $S_t 1 = 1$, so that ξ has infinite lifetime. The superprocesses $X = (\mathcal{W}, g, g_t, \Theta_t, X_t, P^\mu)$ arising from ξ is determined by the following Laplace functional

$$E^\mu[\exp(-\langle X_t, f\rangle)] = \exp(-\langle \mu, u_t\rangle). \tag{1.1}$$

Here u_t is the unique positive solution of the following equation

$$\begin{aligned} &\frac{\partial}{\partial t} u_t = A u_t - u_t^2, \\ &u(0) = f, \end{aligned} \tag{1.2}$$

where A is the infinite generator for ξ, $f \in \mathcal{B}(E)$, $\mu \in M(E)$.

We have known that X_t is an $M(E)$-valued right Markov process (see [5]). From [10, p. 286], it is extinct eventually.

Let T be the extinction time for X_t. For $\mu \in M(E) - \{0\}$, write

$$P_t^\mu[\cdot] = P^\mu\,[\cdot \mid t < T < \infty].$$

Since $P^\mu(T = \infty) = 0$, we have

$$P_t^\mu[\cdot] = P^\mu\,[\cdot \mid t < T]. \tag{1.3}$$

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Under the condition of non-extinction, we will construct a class of superprocesses (which we call conditioned superprocesses), and investigate its property and the limit property of its occupation time process

§2. The Construction of Conditioned Superprocesses

Write E_t^μ as the expectation corresponding to the law P_t^μ, and

$$X_t(f) = \langle X_t, f\rangle.$$

We first give a lemma.

Lemma 2.1 *If* $X_t = 0 \quad P^\mu$ a.s $(t>0)$ *for* $\mu \in M(E)$, *then for* $s > t$

$$X_s = 0 \quad P^\mu \text{ a.s.}$$

Proof. For $\lambda > 0$, by (1.1) and (1.2), we get

$$\begin{aligned} E^\mu[\exp(-\lambda X_s(1))] &= E^\mu[E^\mu(\exp(-\lambda X_s(1) \mid X_t)] \\ &= E^\mu[E^{X_t}\exp(-\lambda X_{s-t}(1))] \\ &= E^\mu\Big[\exp\Big(-\langle X_t, \frac{\lambda}{1+\lambda(s-t)}\rangle\Big)\Big] = 1. \end{aligned}$$

Hence

$$X_s = 0 \quad P^\mu \quad \text{a.s.}$$

Suppose that G is g_t-measurable. For $s, t > 0$, since the path of X_t is right continuous, by Lemma 2.1 and (1.3), we get

$$\begin{aligned} E_{s+t}^\mu(G) &= E^\mu[G \mid s+t < T] \\ &= \frac{E^\mu(G) - E^\mu[G; T \le s+t]}{1 - P^\mu(T \le s+t)} \\ &= \frac{E^\mu(G) - E^\mu[G; X_{s+t} = 0]}{1 - P^\mu(X_{s+t} = 0)} \end{aligned}$$

and by the Markov property of X_t,

$$\begin{aligned} E^\mu[G; X_{s+t} = 0] &= E^\mu[GE^\mu[I_{(X_{s+t}=0)} \mid g_t]] \\ &= E^\mu[GE^\mu[I_{(X_{s+t}=0)} \mid X_t]] \\ &= E^\mu[GE^{X_t}[I_{(X_s=0)}]] \\ &= E^\mu[GP^{X_t}(X_s = 0)] \\ &= E^\mu[G \cdot \exp(-X_t(1) \cdot s^{-1})]. \end{aligned}$$

Hence for $\mu \in M(E) - \{0\}$,

$$E_{s+t}^\mu(G) = \frac{E^\mu[G(1 - \exp(-X_t(1) \cdot s^{-1}))]}{1 - \exp[-\mu(1) \cdot (s+t)^{-1}]}. \tag{2.1}$$

Write

$$E_\infty^\mu(G) = \lim_{s\to\infty} E_{s+t}^\mu(G).$$

Then we get

Theorem 2.1. *For* $\mu \in M(E) - \{0\}$, *there exists a probability measure* P_∞^μ *on* $\mathcal{W}$ *with the expectation* E_∞^μ *such that*

