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THE K-THEORY AND THE CLASSIFICATION OF THE IRRATIONAL ROTATION GROUPOID C^*-ALGEBRAS**

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Abstract

The authors determine the invariants of the irrational rotation C^*-algebras over the L-shaped domain in $\mathbb{C}^2$ by the maximal radical series.

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§0. Introduction

In [7], R. E. Curto and P. S. Muhly discussed the Toeplitz C^*-algebra $C^*(\Omega)$ over the L-shaped domain Ω in $\mathbb{C}^2$ by groupoid appoach, showing that $C^*(\Omega)$ is isomorphic to some groupoid C^*-algebra $C^*(G)$. In our former paper[20], we discussed the irrational rotation C^*-algebra $C^*(G(C_\theta))$ for $C^*(G)$, presenting the maximal radical series of the rotational C^*-algebra.

$$\{0\} \triangleleft C^*(G(C_\theta)'') \triangleleft C^*(G(C_\theta)') \triangleleft C^*(G(C_\theta)),$$

which is invariant under the isomorphism. In this paper, we will use the method in [19] to calculate the K-groups of the series and classify the rotaional C^*-algebra $C^*(G(C_\theta))$.

§1. On the Simplicity of the C^*-Algebra $C^*(\mathbb{R}\times\mathbb{T}\times_\theta\mathbb{Z}^2)$

In our former paper[20], we have seen that the transformation group C^*-algebra $C^*(\mathbb{R}\times\mathbb{T}\times_\theta\mathbb{Z}^2)$, where the action of $\mathbb{Z}^2$ on $\mathbb{R}\times\mathbb{T}$ is $(s,t)+p=(s+2p_2\ln\delta_2-2p_1\ln\delta_1, t\theta_1^{p_1}\theta_2^{p_2})$, occurs frequently in the discussion. A natural problem arises when the C^*-algebra $C^*(\mathbb{R}\times\mathbb{T}\times_\theta\mathbb{Z}^2)$ is simple. In this section we will determine the necessary and sufficient condition to the problem.

In order to solve the problem, we need the following well-known classification theorem of the closed groups in the Euclidian space $\mathbb{R}^N$.

Theorem A. *Any closed subgroup in the Euclidian space $\mathbb{R}^N$ is the orthogonal sum of a linear subspace and a free subgroup finitely generated. The linear subspace is called the linear part of the group, while the free subgroup is called the free part of the group.*

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As a corollary, we have the following criterion for a subgroup to be dense in the Euclidian space.

Theorem B. *Suppose that G is a subgroup in the Euclidian space $\mathbb{R}^N$. G is dense in $\mathbb{R}^N$ if and only if the projection of G in every direction u ($u \in \mathbb{R}^N$) is dense in the line $\mathbb{R}u$.*

Proof. The necessity is obvious.

Conversely, if G is not dense in the whole space, there are linearly independent vectors $u_1, u_2, \cdots, u_n$ in $\mathbb{R}^N$, orthogonal to the linear part of $\overline{G}$, such that

$$\overline{G} = L \oplus \mathbb{Z}u_1 \oplus \mathbb{Z}u_2 \oplus \cdots \oplus \mathbb{Z}u_n,$$

where L is the linear part of the group $\overline{G}$.

Since the vectors $u_1, u_2, \cdots, u_n$ are linearly independent, the matrix $((u_i, u_j))_{n\times n}$ is non singular. Fix any rational n-tuple $(s_1, s_2, \cdots, s_n) \in \mathbb{Q}^n \setminus \{0\}$, there is a unique real n-tuple $(r_1, r_2, \cdots, r_2) \in \mathbb{R}^n$, such that for $i = 1, 2, \cdots, n$, $\sum\limits_{j=1}^{n}(u_i, u_j)r_j = s_i$. Let $u = \sum\limits_{j=1}^{n} r_j u_j$. Then the projection of G in the direction u is $\Big\{\Big(\sum\limits_{j=1}^{n} p_j s_j\Big)\dfrac{u}{\|u\|^2}\Big| p \in \mathbb{Z}^n\Big\}$. However since $s_1, s_2, \cdots, s_n$ are all rational numbers, the coefficient set $\Big\{\sum\limits_{j=1}^{n} p_j s_j \Big| p \in \mathbb{Z}^n\Big\}$ must be of the form $\mathbb{Z}\gamma$ for some $\gamma \in \mathbb{R}_+$, a contradiction. The conclusion follows.

