Chin. Ann. of Math.
18B: 3(1997),265-276.

BERGMAN TYPE OPERATOR ON MIXED NORM SPACES WITH APPLICATIONS

Ren Guangbin* Shi Jihuai*

Abstract

The authors investigate the conditions for the boundedness of Bergman type operators $P_{s,t}$ in mixed norm space $L_{p,q}(\varphi)$ on the unit ball of $\mathbf{C}^n$ $(n \geq 1)$, and obtain a sufficient condition and a necessary condition for general normal function φ, and a sufficient and necessary condition for

$$\varphi(r) = (1-r^2)^{\alpha} \log^{\beta}(2(1-r)^{-1}) \quad (\alpha > 0, \beta \geq 0).$$

This generalizes the result of Forelli-Rudin[3] on Bergman operator in Bergman space. As applications, a more natural method is given to compute the duality of the mixed norm space, solve the Gleason's problem for mixed norm space and obtain the characterization of mixed norm space in terms of partial derivatives. Moreover, it is proved that $f \in L^{(0)}_{\infty,q}(\varphi)$ iff all the functions $(1-|z|^2)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial z^{\alpha}}(z) \in L^{(0)}_{\infty,q}(\varphi)$ for holomorphic function f, $1 \leq q \leq \infty$.

Keywords Bergman projection, Normal function, Mixed norm space
1991 MR Subject Classification 32A30, 47B38
Chinese Library Classification O174.56

§1. Introduction

Let B denote the unit ball of complex vector space C^n. v is the Lebesgue measure on C^n normalized so that $v(B) = 1$, σ is the surface measure on the boundary ∂B of B with $\sigma(\partial B) = 1$.

A positive continuous function φ on [0,1) is normal, if there exist $0 < a < b, 0 \leq r_0 < 1$ such that

(i) $\frac{\varphi(r)}{(1-r)^a}$ is nonincreasing for $r_0 \leq r < 1$ and $\lim\limits_{r\to 1} \frac{\varphi(r)}{(1-r)^a} = 0$;

(ii) $\frac{\varphi(r)}{(1-r)^b}$ is nondecreasing for $r_0 \leq r < 1$ and $\lim\limits_{r\to 1} \frac{\varphi(r)}{(1-r)^b} = \infty$.

For $0 < p \leq \infty, 0 < q \leq \infty$, and a normal function φ, let $L_{p,q}(\varphi)$ denote the space of measurable complex function on B with

$$||f||_{p,q,\varphi} = \left\{ \int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r) M_q^p(r,f)dr \right\}^{1/p} < \infty, \quad 0 < p < \infty,$$
$$||f||_{\infty,q,\varphi} = \sup_{0<r<1} \varphi(r) M_q(r,f) < \infty,$$

Manuscript received October 25, 1995. Revised March 20, 1996.

*Department of Mathematics, University of Science and Technology of China, Hefei 230026, China.

**Project supported by the National Natural Science Foundation of China and the Doctoral Program Fundation of the State Education Commission of China.

where

$$M_q(r,f)=\left\{\int_{\partial B}|f(r\zeta)|^q d\sigma(\zeta)\right\}^{\frac{1}{q}},\quad 0<q<\infty,$$
$$M_\infty(r,f)=\sup\{|f(r\zeta)|:\zeta\in\partial B\}.$$

Denote

$$L^{(0)}_{\infty,q}(\varphi)=\left\{f\in L_{\infty,q}(\varphi):\lim_{r\to 1}\varphi(r)M_q(r,f)=0\right\}.$$