(1) *on* g_t

$$P^{\mu}_{\infty}[\cdot] = \mu(1)^{-1} P^{\mu}[\cdot X_t(1)];$$

(2) *if* G *is nonnegative* g_t*-measurable, and* $E^{\mu}[GX_t(1)] < \infty$*, then*

$$E^{\mu}_{\infty}(G) = \mu(1)^{-1} E^{\mu}[GX_t(1)].$$

Proof. Applying the inequality

$$x - \frac{x^2}{2} < 1 - e^{-x} < x \quad (x > 0),$$

for fixed $t > 0$, and $s > \max(\frac{\mu(1)}{2}, t)$, we get

$$\begin{aligned} 0 &\le G \frac{1 - \exp(-X_t(1)s^{-1})}{1 - \exp[-\mu(s+t)^{-1}]} \\ &\le G \frac{X_t(1)s^{-1}}{\frac{\mu(1)}{s+t}\left[1 - \frac{\mu(1)}{2(s+t)}\right]} \\ &\le 4\mu(1)^{-1} G X_t(1). \end{aligned}$$

Let $s \to \infty$ in (2.1). By the dominated convergence theorem, we get (2) immediately. Hence there exists a probability measure P^{μ}_{∞} with the corresponding expectation E^{μ}_{∞} such that (1) holds (see [5]).

Remark. From Theorem 2.1 we have constructed a new class of superprocesses—conditioned superprocesses, and it is the original superprocess X_t under the condition of probability measure P^{μ}_{∞}.

Theorem 2.2. *For* $\mu \in M(E) - \{0\}, f \in \mathcal{B}(E)$*, we have*

$$E^{\mu}_{\infty}[X_t(f)]^n = \mu(1)^{-1} E^{\mu}[(X_t(f))^n \cdot X_t(1)], \quad t > 0, \quad n \in N$$

and the distribution of $X_t(f)$ *under* P^{μ}_{∞} *is determined by the moments.*

Proof. Denote $\|f\| = \sup_{x \in E} |f(x)|$. Then

$$X_t(f) \le \|f\| X_t(1).$$

Since $EX_t^n(1)$ exists, by Theorem 2.1(2), the equality in Theorem 2.2 holds.

Applyling the property of series of power which is absolutely continuous and uniformly convergent in the radius of convergence, we easily get that the radius of convergence of series of power for $E^{\mu}_{\infty}[X_t(f)]^n$ is larger than zero. By [11, Theorem 1], the distribution of $X_t(f)$ under P^{μ}_{∞} is determined by all the moments.

Theorem 2.3. *For* $\mu \in M(E) - \{0\}$*, we have*

$$P^{\mu}_{\infty} \ll P^{\mu} \quad \text{on} \quad g_t \ (t > 0).$$

Proof. Since X_t is extinct, there exists an r such that $r > t$, so that

$$X_r(1) \le 1 \ P^{\mu} \quad \text{a.s.}$$

For $A \in g_t \subset g_r$, by Theorem 2.1

$$\begin{aligned} P^{\mu}_{\infty}(A) &= E^{\mu}_{\infty}(I_A) = \mu(1)^{-1} E^{\mu}[I_A X_r(1)] \\ &\le \mu(1)^{-1} E^{\mu}(I_A) = \mu(1)^{-1} P^{\mu}(A). \end{aligned}$$

Hence on g_t

$$P^{\mu}_{\infty} \ll P^{\mu}.$$

If ξ is a d–dimensional Browinan motion on R^d, and $\mathcal{B}(R^d)$ is the Borel σ-field, then the superprocess arising from ξ is a super–Brownian motion. By Theorem 2.3 and [8, Theorem 7.3], we get

Corollary 2.1. *Suppose that X_t is a super-Brownian motion. For $A \in \mathcal{B}(R^d), \mu \in M(R^d) - \{0\}$, if $\lambda(A) = 0$ (λ is Lebesgue measure), we have*

$$X_t(A) = 0 \quad P^\mu_\infty \text{ a.s } (t > 0).$$

§3. Some Limit Theorems

Theorem 3.1. *For $\mu \in M(E) - \{0\}$, we have*

(1) *for $s > 0$, under P^μ_{s+t}*

$$t^{-1}X_t(1) \xrightarrow{W} \xi,$$

where $\xrightarrow{W}$ denotes convergence in distribution, ξ is an exponential distribution with parameter 1;

(2) *under P^μ_∞*

$$t^{-1}X_t(1) \xrightarrow{W} \xi + \eta,$$

where η is a copy of ξ;

(3) *for $a \in (0, 1)$, under P^μ_t*

$$t^{-1}X_{ta}(1) \xrightarrow{W} U + V,$$

where U and V are exponential distributions with parameters $\frac{1}{a}$ and $\frac{1}{a(1-a)}$ respectively ,and are independent of each other.