Theorem 1.1 *The groupoid C^*-algebra $C^*(\mathbb{R} \times \mathbb{T} \times_\theta \mathbb{Z}^2)$ is simple if and only if the real numbers $\dfrac{1}{\ln\delta_1}, \dfrac{1}{\ln\delta_2}, \dfrac{\psi_1}{2\pi\ln\delta_1} + \dfrac{\psi_2}{2\pi\ln\delta_2}$ are linearly independent over the field $\mathbb{Q}$ of the rational numbers, where ψ_1 and ψ_2 are the arguments of θ_1 and θ_2, respectively.*

Proof. Since the groupoid $\mathbb{R} \times \mathbb{T}_\theta \times \mathbb{Z}^2$ is r-discrete and principal (because $\ln\delta_1/\ln\delta_2$ is irrational), there is an order-preserving isomorphism between the family of the invariant open subsets of the unit space $\mathbb{T} \times \mathbb{R}$ and the lattice of the closed ideals in the groupoid C^*-algebra $C^*(\mathbb{R} \times \mathbb{T}_\theta \times \mathbb{Z}^2)$.

Define the distance function d on $\mathbb{R} \times \mathbb{T}$ by $d((s,t),(s',t')) = |s - s'| + |t - t|'$. Then the distance is invariant under the action of $\mathbb{Z}^2$ on $\mathbb{R} \times \mathbb{T}$. For each $v \in \overline{[u]}$ there is a sequence $\{p_m\}_{m=1}^\infty$ in $\mathbb{Z}^2$ such that $v = \lim\limits_{m\to\infty}(u + p_m)$. However

$$\lim_{m\to\infty} d(u, v - p_m) = \lim_{m\to\infty} d(u + p_m, v) = 0.$$

It follows that the closed orbits in $\mathbb{R} \times \mathbb{T}$ are either disjoint or identical and therefore they are the minimal invariant closed subsets in the unit space. Consequently one orbit's being dense in $\mathbb{R} \times \mathbb{T}$ implies other's same property.

By the above discussion, it suffices to determine when there is a dense orbit and when there is not.

Define the map $\varphi : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{T}$ by $\varphi(s_1, s_2) = (s_1, \exp(2\pi\sqrt{-1}s_2))$. Then φ is a continuous and open homomorphism. The orbit containing (0,1) in $\mathbb{R} \times \mathbb{T}$ is $\{(2p_2\ln\delta_2 - 2p_1\ln\delta_1, \theta^p) | p \in \mathbb{Z}^2\}$, denoted by [(0,1)]. Then

$$\varphi^{-1}([(0,1)]) = \Big\{\Big(2p_2\ln\delta_2 - 2p_1\ln\delta_1, p_1\frac{\psi_1}{2\pi} + p_2\frac{\psi_2}{2\pi} + q\Big)\Big| p \in \mathbb{Z}^2, q \in \mathbb{Z}\Big\},$$

which is a subgroup of $\mathbb{R}^2$. Given any direction $u \in \mathbb{R}^2$, the projection of the group

$\varphi^{-1}([(0,1)])$ in it is a group

$$G_u = \Big\{ \Big(p_1\Big(\frac{\psi_1}{2\pi}u_2 - 2u_1 \ln\delta_1\Big) + p_2\Big(\frac{\psi_2}{2\pi}u_2 + 2u_1\ln\delta_2\Big) + qu_2 \Big) \frac{u}{|u|^2} \Big| p \in \mathbb{Z}^2, q \in \mathbb{Z} \Big\}.$$

Suppose that $\dfrac{1}{\ln\delta_1}, \dfrac{1}{\ln\delta_2}, \dfrac{\psi_1}{2\pi\ln\delta_1} + \dfrac{\psi_2}{2\pi\ln\delta_2}$ are linearly independent over the field of rational numbers.

If $u_2 = 0$, then G_u is dense in the line $\mathbb{R}u$ since $\dfrac{\ln\delta_1}{\ln\delta_2}$ is irrational.

If $u_2 \neq 0$, assume without loss of generality that $u_2 = 1$.

(1) If $\dfrac{\psi_1}{2\pi} - 2u_1\ln\delta_1$ is irrational, then G_u is dense in the line $\mathbb{R}u$.

(2) If $\dfrac{\psi_1}{2\pi} - 2u_1\ln\delta_1$ is rational, then for some $r \in \mathbb{Q}$, $u_1 = \left(\dfrac{\psi_1}{2\pi} + r\right)\dfrac{1}{2\ln\delta_1}$. The coefficient of p_2 in the group G_u is $\dfrac{\psi_2}{2\pi} + \left(\dfrac{\psi_1}{2\pi} + r\right)\dfrac{\ln\delta_2}{\ln\delta_1}$, which is irrational since $\dfrac{1}{\ln\delta_1}, \dfrac{1}{\ln\delta_2}$, $\dfrac{\psi_1}{2\pi\ln\delta_1} + \dfrac{\psi_2}{2\pi\ln\delta_2}$ are linearly independent over $\mathbb{Q}$. Hence G_u is dense in the line $\mathbb{R}\,u$.