If for $dv_\alpha(z)=(1-|z|^2)^\alpha dv(z)$ $(\alpha>-1)$, let $L^p(v_\alpha)$ $(0<p<\infty)$ denote the space of measurable complex function on B with

$$\int_B|f(z)|^p dv_\alpha(z)=\int_B|f(z)|^p(1-|z|^2)^\alpha dv(z)<\infty,$$

then from the integral formula in polar coordinates

$$\int_B|f(z)|^p(1-|z|^2)^\alpha dv(z)=2n\int_0^1 r^{2n-1}(1-r^2)^\alpha M_p^p(r,f)dr,$$

we obtain

$$L^p(dv_\alpha)=L_{p,p}((1-r^2)^{(\alpha+1)/p}).$$

We use $H(B)$ to denote the class of all holomorphic functions on B. Let

$$H_{p,q}(\varphi)=L_{p,q}(\varphi)\cap H(B),$$

$$H^{(0)}_{\infty,q}(\varphi)=L^{(0)}_{\infty,q}(\varphi)\cap H(B)$$

denote holomorphic mixed norm space. By the monotonicity of the integral means $M_q(r,f)$ of holomorphic function, the norm in $H_{p,q}(\varphi)$ is equivalent to

$$||f||_{p,q,\varphi}=\left\{\int_0^1(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\right\}^{1/p}\quad(0<p<\infty,0<q\le\infty).$$

For $s\in\mathbf{R},t>0$, let $P_{s,t}$ be the Bergman type operator defined by

$$P_{s,t}f(z)=c_{n,t}(1-|z|^2)^s\int_B\frac{(1-|w|^2)^{t-1}f(w)}{(1-\langle z,w\rangle)^{n+t+s}}dv(z),$$

where the complex power is understood to be principal branches,

$$c_{n,t}=\binom{n+t-1}{n}=\frac{\Gamma(n+t)}{\Gamma(t)\Gamma(n+1)},$$

$\langle z,w\rangle=\sum\limits_{i=1}^n z_i\overline{w_i}, z=(z_1,\cdots,z_n), w=(w_1,\cdots,w_n)$.

For $s=0$, Forelli-Rudin[3] first obtained a sufficient and necessary condition on the boundedness of $P_{0,t}$ for $L^p(v)$ $(1\le p<\infty)$, and subsequently Kolaski[6], Gadbois[4], Choe[2] and Zhu[13] investigated the operator $P_{s,t}$ in their interesting problems and got the corresponding results. In this paper we will consider the action of the operator $P_{s,t}$ on the space $L_{p,q}(\varphi)$. Our results contain those of the above-mentioned authors. Our main result is

Theorem A. *Let φ be a normal function with constants a, b as in the definition of normal function.*

(i) *If $P_{s,t}:L_{p,q}(\varphi)\longrightarrow L_{p,q}(\varphi)$ $(1\le p<\infty,1\le q<\infty)$ is a bounded operator, then $s>-b,\ t>a$.*

(ii) *If* $t > b > a > -s$, *then* $P_{s,t} : L_{p,q}(\varphi) \longrightarrow L_{p,q}(\varphi)$ $(1 \le p \le \infty, 1 \le q \le \infty)$ *and* $P_{s,t} : L^{(0)}_{\infty,q}(\varphi) \longrightarrow L^{(0)}_{\infty,q}(\varphi)$ $(1 \le q \le \infty)$ *are bounded operators.*

(iii) *If* $\varphi(r) = (1-r^2)^\alpha \log^\beta(2(1-r)^{-1})$ $(\alpha > 0, \beta \ge 0)$, *then* $P_{s,t} : L_{p,q}(\varphi) \longrightarrow L_{p,q}(\varphi)$ $(1 \le p < \infty, 1 \le q < \infty)$ *is a bounded operator iff* $t > \alpha > -s$.

(iv) *If* $t > b$, *then* $P_{0,t} : L_{p,q}(\varphi) \longrightarrow H_{p,q}(\varphi)$ $(1 \le p \le \infty, 1 \le q \le \infty)$ *is a bounded linear operator. Moreover* $P_{0,t}f = f$ *for any* $f \in H_{p,q}(\varphi)$ $(1 \le p \le \infty, 1 \le q \le \infty)$.

Before proving Theorem A, we first give some of its simple corollaries.

Corollary 1.1. *For* $1 \le p < \infty, \alpha > -1$, $P_{s,t}$ *is a bounded operator on* $L^p(dv_\alpha)$ *iff* $-sp < \alpha + 1 < pt$.