Proof. For $s > 0, \mu \in M(E) - \{0\}$, by (2.1)

$$\begin{aligned}
&E^\mu_{s+t}[\exp(-\lambda t^{-1}X_t(1))] \\
&= \frac{E^\mu[\exp(-\lambda t^{-1}X_t(1))] - E^\mu[\exp(-(\lambda t^{-1} + s^{-1})X_t(1))]}{1 - \exp[-\mu(1)(t + s)^{-1}]} \\
&= \frac{\exp[-\mu(1)\lambda(\lambda + 1)^{-1}t^{-1}] - \exp[-\mu(1)(\lambda t^{-1} + s^{-1})(1 + \lambda + ts^{-1})^{-1}]}{1 - \exp[-\mu(1)(t + s)^{-1}]}.
\end{aligned}$$

(1) For a fixed $s > 0$, we have

$$E^\mu_{t+s}\left[\exp(-\lambda t^{-1}X_t(1))\right] \overset{t\to\infty}{\longrightarrow} \frac{1}{1 + \lambda}.$$

We note that $\frac{1}{1+\lambda}$ is the Laplace functional of ξ. Hence under P^μ_{t+s}

$$t^{-1}X_t(1) \xrightarrow{W} \xi.$$

(2)

$$\begin{aligned}
&E^\mu_{t+s}\left[\exp(-\lambda t^{-1}X_t(1))\right] \\
&= \frac{\exp[-\mu(1)\lambda(\lambda + 1)^{-1}t^{-1}] \cdot [1 - \exp[-\mu(1)(1 + \lambda)^{-1}[(1 + \lambda)s + t]^{-1}]]}{1 - \exp(-\mu(1)(t + s)^{-1})} \\
&\overset{s\to\infty}{\longrightarrow} \exp[-\mu(1)\lambda(1 + \lambda)^{-1}t^{-1}] \cdot \frac{1}{(1 + \lambda)^2} \\
&\overset{t\to\infty}{\longrightarrow} \frac{1}{(1 + \lambda)^2},
\end{aligned}$$

hence

$$E^{\mu}_{\infty}\left[\exp(-\lambda t^{-1}X_t(1))\right] \xrightarrow{t\to\infty} \frac{1}{(1+\lambda)^2}.$$

Since the Laplace functional of $\xi+\eta$ is $\frac{1}{(1+\lambda)^2}$, Theorem 3.1(2) follows.

(3) Similarly, as (2.1), we get

$$\begin{aligned}
&E^{\mu}_{t}\left[\exp(-\lambda t^{-1}X_{ta}(1))\right]\\
=&\frac{E^{\mu}[\exp(-\lambda t^{-1}X_{ta}(1))]-E^{\mu}[\exp[-(\frac{\lambda}{t}+\frac{1}{t(1-a)})X_{ta}(1)]]}{1-P^{\mu}(X_t=0)}\\
=&\frac{\exp[-\mu(1)\lambda t^{-1}(1+\lambda a)^{-1}]-\exp[-\mu(1)t^{-1}(\lambda+\frac{1}{1-a})(1+\lambda a+\frac{a}{1-a})^{-1}]}{1-\exp(-\mu(1)t^{-1})}\\
\xrightarrow{t\to\infty}&(1+\lambda a)^{-1}\cdot[1+a(1-a)\lambda]^{-1}.
\end{aligned}$$

Similarly, as (1.2), under P^{μ}_{t}

$$t^{-1}X_{ta}(1)\xrightarrow{W} U+V.$$

Remark. Theorem 3.1(2) and (3) is similar to [1, Theorem I.14.3 and I.15.1].

We have known that X_t is extinct for $\mu\in M(E)$, but for the occupation time process Y_t:

$$\langle Y_t, f\rangle=\int_0^t\langle X_s,f\rangle ds,\ \ f\in\mathcal{B}(E).$$

We will investigate its limit property.