By Theorem B, the group $\varphi^{-1}([(0,1)])$ is dense in $\mathbb{R}\times\mathbb{R}$ and by Lemma 1.1 below the orbit $[(1,0)]$ is dense in $\mathbb{R}\times\mathbb{T}$.

If $\dfrac{1}{\ln\delta_1}, \dfrac{1}{\ln\delta_2}, \dfrac{\psi_1}{2\pi\ln\delta_1} + \dfrac{\psi_2}{2\pi\ln\delta_2}$ are linearly dependent over the rational numbers $\mathbb{Q}$, then for some $r, r_1, r_2 \in \mathbb{Q}$,

$$\frac{r_1}{\ln\delta_1} + \frac{r_2}{\ln\delta_2} + r\left(\frac{\psi_1}{2\pi\ln\delta_1} + \frac{\psi_2}{2\pi\ln\delta_2}\right) = 0.$$

It follows that $r \neq 0$ since $\dfrac{\ln\delta_2}{\ln\delta_1}$ is irrational. So we can choose $r = 1$. Set

$$u = (u_1, u_2) = \Big(\Big(\frac{\psi_1}{2\pi} + r_1\Big)\frac{1}{2\ln\delta_1}, 1\Big).$$

Then the coefficient of p_2 in G_u is

$$\frac{\psi_2}{2\pi} + 2u_1\ln\delta_2 = \frac{\psi_2}{2\pi} + \Big(\frac{\psi_1}{2\pi} + r_1\Big)\frac{\ln\delta_2}{\ln\delta_1} = \ln\delta_2\Big(\frac{\psi_2}{2\pi\ln\delta_2} + \frac{\psi_1}{2\pi\ln\delta_1} + \frac{r_1}{\ln\delta_1}\Big) = -r_2,$$

which is rational and the coefficient of p_1 in G_u is

$$\frac{\psi_1}{2\pi} - 2u_1\ln\delta_1 = \frac{\psi_1}{2\pi} - \left(\frac{\psi_1}{2\pi} + r_1\right) = -r_1,$$

which is also rational. Therefore the group G_u is not dense in the line $\mathbb{R}u$. By Theorem B, the group $\varphi^{-1}([(1,0)])$ is not dense in $\mathbb{R}\times\mathbb{R}$ and therefore the orbit [(1,0)] is not dense in the unit space $\mathbb{R}\times\mathbb{T}$ by Lemma 1.1 below.

Lemma 1.1. *Any continuous surjective homomorphism between the locally compact and second countable groups is open.*

If $f : X \to Y$ is a surjective continuous map, the following conditions are equivalent.

(1) *The map f is open.*

(2) *For any subset B of Y, $f^{-1}(\mathrm{int}(B)) = \mathrm{int} f^{-1}(B)$.*

(3) *For any subset F of Y, $f^{-1}(\overline{F}) = \overline{f^{-1}(F)}$.*

§2. Main Result

From now on we will concentrate on the calssification of $C^*(G(C_\theta))$. We will mainly consider the ideals

$$J_1 = \mathbf{I}(\alpha(\mathbb{Z}_+^2)), \quad J_2 = \mathbf{I}(\alpha(\mathbb{Z}_+^2 \cup \{\infty\} \times \mathbb{Z}_+ \cup \mathbb{Z}_+ \times \{\infty\}),$$
$$J_2' = \mathbf{I}(X \setminus \beta\{-\infty, +\infty\}),$$

and the quotient

$$\begin{aligned} J_2'/J_1 &= C^*(G|_{\alpha(\mathbb{Z}_+ \times \{\infty\})}) \oplus C^*(G|_{\alpha(\{\infty\}\times\mathbb{Z}_+}) \oplus C^*(\beta(\mathbb{R}) \times \mathbb{Z}^2) \\ &\cong \mathbf{K}(l^2(\mathbb{Z}_+)) \otimes C(\mathbb{T}) \oplus C(\mathbb{T}) \otimes \mathbf{K}(l^2(\mathbb{Z}_+)) \oplus C^*(\mathbb{R} \times \mathbb{Z}^2) \\ &\cong \mathbf{K}(\mathrm{H}^2(\mathbb{T})) \otimes C(\mathbb{T}) \oplus C(\mathbb{T}) \otimes \mathbf{K}(\mathrm{H}^2(\mathbb{T})) \oplus C^*(\mathbb{R} \times \mathbb{Z}^2). \end{aligned}$$

Let u denote the function in $C(\mathbb{T})$ given by $\chi(z) = z$ and s the unilateral shift on $\mathrm{H}^2(\mathbb{T})$. Then $e = 1 - ss^*$ is a projection of rank 1 in $\mathbf{K}(\mathrm{H}^2(\mathbb{T}))$. Now we set

$$e_1 = 1 \otimes e, \quad e_2 = e \otimes 1, \quad u_1 = e \otimes u,$$
$$u_2 = u \otimes e, \quad s_1 = s \otimes e, \quad s_2 = e \otimes s.$$

It is easy to calculate the K-groups in the Proposition 2.1.