Proof. Taking $p = q$, $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ in Theorem A(iii) gives Corollary 1.1.

For $n = 1$, Zhu[13] proved Corollary 1.1 on the unit disc.

Corollary 1.2. *For* $1 \le p < \infty, 1 \le q < \infty, \alpha > -1$, *then* $P_{0,t} : L_{p,q}((1-r^2)^\alpha) \longrightarrow H_{p,q}((1-r^2)^\alpha)$ *is a bounded linear operator iff* $pt > \alpha + 1$.

Proof. Taking $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ in Theorem A(iii), (iv) gives Corollary 1.2.

Gadbois[4] proved the necessity of Corollary 1.2 for $1 < p < \infty, 1 \le q < \infty, \alpha > -1$. Forelli-Rudin[3] and Choe[2] proved Corollary 1.2 for $p = q, \alpha = 0$ and $p = q, \alpha > -1$ respectively.

It is worth to point out that the proofs of the above authors depend on the following important facts:

$$\int_B |K_t(z,w)|(1-|w|^2)^{-a} dV(w) \le C(1-|z|^2)^{-a}, \qquad \forall z \in B, 0 < a < t,$$

$$\int_B |K_t(z,w)|(1-|z|^2)^{-a} dV(z) \le C(1-|w|^2)^{-a}, \qquad \forall w \in B, 0 < a + t - 1 < t,$$

where

$$K_t(z,w) = \frac{(1-|w|^2)^{t-1}}{(1-\langle z,w \rangle)^{n+t}}, \qquad z, w \in B,$$

and our method is different from this.

The proof of Theorem A will be given in section two.

As the first application of Theorem A, we investigate the Gleason problem on $H_{p,q}(\varphi)$ and $H^{(0)}_{\infty,q}(\varphi)$, and obtain the following

Theorem B. *Gleason problem can be solved on* $H_{p,q}(\varphi)$ $(1 \le p, q \le \infty)$ *and* $H^{(0)}_{\infty,q}(\varphi)$ $(1 \le q \le \infty)$. *More precisely , for any integer* $m \ge 1$, *there exists bounded linear operators* A_α *on* $H_{p,q}(\varphi)$ $(H^{(0)}_{\infty,q}(\varphi))$ *such that if* $f \in H_{p,q}(\varphi)$ $(H^{(0)}_{\infty,q}(\varphi)), D^\alpha f(0) = 0$ $(|\alpha| \le m-1)$, *then* $f(z) = \sum\limits_{|\alpha|=m} z^\alpha A_\alpha f(z)$ *on* B, *where* $\alpha = (\alpha_1, \cdots, \alpha_n)$ *is multiindex,* $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For $p = q$, $1 \le p < \infty$, Zhu[12] and Choe[2] proved Theorem B in the cases $\varphi(r) = (1-r^2)^{1/p}$ and $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ respectively.

Using Theorem A and Theorem B, we obtain the characterization of $H_{p,q}(\varphi)$ in terms of partial derivatives. That is the following

Theorem C. *Let* φ *be a normal function,* $f \in H(B)$, m *be an integer.*

(i) $f \in L_{p,q}(\varphi)$ $(1 \le p, q \le \infty)$ *iff* $(1-|z|^2)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \in L_{p,q}(\varphi)$ *for all* α *with* $|\alpha| = m$.

(ii) $f \in L^{(0)}_{\infty,q}(\varphi)$ $(1 \le q \le \infty)$ *iff* $(1-|z|^2)^{|\alpha|}\frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \in L^{(0)}_{\infty,q}(\varphi)$ *for all* α *with* $|\alpha| = m$.

For $p = q$, $1 \le p < \infty$, Zhu[12] and Choe[2] proved Theorem C(i) in the cases of $\varphi(r) = (1-r^2)^{1/p}$ and $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ respectively. Shi[8] proved Theorem C(i) in the case of $\varphi(r) = (1-r^2)^{(\alpha+1)/p}$ and $p = q,\ 0 < p < \infty$.