Theorem 3.2. *For $\mu\in M(E)-\{0\}$, we have*

(1) *For any $s>0$, under P^{μ}_{t+s}*

$$t^{-2}Y_t(1)\xrightarrow{W}\xi_1.$$

Here ξ_1 has the following Laplace functional

$$\phi_1(\lambda)=\frac{4\sqrt{\lambda}e^{2\sqrt{\lambda}}}{e^{4\sqrt{\lambda}}-1}\quad(\lambda>0).$$

(2) *Under P^{μ}_{∞}*

$$t^{-2}Y_t(1)\xrightarrow{W}\xi_2.$$

Here ξ_2 has the following Laplace functional

$$\phi_2(\lambda)=\frac{4e^{2\sqrt{\lambda}}}{(e^{2\sqrt{\lambda}}+1)^2}\quad(\lambda>0).$$

Proof. By [3, (1.43) and (1.44)], for $r\geq 0,\lambda>0,\mu\in M(E)$,

$$E^{\mu}[\exp(-rX_t(1)-\lambda Y_t(1))]=\exp(-\langle\mu,u_t\rangle),\tag{3.1}$$

where u_t is the unique positive solution of the following evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t)&=Au(t)-u^2(t)+\lambda,\\ u(0)&=r.\end{aligned}\tag{3.2}$$

It is easy to see that the solution is

$$u_t=\frac{\sqrt{\lambda}[(\sqrt{\lambda}+r)e^{2\sqrt{\lambda}t}-(\sqrt{\lambda}-r)]}{(\sqrt{\lambda}+r)e^{2\sqrt{\lambda}t}+(\sqrt{\lambda}-r)}.\tag{3.3}$$

For $s>0$, by (2.1), (3.1) and (3.3), we get

$$\begin{aligned}
&E^{\mu}_{s+t}\,[\exp(-\lambda t^{-2}Y_t(1))]\\
&=\frac{E^{\mu}[\exp(-\lambda t^{-2}Y_t(1))]-E^{\mu}[\exp-(\lambda t^{-2}Y_t(1)+s^{-1}X_t(1))]}{1-\exp[-\mu(1)(t+s)^{-1}]}\\
&=\frac{\exp[-\mu(1)\frac{\sqrt{\lambda}}{t}\cdot\frac{e^{2\sqrt{\lambda}}-1}{e^{2\sqrt{\lambda}}+1}]-\exp[-\mu(1)\frac{\sqrt{\lambda}}{t}\cdot\frac{(\sqrt{\lambda}t^{-1}+s^{-1})e^{2\sqrt{\lambda}}-(\sqrt{\lambda}t^{-1}-s^{-1})}{(\sqrt{\lambda}t^{-1}+s^{-1})e^{2\sqrt{\lambda}}+(\sqrt{\lambda}t^{-1}-s^{-1})}]}{1-\exp[-\mu(1)(t+s)^{-1}]}\\
&\xrightarrow{t\to\infty}\frac{4\sqrt{\lambda}e^{2\sqrt{\lambda}}}{e^{4\sqrt{\lambda}}-1};
\end{aligned}$$

this proves Theorem 3.2(1).

Similarly, as the proof of Theorem3.1 (2), we have

$$\begin{aligned}
E^{\mu}_{s+t}\,[\exp(-\lambda t^{-2}Y_t(1))]&\xrightarrow{s\to\infty}\frac{4e^{2\sqrt{\lambda}}}{(e^{2\sqrt{\lambda}}+1)^2}\cdot\exp[-\mu(1)\frac{\sqrt{\lambda}}{t}\cdot\frac{e^{2\sqrt{\lambda}}-1}{e^{2\sqrt{\lambda}}+1}]\\
&\xrightarrow{t\to\infty}\frac{4e^{2\sqrt{\lambda}}}{(e^{2\sqrt{\lambda}}+1)^2}.
\end{aligned}$$

Hence

$$E^{\mu}_{\infty}\,\exp(-\lambda t^{-2}Y_t(1))]\xrightarrow{t\to\infty}\frac{4e^{2\sqrt{\lambda}}}{(e^{2\sqrt{\lambda}}+1)^2}.$$