Proposition 2.1. (1) *$K_0(J_1) = \mathbb{Z}[e \otimes e]$ and $K_1(J_1) = 0$.*

(2) *$K_0(J_2'/J_1) = K_0(C(\mathbb{T}) \otimes \mathbf{K} \oplus \mathbf{K} \otimes C(\mathbb{T}) \oplus C^*(\mathbb{R} \times \mathbb{Z}^2))$ and*

$$K_0(C(\mathbb{T}) \otimes \mathbf{K}) = \mathbb{Z}[e_1], \quad K_0(\mathbf{K} \otimes C(\mathbb{T})) = \mathbb{Z}[e_2].$$

By Connes' Thom isomorphism the group $K_0(C^*(\mathbb{R} \times \mathbb{Z}^2)$ is free abelian of rank 2 (see the later part of the proof of Lemma 2.9) with geneators $[u] \otimes [e]$ and $[e] \otimes [u]$.

(3) *$K_1(J_2'/J_1) = K_1((C(\mathbb{T}) \otimes \mathbf{K} \oplus \mathbf{K} \otimes C(\mathbb{T}) \oplus C^*(\mathbb{R} \times \mathbb{Z}))$ and*

$$K_1(C(\mathbb{T}) \otimes \mathbf{K}) = \mathbb{Z}[u_1], \quad K_1(\mathbf{K} \otimes C(\mathbb{T})) = \mathbb{Z}[u_2].$$

And again by Connes' Thom isomorphism $K_1(C^*(\mathbb{R} \times \mathbb{Z}^2))$ is free abelian of rank 2 with generators $[e] \otimes [e]$ and $[u] \otimes [u]$.

Proposition 2.2. *$K_0(J_2') = K_0(J_2'/J_1)$ and $K_1(J_2'/J_1)$ is free abelian with rank 3. $K_1(J_2')$ has a generator $[v_1]$ such that $K_1(\pi)([v_1]) = [u_1] - [u_2]$.*

Proof. Applying the six-term exact sequence to the short exact sequence

$$0 \to J_1 \to J_2' \to J_2'/J_1 \to 0,$$

we obtain the exact sequence

$$\begin{array}{ccccc} K_0(J_1) & \xrightarrow{j_{*0}} & K_0(J_2') & \xrightarrow{\pi_{*0}} & K_0(J_2'/J_1) \\ \uparrow & & & & \downarrow{\scriptstyle 0} \\ K_1(J_2'/J_1) & \xleftarrow{\pi_{*1}} & K_1(J_2') & \xleftarrow{j_{*1}} & K_1(J_1). \end{array}$$

Since $K_1(J_1) = 0$ and the inclusion $j : J_1 \to J_2'$ is the composition of the inclusion $i : J_2 \to J_2'$ with the inclusion $k : J_1 \to J_2$ while the latter's K_0-level homomorphism k_{*0} is zero as shown in [19], we get that $j_{*0} = 0$ and therefore π_{*0} is isomorphic, i.e., $K_0(J_2') \cong K_0(J_2'/J_1)$.

Next we will determine one generator of the group $K_1(J_2')$, which is important in the calculation of other C^*-algebras' K-groups.

The subset $\alpha((\mathbb{N} \cup \{\infty\}) \times \{0\})$ is both open and compact in the unit space X and therefore the characteristic function $\chi_{\alpha((\mathbb{N}\cup\{\infty\})\times\{0\})\times\{(-1,0)\}}$, denoted by f, is in $C_c(G)$. However $\mathrm{ind}\delta_{\alpha(0)}(f) = s_1$, hence s_1 is in $C^*(G)$ and a fortiori s_2 is in $C^*(G)$.

Now that $1-e_i+u_i$'s belong to $U_1(J_2'/J_1)$, it follows that if we denote $[1-e_i+u_i\}$ briefly by $[u_i]$ and let

$$v_1=\begin{pmatrix}1-e_1+s_1 & e\otimes e\\ & 1-e_2+s_2^*\end{pmatrix},$$

then v_1 belongs to $U_2(J_2')$ and $K_1(\pi)([v_1])=[u_1]-[u_2]$.