In the proof of Theorem C, we will use the following result of Jevtic[5] on the duality of $H_{p,q}(\varphi)$ $(1 \le p < \infty, 1 \le q \le \infty)$.

Theorem D. *For* $1 \le p < \infty, 1 \le q \le \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$, *let* φ, ψ *be normal functions,* $\varphi(r)\psi(r) = (1-r^2)^\beta$. *Then* $(H_{p,q}(\varphi))^* = H_{p',q'}(\psi)$, *with pairing*

$$(f, g) = \int_B f(z)\overline{g(z)}(1-|z|^2)^{\beta-1}dv(z).$$

Using Theorem A, we will give a new proof of Theorem D in the case of $1 \le p < \infty, 1 \le q < \infty$, which seems to be more natural.

The proofs of Theorems B–D will be given in section three.

In what follows, C denotes a finite positive constant, not necessarily the same at each occurrence.

§2. The Proof of Theorem A

The following lemmas will be needed in the proof of Theorem A.

Lemma 2.1. *If* $1 \le q \le \infty, s + t > 0$, *then*

$$M_q(\rho, P_{s,t}f) \le C(1-\rho^2)^s \int_0^1 \frac{r^{2n-1}(1-r^2)^{t-1}}{(1-r\rho)^{t+s}} M_q(r, f)dr.$$

Proof. By the integral formula in polar coordinates, we have

$$P_{s,t}f(z) = 2nc_{n,t}(1-|z|^2)^s \int_0^1 r^{2n-1}dr \int_{\partial B} \frac{(1-r^2)^{t-1}f(r\zeta)}{(1-\langle z, r\zeta\rangle)^{n+t+s}}d\sigma(\zeta).$$

Taking $z = \rho\xi,\ \xi \in \partial B$, gives

$$\begin{aligned} |(P_{s,t}f)(\rho\xi)| &\le C(1-\rho^2)^s \int_0^1 r^{2n-1}(1-r^2)^{t-1}dr \int_{\partial B} \frac{|f(r\zeta)|d\sigma(\zeta)}{|1-\langle \rho\xi, r\zeta\rangle|^{n+t+s}} \\ &= C(1-\rho^2)^s \int_0^1 r^{2n-1}(1-r^2)^{t-1}\tilde{f}(r, \rho, \xi)dr, \end{aligned} \tag{2.1}$$

where

$$\tilde{f}(r, \rho, \xi) = \int_{\partial B} \frac{|f(r\zeta)|}{|1-\langle \rho\xi, r\zeta\rangle|^{n+t+s}}d\sigma(\zeta). \tag{2.2}$$

If $1 < q < \infty$, then by Holder's inequality and the formula in [7, 1.4.10] we have

$$\begin{aligned} \tilde{f}(r, \rho, \xi) &\le \left\{\int_{\partial B} \frac{|f(r\zeta)|^q d\sigma(\zeta)}{|1-\langle \rho\xi, r\zeta\rangle|^{n+s+t}}\right\}^{1/q} \left\{\int_{\partial B} \frac{d\sigma(\zeta)}{|1-\langle \rho\xi, r\zeta\rangle|^{n+s+t}}\right\}^{1/q'} \\ &\le C\frac{1}{(1-r\rho)^{\frac{s+t}{q'}}} \left\{\int_{\partial B} \frac{|f(r\zeta)|^q d\sigma(\zeta)}{|1-\langle \rho\xi, r\zeta\rangle|^{n+s+t}}\right\}^{1/q}, \end{aligned}$$

where $\frac{1}{q} + \frac{1}{q'} = 1$.