Theorem 3.3. *For $\mu\in M(E)-\{0\}$, under P^{μ}*

$$Y_t(1)\xrightarrow{W}\xi,$$

where ξ has the probability density

$$\phi(x)=\frac{\mu(1)}{2\sqrt{\pi}x^{\frac{3}{2}}}\exp\Big[-\frac{\mu(1))^2}{4x}\Big],\ x>0.$$

Proof. From (3.1), (3.2) and (3.3), we have

$$E^{\mu}[\exp(-\lambda Y_t(1))]=\exp(-\langle\mu,u_t\rangle),\quad\lambda>0,\tag{3.4}$$

where u_t is the unique positive solution of the following evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t}u(t)&=Au(t)-u^2(t)+\lambda,\\
u(0)&=0.
\end{aligned}\tag{3.5}$$

Its solution is

$$u_t=\sqrt{\lambda}\cdot\frac{e^{2t\sqrt{\lambda}}-1}{e^{2t\sqrt{\lambda}}+1}.$$

Hence

$$\begin{aligned}
E^{\mu}[\exp(-\lambda Y_t(1))]&=\exp(-\mu(1)\sqrt{\lambda}\cdot\frac{e^{2t\sqrt{\lambda}}-1}{e^{2t\sqrt{\lambda}}+1})\\
&\xrightarrow{t\to\infty}\exp(-\mu(1)\sqrt{\lambda}).
\end{aligned}$$

Since the r.v.ξ with the probability density $\phi(x)$ has the representation $\exp(-\mu(1)\sqrt{\lambda})$ of Laplace functional, we have under P^{μ}

$$Y_t(1)\xrightarrow{W}\xi.$$

For a particular class of superprocesses, we can investigate the limit property for Y_t. In the following we give an example.

Example. Suppose that E is discrete. If the motion process ξ has the semigroup (P_t) such that $P_t(x,\{y\})=v(\{y\}), x,y\in E, \forall t$, then

$$Y_t \xrightarrow{W} \xi\cdot v.$$

Proof. It is easy to see that v is a probability measure on E.

We note that for $t>0$

$$P^\mu(Y_t=0)=\lim_{\lambda\to+\infty}E^\mu[\exp(-\lambda Y_t(1))]=0.$$

Hence $Y_t(f)/Y_t(1)$ is well defined under P^μ.

For $\varepsilon,\ \delta>0$

$$P^\mu\Big(\Big|\frac{Y_t(f)}{Y_t(1)}-v(f)\Big|>\varepsilon\Big)\le P^\mu(|Y_t(f)-v(f)Y_t(1)|>\delta\varepsilon)+P^\mu(Y_t(1)\le\delta),$$

where $f\in C(E)$ (the space of continuous functions).

By Theorem 3.3

$$\begin{aligned}\lim_{t\to\infty}P^\mu(Y_t(1)\le\delta)&=\int_0^\delta\frac{\mu(1)}{2\sqrt{\pi}x^{\frac32}}\exp\Big(-\frac{(\mu(1))^2}{4x}\Big)dx\\&\longrightarrow 0 \text{ as } \delta\to0,\\ E^\mu[Y_t(f)-v(f)Y_t(1)]^2&=E^\mu[\langle Y_t,f-v(f)\rangle]^2\\&=-\langle u''(t),\mu\rangle+\langle u'(t),\mu\rangle^2.\end{aligned}$$

By [6]

$$u'(t)=\int_0^t(P_sf-v(f))ds,$$

$$\langle u''(t),\mu\rangle=-2\int_0^t\langle P_{t-s}[u'(s)]^2,\mu\rangle ds$$

but

$$P_sf(x)=\int_E P_s(x,dy)f(y)=\int_E v(dy)f(y)=v(f).$$

Hence

$$u'(t)=0,\qquad \langle u''(t),\mu\rangle=0.$$

We get

$$E^\mu[Y_t(f)-v(f)Y_t(1)]^2=0$$

and

$$P^\mu(|Y_t(f)-v(f)Y_t(1)|>\delta\varepsilon)=0.$$

Therefore

$$\lim_{t\to\infty}P^\mu\Big(\Big|\frac{Y_t(f)}{Y_t(1)}-v(f)\Big|>\varepsilon\Big)=0,$$

under P^μ

$$\frac{Y_t(f)}{Y_t(1)}\xrightarrow{P}v(f).$$

By Theorem 3.3 and [7, Theorem 4.2]

$$Y_t \xrightarrow{W} \xi \cdot v.$$

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