Proposition 2.3. *Let p_θ be the Rieffel projection of the rotation C^*-algebra A_θ, then $K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z})=\mathbb{Z}^8$ with base $\{[e_1],[p_{\theta_1}],[e_2],[p_{\theta_2}],[e]\otimes[u],[u]\otimes[e],[e]\otimes[e]\}$.*

Proof. Similar to [4.2, 19]. We just point out that α_θ preserves the direct sum.

Proposition 2.4. *$K_0(J_2'\times_{\alpha_\theta}\mathbb{Z})$ is free abelian with rank 7 and base $\{[e_1],[e_2],d_i|1\le i\le 5\}$.*

(1) *The boundary map $\partial: K_0(J_2'\times_{\alpha_\theta}\mathbb{Z})\to K_1(J_2')$ in the Pimsner-Voiculescu sequence is given by $\partial([e_k])=0$ for $1\le k\le 2$ and $\partial(d_1)=v_1$.*

(2) *The induced K-group homomorphism $\pi_{*0}: K_0(J_2'\times_{\alpha_\theta}\mathbb{Z})\to K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z})$ is partially given by $\pi_{*0}([e_k])=[e_k]$, for $k=1,2$, and $\pi_{*0}(d_1)=-[p_{\theta_1}]+[p_{\theta_2}]$.*

Proof. The short exact sequences

$$0\to J_1\to J_2'\to J_2'/J_1\to 0,\quad 0\to J_1\times_{\alpha_\theta}\mathbb{Z}\to J_2'\times_{\alpha_\theta}\mathbb{Z}\to J_2'/J_1\times_{\alpha_\theta}\mathbb{Z}\to 0$$

yield the communicative diagram of the K-groups

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(J_1) & \xrightarrow{K_1(i)} & K_1(J_1\times_{\alpha_\theta}\mathbb{Z}) & \xrightarrow{\partial_t} & K_0(J_1) & \longrightarrow & 0\\
 & & \partial_0\Big\uparrow & & \partial_0\Big\uparrow & & \partial_1\Big\uparrow & & \\
0 & \longrightarrow & K_0(J_2'/J_1) & \xrightarrow{K_0(i)} & K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z}) & \xrightarrow{\partial_w} & K_1(J_2'/J_1) & \longrightarrow & 0\\
 & & \Big\uparrow & & K_0(\pi)\Big\uparrow & & K_1(\pi)\Big\uparrow & & \\
0 & \longrightarrow & K_0(J_2') & \longrightarrow & K_0(J_2'\times_{\alpha_\theta}\mathbb{Z}) & \xrightarrow{\partial_b} & K_1(J_2') & \longrightarrow & 0.
\end{array}\tag{2.1}$$

Let d_1' be the lift of $[v_1]$, i.e., $\partial_b d_1'=[v_1]$, and $K_0(\pi)d_1'=m_1e_1+m_2e_2+l_1p_{\theta_1}+l_2p_{\theta_2}$.

Since functor K_i's preserve the direct sum[21] and each component in the direct sum J_2'/J_1 is invariant under the isomorphism α_θ, the homomorphism $\partial_w: K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z})\to K_1(J_2'/J_1)$ preserves the direct sum.

Let $d_1'=d_1-m_1[e_1]-m_2[e_2]$. Since $K_0(\pi)[e_i]=[e_i]$, $K_0(\pi)(d_1)=l_1p_{\theta_1}+l_2p_{\theta_2}$ and d_1 is the lift of v_1, we have

$$K_1(\pi)\partial_b(d_1)=[u_1]-[u_2],\quad \partial_w[p_{\theta_i}]=-[u_i].$$

If we set $K_0(\pi)(d_1)=l_1p_{\theta_1}+l_2p_{\theta_2}+*$, then $\partial_w*=0$, and therefore $*=0$. All these imply $l_1=-1$ and $l_2=1$, i.e., $K_0(\pi)(d_1)=-p_{\theta_1}+p_{\theta_2}$.

Proposition 2.5. $\partial_0: K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z})\to K_1(J_1\times_{\alpha_\theta}\mathbb{Z})=\mathbb{Z}$. $\partial_0([e_k])=0$, $\partial_0[p_{\theta_k}]=1$, *the generator of $K_1(J_1\times_{\alpha_\theta}\mathbb{Z})$, pre-image of $[e\otimes e]$.*

Proof.

$$\partial_0([e_k])=\partial_0K_0(i)([e_k])=K_1(i)\partial_0([e_k])=0.$$
$$\partial_t\partial_0(p_{\theta_k})=\partial_l\partial_m(p_{\theta_k})=\partial_l(-u_k)=-[e\otimes e].$$

Since ∂_t is isomorphic, $\partial_0(p_{\theta_k})=1$.