Thus using Minkowski's inequality we obtain

$$\begin{aligned}
&M_q(\rho, P_{s,t}f)\\
&\le C(1-\rho^2)^s\left\{\int_{\partial B}\left(\int_0^1 r^{2n-1}(1-r^2)^{t-1}\tilde{f}(r,\rho,\xi)dr\right)^q d\sigma(\xi)\right\}^{\frac{1}{q}}\\
&\le C(1-\rho^2)^s\int_0^1 r^{2n-1}(1-r^2)^{t-1}\left\{\int_{\partial B}(\tilde{f}(r,\rho,\xi))^q d\sigma(\xi)\right\}^{1/q}dr\\
&\le C(1-\rho^2)^s\int_0^1 \frac{r^{2n-1}(1-r^2)^{t-1}}{(1-r\rho)^{\frac{t+s}{q'}}}\left\{\int_{\partial B}\left(\int_{\partial B}\frac{|f(r\zeta)|^q d\sigma(\zeta)}{|1-\langle\rho\xi, r\zeta\rangle|^{n+s+t}}\right)d\sigma(\xi)\right\}^{1/q}dr\\
&\le C(1-\rho^2)^s\int_0^1 \frac{r^{2n-1}(1-r^2)^{t-1}}{(1-r\rho)^{t+s}}M_q(r,f)dr.
\end{aligned}$$

If $q=1,\ \infty$, then the lemma follows from (2.1) and (2.2) directly. This proves the assertion made about $M_q(\rho, P_{s,t}f)$.

Remark 2.1 If $1\le q\le\infty$, $s+t>0$, then we have actually proved that

$$M_q(\rho,\tilde{P}_{s,t}f)\le C(1-\rho^2)^s\int_0^1\frac{r^{2n-1}(1-r^2)^{t-1}}{(1-r\rho)^{t+s}}M_q(r,f)dr,$$

where

$$\tilde{P}_{s,t}f(z)=c_{n,t}(1-|z|^2)^s\int_B\frac{(1-|w|^2)^{t-1}|f(w)|}{|1-\langle z,w\rangle|^{n+t+s}}dv(w).$$

Lemma 2.2.[11,Lemma 6] *For $0<\rho<1, s_1>s_2>0$,*

$$\int_0^1\frac{(1-r)^{s_2-1}}{(1-r\rho)^{s_1}}dr\le C\frac{1}{(1-\rho)^{s_1-s_2}}.$$

Lemma 2.3. *Let φ be a normal function. If $s+t>b>a>s$, then*

$$\int_0^1\frac{\varphi^p(\rho)d\rho}{(1-\rho)^{ps+1}(1-r\rho)^{pt}}\le C\frac{\varphi^p(r)}{(1-r)^{p(s+t)}}\quad(0\le r<1, p>0). \tag{2.3}$$

Proof. From the definition of normal function, there exists $0\le r_0<1$ such that for $r_0\le r<1, r\to 1$,

$$\frac{\varphi(r)}{(1-r)^a}\searrow 0,\quad \frac{\varphi(r)}{(1-r)^b}\nearrow\infty.$$

By the assumption $s+t>b$, using Lemma 2.2 gives

$$\begin{aligned}
&\int_0^1\frac{\varphi^p(\rho)d\rho}{(1-\rho)^{ps+1}(1-r\rho)^{pt}}=\int_0^{r_0}+\int_{r_0}^r+\int_r^1\\
&\le C+C\frac{\varphi^p(r)}{(1-r)^{pb}}\frac{1}{(1-r)^{pt-pb+ps}}+C\frac{\varphi^p(r)}{(1-r)^{pa}}\frac{1}{(1-r)^{pt-pa+ps}}\\
&\le C\frac{\varphi^p(r)}{(1-r)^{p(s+t)}}.
\end{aligned}$$

The last step is because $\frac{\varphi^p(r)}{(1-r)^{p(s+t)}}$ has a positive minimum in [0,1]. Lemma 2.3 is proved.

Lemma 2.4. *Let $1<p<\infty, 1\le q<\infty, \frac{1}{p}+\frac{1}{p'}=1, \frac{1}{q}+\frac{1}{q'}=1, \varphi$ be a normal function. Then $((L_{p,q}(\varphi))^*=L_{p',q'}(\varphi^{\frac{p}{p'}})$. The pairing is given by*

$$(f,g)=\frac{1}{2n}\int_B f(z)\overline{g(z)}(1-|z|)^{-1}\varphi^p(|z|)dv(z). \tag{2.4}$$

More precisely, $T \in (L_{p,q}(\varphi))^*$ *iff there is a unique function* $g \in L_{p',q'}(\varphi^{\frac{p}{p'}})$ *such that for any* $f \in L_{p,q}(\varphi)$, $Tf = (f,g)$ *and* $||T|| = ||g||_{p',q',\varphi^{\frac{p}{p'}}}$.

Proof. With the notation of [1], we can write

$$||f||_{p,q,\varphi} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_q^p(r,f)dr\right\}^{\frac{1}{p}} = ||\ ||f_r||_{L^q(\partial B,d\sigma)}||_{L^p(I,d\mu)}, \tag{2.5}$$

where $f_r(z) = f(rz), I = [0,1], d\mu = r^{2n-1}(1-r)^{-1}\varphi^p(r)dr$.

Thus, according to [1], the norm of $(L_{p,q}(\varphi))^*$ is

$$||\ ||g_r||_{L^{q'}(\partial B,d\sigma)}||_{L^{p'}(I,d\mu)} = \left\{\int_0^1 r^{2n-1}(1-r)^{-1}\varphi^p(r)M_{q'}^{p'}(r,g)dr\right\}^{\frac{1}{p'}} = ||g||_{p',q',\varphi^{\frac{p}{p'}}}.$$

Hence $(L_{p,q}(\varphi))^* = L_{p',q'}(\varphi^{\frac{p}{p'}})$.

Again according to [1], the pairing is

$$(f,g) = \int_I \left(\int_{\partial B} f(r\zeta)\overline{g(r\zeta)}d\sigma(\zeta)\right)d\mu = \frac{1}{2n}\int_B f(z)\overline{g(z)}(1-|z|)^{-1}\varphi^p(|z|)dv(z).$$

The rest of the proof is a direct corollary of Theorem 1 in [1]. Lemma 2.4 is proved.

We are now ready to give the

Proof of Theorem A. (i) Assume that $P_{s,t}$ is bounded on $L_{p,q}(\varphi)$. Let N be a positive integer large enough such that $f_N(z) = (1-|z|^2)^N \in L_{p,q}(\varphi)$. Thus

$$(P_{s,t}f_N)(z) = c_{n,t}(1-|z|^2)^s \int_B \frac{(1-|w|^2)^{N+t-1}}{(1-\langle z,w\rangle)^{n+t+s}}dv(w).$$

Write the integrand in terms of its Taylor series. It follows easily from the orthogonality of $\{w^\alpha\}$ (α be multiindex) in B that the above integral is a constant, that is $(P_{s,t}f_N)(z) = C(1-|z|^2)^s$. Hence the boundedness of $P_{s,t}$ on $L_{p,q}(\varphi)$ implies

$$\begin{aligned}\infty > ||P_{s,t}f_N||_{p,q,\varphi}^p &\geq C\int_0^1 (1-r)^{sp-1}r^{2n-1}\varphi^p(r)dr \\ &\geq C\int_\epsilon^1 (1-r)^{sp-1+pb}dr,\ (r_0 < \epsilon < 1),\end{aligned} \tag{2.6}$$

thus $s > -b$.

On the other hand, if $1 < p < \infty$, then from Lemma 2.4 the boundedness of $P_{s,t}$ on $L_{p,q}(\varphi)$ is equivalent to that of $P_{s,t}^*$ on $L_{p',q'}(\varphi^{\frac{p}{p'}})$, where $P_{s,t}^*$ is the adjoint operator of $P_{s,t}$. It is easy to compute that

$$P_{s,t}^*f(z) = c_{n,t}\frac{(1-|z|^2)^t}{\varphi^p(|z|)}\int_B \frac{(1-|w|^2)^{s-1}\varphi^p(|w|)f(w)}{(1-\langle z,w\rangle)^{n+t+s}}dv(w).$$