From now on, we will determine the invariants of the irrational rotation C^*-algebra $C^*(\Omega) \times_{\alpha_\theta} \mathbb{Z}$.

Suppose that the irrational rotation C^*-algebras $C^*(\Omega) \times_{\alpha_\theta} \mathbb{Z}$ and $C^*(\Omega) \times_{\alpha_\rho} \mathbb{Z}$ are isomorphic with isomorphism h. Then we have the communicative diagram

$$\begin{array}{ccccc} C^*(G''(C_\theta)) & \longrightarrow & C^*(G'(C_\theta)) & \longrightarrow & C^*(G(C_\theta)) \\ \text{isom.}\big\downarrow h_0 & & \text{isom.}\big\downarrow h_1 & & \text{isom.}\big\downarrow h \\ C^*(G''(C_\rho)) & \longrightarrow & C^*(G'(C_\rho)) & \longrightarrow & C^*(G(C_\rho)), \end{array}$$

i.e.

$$\begin{array}{ccccc} J_1 \times_{\alpha_\theta} \mathbb{Z} & \longrightarrow & J_2' \times_{\alpha_\theta} \mathbb{Z} & \longrightarrow & C^*(G) \times_{\alpha_\theta} \mathbb{Z} \\ \text{isom.}\big\downarrow h_0 & & \text{isom.}\big\downarrow h_1 & & \text{isom.}\big\downarrow h \\ J_1 \times_{\alpha_\rho} \mathbb{Z} & \longrightarrow & J_2' \times_{\alpha_\rho} \mathbb{Z} & \longrightarrow & C^*(G) \times_{\alpha_\rho} \mathbb{Z}. \end{array} \tag{2.2}$$

The isomorphism h induces an isomorphism between the first quotient algebras

$$h' : J_2'/J_1 \times_{\alpha_\theta} \mathbb{Z} \to J_2'/J_1 \times_{\alpha_\rho} \mathbb{Z}. \tag{2.3}$$

Equivalently we have

$$\begin{aligned} &(\mathbf{K}(l^2(\mathbb{Z}_+)) \otimes C(\mathbb{T})) \times_{\alpha_\theta} \mathbb{Z} \oplus (C(\mathbb{T}) \otimes \mathbf{K}(l^2(\mathbb{Z}_+))) \times_{\beta_\theta} \mathbb{Z} \oplus C^*(\mathbb{R} \times_\theta \mathbb{Z}^2) \times_{\gamma_\theta} \mathbb{Z} \\ \cong\ &(\mathbf{K}(l^2(\mathbb{Z}_+)) \otimes C(\mathbb{T})) \times_{\alpha_\rho} \mathbb{Z} \oplus (C(\mathbb{T}) \otimes \mathbf{K}(l^2(\mathbb{Z}_+))) \times_{\beta_\rho} \mathbb{Z} \oplus C^*(\mathbb{R} \times_\rho \mathbb{Z}^2) \times_{\gamma_\rho} \mathbb{Z}, \end{aligned} \tag{2.4}$$

where $\alpha_\theta(x \otimes f) = u_{\theta_1} x u_{\theta_1}^* \otimes f\varphi_{\theta_2}$, with the unitary operator u_θ defined by $u_{\theta_1}(\epsilon_n) = \theta_1^n \epsilon_n$ and the homeomorphism φ_{θ_2} defined by $\varphi_{\theta_2}(\lambda) = f(\lambda\theta_2)$, $\beta_\theta(f \otimes x) = f\varphi_{\theta_1} \otimes u_{\theta_2} x u_{\theta_2}^*$, with u_{θ_2} and φ_{θ_1} defined in the same ways as u_{θ_1} and φ_{θ_2}, respectively, and $\gamma_\theta(f)(x,p) = \theta^p f(x,p)$ for $f \in C^*(\mathbb{R} \times \mathbb{Z}^2) \subset C_0(\mathbb{R} \times \mathbb{Z}^2)$ (see Propositions II. 4.2, II. 5.1 and II. 5.7 in [1]).

Definition 2.1. *The C^*-algebra A is non-complementary if for any proper closed ideal I of A there is no proper closed ideal J of A such that $I \cap J = 0$ and $I + J = A$.*

Lemma 2.1. *Suppose that the C^*-algebras A_i's and B_i's are non-complementary. Then $A_1 \oplus \cdots \oplus A_n \cong B_1 \oplus \cdots B_n$ if and only if there is a permutation σ on $\{1, 2, \cdots, n\}$ such that $A_i \cong B_{\sigma(i)}$ for $i = 1, 2, \cdots, n$.*

Proof. Suppose that $\varphi : \bigoplus\limits_{i=1}^n A_i \to \bigoplus\limits_{i=1}^n B_i$ is an isomorphism.

For $a \in A_1$, $\varphi(a) = (b_1, \cdots, b_n)$. Since $\varphi(A_1)$ is a closed ideal in $\bigoplus\limits_{i=1}^n B_i$, for the approximate unit u_j of B_1, $(u_j b_1, 0, \cdots, 0)$ is in $\varphi(A)$. We get that $(b_1, 0, \cdots, 0)$ is in $\varphi(A_1)$ and a fortiori $(0, b_2, 0, \cdots, 0)$, $\cdots$, $(0, 0, \cdots, b_n)$ are in $\varphi(A_1)$. And now we have the decomposition $\varphi(A_1) = I_{11} \oplus I_{12} \oplus \cdots \oplus I_{1n}$, where I_{1j} is the closed ideal of B_j for $j = 1, 2, \cdots, n$. Repeat the procedure above we get for $i = 1, 2, \cdots, n$, $\varphi(A_i) = I_{i1} \oplus I_{i2} \oplus \cdots \oplus I_{1n}$, where I_{ij} is the closed ideal of B_j for $j = 1, 2, \cdots, n$. And now it is easy to verify that for $j = 1, 2, \cdots, n$, $B_j = I_{1j} \oplus I_{2j} \oplus \cdots \oplus I_{nj}$. Since A_i's and B_j's are non-complementary, we find a permutation σ on $\{1, 2, \cdots, n\}$ such that $\varphi(A_i) = B_{\sigma(i)}$.

Remark 2.1. Being connected, $\mathbb{R} \times \mathbb{T}$ has no disjoint pair of invariant proper open subsets U_1 and U_2 such that $U_1 \dot{\cup} U_2 = \mathbb{R} \times \mathbb{T}$. Hence the C^*-algebras $C^*(\mathbb{R} \times \mathbb{T} \times_\theta \mathbb{Z}^2)$ is non-complementary. The C^*-algebras $C^*(G(C_\theta)|_{\alpha(\mathbb{Z}_+ \times \{\infty\}) \times \mathbb{T}})$ and $C^*(G(C_\theta)|_{\alpha(\{\infty\} \times \mathbb{Z}_+) \times \mathbb{T}})$ are simple and therefore non-complementary.

Lemma 2.2. *The K-groups $K_*((\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}) = \mathbb{Z}^2$ and $K_*((C^*(\mathbb{R}\times\mathbb{Z}^2)\times_{\gamma_\rho}\mathbb{Z}) = \mathbb{Z}^4$. As a corollary, the C^*-algebras $(\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}$ and $(C^*(\mathbb{R}\times\mathbb{Z}^2)\times_{\gamma_\rho}\mathbb{Z}$ are not isomorphic.*

Proof. Apply the Pimsner-Voiculescu exact sequence to the crossed product $(\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}$**,** we obtain

$$\begin{array}{ccccc} K_0(\mathbf{K}\otimes C(\mathbb{T})) & \xrightarrow{1-\alpha_{\theta *}} & K_0(\mathbf{K}\otimes C(\mathbb{T})) & \xrightarrow{l_*} & K_0((\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1((\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}) & \xleftarrow{l_*} & K_1(\mathbf{K}\otimes C(\mathbb{T})) & \xleftarrow{1-\alpha_{\theta *}} & K_1(\mathbf{K}\otimes C(\mathbb{T})) \end{array}$$

Now that $K_*(\mathbf{K}\otimes C(\mathbb{T})) = \mathbb{Z}$ and the isomorphism α_θ is homotopic to the identity 1 and hence $\alpha_{\theta *} = 1$, we get $K_*((\mathbf{K}\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}) = \mathbb{Z}^2$.

Since $C^*(\mathbb{R}\times\mathbb{Z}^2)\cong C(\mathbb{T}^2)\times_\gamma\mathbb{R}$ (see the remark below), the Connes' Thom isomorphisms yield $K_i(C^*(\mathbb{R}\times\mathbb{Z}^2)) = K_{1-i}(C(\mathbb{T}^2)) = \mathbb{Z}^2$, and now again applying the Pimsner-Voiculescu exact sequence to the crossed product $C^*(\mathbb{R}\times\mathbb{Z}^2)\times_{\alpha_\theta}\mathbb{Z}$, we get that $K_*(C^*(\mathbb{R}\times\mathbb{Z}^2)\times_{\alpha_\theta}\mathbb{Z}) = \mathbb{Z}^4$.