The fact that $g_N(z) = \frac{(1-|z|^2)^N}{\varphi^p(|z|)} \in L_{p',q'}(\varphi^{\frac{p}{p'}})$ for sufficiently large N and the boundedness of $P_{s,t}^*$ on $L_{p',q'}(\varphi^{\frac{p}{p'}})$ give

$$\infty > ||P_{s,t}^*g_N||_{p',q',\varphi^{\frac{p}{p'}}}^{p'} \geq C\int_\epsilon^1 (1-r)^{p't-1-p'a}dr \quad (r_0 < \epsilon < 1). \tag{2.7}$$

Hence $t > a$

If $p = 1$, then from [1] and (2.5), the norm of $(L_{p,q}(\varphi))^*$ is given by

$$||\ ||\cdot||_{L^{q'}(\partial B,d\sigma)}||_{L^\infty(I,d\mu)},$$

and the pairing is given by

$$(f,g)=\int_I\left(\int_{\partial B} f(r\zeta)\overline{g(r\zeta)}d\sigma(\zeta)\right)d\mu=\frac{1}{2n}\int_B f(z)\overline{g(z)}(1-|z|)^{-1}\varphi(|z|)dv(z).$$

Thus in this case the adjoint operator of $P_{s,t}$ is given by

$$P^*_{s,t}f(z)=c_{n,t}\frac{(1-|z|^2)^t}{\varphi(|z|)}\int_B\frac{(1-|w|^2)^{s-1}\varphi(|w|)f(w)}{(1-\langle z,w\rangle)^{n+t+s}}dv(w).$$

As in the proof of the case $p>1$, since $g_N(z)=\frac{(1-|z|^2)^N}{\varphi(|z|)}\in(L_{1,q}(\varphi))^*$ for sufficiently large N, the boundedness of $P^*_{s,t}$ on $(L_{p,q}(\varphi))^*$ gives

$$\infty>||\ ||(P^*_{s,t}g_N)_r||_{L^{q'}(\partial B,d\sigma)}||_{L^\infty(I,d\mu)}=\sup_{0<r<1}C\frac{(1-r^2)^t}{\varphi(r)}.$$

Thus the condition $\lim_{r\to1}\frac{\varphi(r)}{(1-r)^a}=0$ implies $t>a$.

(ii) We now prove that $P_{s,t}:L_{p,q}(\varphi)\to L_{p,q}(\varphi),\ 1\le p,q\le\infty$ is a bounded operator.

Write t as $t=t_1+t_2=t_3+t_4$, which satisfies

(a) $t_i>0, i=1,2,3,4,$ (b) $a+t_1>t_3>t_1$,

(c) $t_3+s>t_1$, (d) $t_2>b$.

For example, taking a sufficiently small $\epsilon>0$, and assuming

$$t_1=t-(1+\epsilon)b,\ t_2=(1+\epsilon)b,\ t_3=t-(1+\epsilon)b+(1-\epsilon)a,\ t_4=(1+\epsilon)b-(1-\epsilon)a,$$

we see that t_1,t_2,t_3 and t_4 satisfy the above conditions.

We first prove that for $1<p<\infty, 1\le q\le\infty$,

$$M_q(\rho,P_{s,t}f)\le\frac{C}{(1-\rho)^{t_3-t_1}}\left\{\int_0^1\frac{(1-r^2)^{pt_2-1}r^{p(2n-1)}}{(1-r\rho)^{pt_4}}M_q^p(r,f)dr\right\}^{\frac1p}. \tag{2.8}$$

In fact, using Lemma 2.1, Lemma 2.2 and Holder's inequality we obtain

$$\begin{aligned}&M_q(\rho,P_{s,t}f)\\&\le C(1-\rho^2)^s\left\{\int_0^1\frac{r^{(2n-1)p}(1-r^2)^{pt_2-1}}{(1-r\rho)^{pt_4}}M_q^p(r,f)dr\right\}^{\frac1p}\left\{\int_0^1\frac{(1-r^2)^{p't_1-1}}{(1-r\rho)^{p'(t_3+s)}}dr\right\}^{\frac1{p'}}\\&\le\frac{C}{(1-\rho)^{t_3-t_1}}\left\{\int_0^1\frac{r^{p(2n-1)}(1-r^2)^{pt_2-1}}{(1-r\rho)^{pt_4}}M_q^p(r,f)dr\right\}^{\frac1p}.\end{aligned}$$