Remark 2.2. Suppose that G and G' are locally compact and second countable abelian groups and there is a continuous homomorphism $\varphi: G\to G'$. Then the transformation group (G', G) with action $g' + g = g' + \varphi(g)$ is just the skew product groupoid $G\times_\varphi G'$, while the latter's groupoid C^*-algebra, by the Proposition 5.7 in [1], is isomorphic to the crossed product $C^*(G)\times_\gamma \hat{G}'$ where $\gamma_\sigma(f)(g) = \sigma(\varphi(g))f(g)$ for $f\in C_c(G)$.

By Lammas 2.1, 2.2 and Remark 2.1, we get that

$$\begin{cases} h'(C^*(\mathbb{R}\times\mathbb{T}\times_\theta\mathbb{Z}^2)) = C^*(\mathbb{R}\times\mathbb{T}\times_\rho\mathbb{Z}^2), \\ h'((\mathbf{K}(l^2(\mathbb{Z}_+))\otimes C(\mathbb{T}))\times_{\alpha_\theta}\mathbb{Z}\oplus(C(\mathbb{T})\otimes\mathbf{K}(l^2(\mathbb{Z}_+)))\times_{\beta_\theta}\mathbb{Z}) \\ \ =(\mathbf{K}(l^2(\mathbb{Z}_+))\otimes C(\mathbb{T}))\times_{\alpha_\rho}\mathbb{Z}\oplus(C(\mathbb{T})\otimes\mathbf{K}(l^2(\mathbb{Z}_+)))\times_{\beta_\rho}\mathbb{Z}. \end{cases} \tag{2.5}$$

Now applying the exact sequence of the K-groups to the the following communicative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & J_1\times_{\alpha_\theta}\mathbb{Z} & \longrightarrow & J_2'\times_{\alpha_\theta}\mathbb{Z} & \longrightarrow & J_2'/J_1\times_{\alpha_\theta}\mathbb{Z} & \longrightarrow & 0 \\ \downarrow & & {\scriptstyle h_0}\downarrow & & {\scriptstyle h_1}\downarrow & & {\scriptstyle h'}\downarrow & & \downarrow \\ 0 & \longrightarrow & J_1\times_{\alpha_\rho}\mathbb{Z} & \longrightarrow & J_2'\times_{\alpha_\rho}\mathbb{Z} & \longrightarrow & J_2'/J_1\times_{\alpha_\rho}\mathbb{Z} & \longrightarrow & 0 \end{array}$$

we obtain the communicative diagram

$$\begin{array}{ccccc} & & K_1(J_1\times_\rho\mathbb{Z}) & & \\ & & {\scriptstyle \partial'}\uparrow & & \\ K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z}) & \xrightarrow{h_u} & K_0(J_2'/J_1\times_{\alpha_\rho}\mathbb{Z}) & \longrightarrow & 0 \\ {\scriptstyle K_0(\pi)}\uparrow & & {\scriptstyle K_0(\pi)'}\uparrow & & \\ K_0(J_2'\times_{\alpha_\theta}\mathbb{Z}) & \xrightarrow{h_d} & K_0(J_2\times_{\alpha_\rho}\mathbb{Z}) & \longrightarrow & 0. \end{array} \tag{2.6}$$

From the above, we know that $\{[e_i], [p_{\theta_i}], l_i\}$ is the base of $K_0(J_2'/J_1\times_{\alpha_\theta}\mathbb{Z})$ and

$$h_u = \begin{pmatrix} x_1 & y_1 \\ z_1 & w_1 \end{pmatrix}\oplus\begin{pmatrix} x_2 & y_2 \\ z_2 & w_2 \end{pmatrix}\oplus A,$$

where we do not need the action of A. By a similar argument as in [19], we get that $z_i = 0$ and $w_1 = w_2$ by equalities $0 = \partial' h_u K_0(\pi)([e_k]) = z_k, \quad 0 = \partial' h_u K_0(\pi)(d_1) = -w_1 + w_2$. The same analysis as in [19] yields

$$\begin{cases} x_1 = x_2 = 1, \\ w_k = 1 \implies y_k = 0. \end{cases} \tag{2.7}$$

or

$$\begin{cases} x_1 = x_2 = 1, \\ w_k = -1 \implies y_k = 1. \end{cases} \tag{2.8}$$

(2.7) implies $\theta_i = \rho_i$ for $i = 1, 2$ and (2.8) implies $\theta_i = \overline{\rho_i}$ for $i = 1, 2$.

Finally we can state our main result.

Theorem 2.1. *If the irrational rotation C^*-algebras $C^*(G) \times_{\alpha_\theta} \mathbb{Z}$ and $C^*(G) \times_{\alpha_\rho} \mathbb{Z}$ are isomorphic, then up to a permutation $\theta_1 = \rho_1$ and $\theta_2 = \rho_2$, or $\theta_1 = \overline{\rho_1}$ and $\theta_2 = \overline{\rho_2}$.*

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