Thus, when $1<p<\infty, 1\le q\le\infty$, from (2.8) and Lemma 2.3 we have

$$\begin{aligned}||P_{s,t}f||^p_{p,q,\varphi}&\le C\int_0^1(1-r)^{pt_2-1}r^{2n-1}M_q^p(r,f)dr\int_0^1\frac{\varphi^p(\rho)}{(1-\rho)^{p(t_3-t_1)+1}(1-r\rho)^{pt_4}}d\rho\\&\le C\int_0^1(1-r)^{-1}r^{2n-1}\varphi^p(r)M_q^p(r,f)dr=C||f||^p_{p,q,\varphi}.\end{aligned}$$

When $p=1$, the result follows from Lemma 2.1 and Lemma 2.3 directly.

When $p=\infty$, from $t>b>a>-s$, there exists a positive number β such that $\beta>b$ and $\beta+s>b>a>\beta-t$, for example $\beta=(1-\epsilon)b+a$ (ϵ be sufficiently small). Let $\psi(r)=\frac{(1-r)^\beta}{\varphi(r)}$. Then ψ is a normal function as well. From Lemma 2.1 and Lemma 2.3 we obtain

$$\begin{aligned}||P_{s,t}f||_{\infty,q,\varphi} &\le C\sup_{0<\rho<1}\varphi(\rho)(1-\rho)^s\int_0^1\frac{(1-r)^{t-\beta-1}\psi(r)}{(1-r\rho)^{t+s}}\varphi(r)M_q(r,f)dr\\ &\le C||f||_{\infty,q,\varphi}\sup_{0<\rho<1}\varphi(\rho)(1-\rho)^s\int_0^1\frac{\psi(r)}{(1-r)^{\beta+1-t}(1-r\rho)^{t+s}}dr\\ &\le C||f||_{\infty,q,\varphi}.\end{aligned}$$

Next we prove that $P_{s,t}$ is also a bounded linear operator on $L^{(0)}_{\infty,q}(\varphi)$ $(1\le q\le\infty)$.

For any $0<\delta<1$,

$$\begin{aligned}\varphi(\rho)M_q(\rho,P_{s,t}f) &\le C\varphi(\rho)(1-\rho)^s\int_0^1\frac{(1-r)^{t-\beta-1}\psi(r)}{(1-r\rho)^{t+s}}\varphi(r)M_q(r,f)dr\\ &\le C\varphi(\rho)(1-\rho)^s\left(\int_\delta^1+\int_0^\delta\right)\\ &\le C\varphi(\rho)(1-\rho)^s\sup_{\delta<r<1}\varphi(r)M_q(r,f)\int_0^1\frac{(1-r)^{t-\beta-1}\psi(r)}{(1-r\rho)^{t+s}}dr\\ &\quad+C\varphi(\rho)(1-\rho)^s||f||_{\infty,q,\varphi}\int_0^\delta\frac{(1-r)^{t-\beta-1}\psi(r)}{(1-r\rho)^{t+s}}dr\\ &\le C\sup_{\delta<r<1}\varphi(r)M_q(r,f)+C(\delta)||f||_{\infty,q,\varphi}(1-\rho)^{a+s}=I_1+I_2.\end{aligned}$$

Since $f\in L^{(0)}_{\infty,q}(\varphi)$ and $a>-s$, we can choose δ so that the first term is less than the given ϵ. Then the second term goes to zero as $\rho\to1$, thus $P_{s,t}$ is a bounded linear operator on $L^{(0)}_{\infty,q}(\varphi)$.

Remark 2.2. From Remark 2.1, we have actually shown that

$$||\tilde{P}_{s,t}f||_{p,q,\varphi}\le C||f||_{p,q,\varphi},\quad 1\le p\le\infty,1\le q\le\infty. \tag{2.9}$$

This fact will be needed in the proof of Gleason's problem below.

(iii) If $\varphi(r)=(1-r^2)^\alpha\log^\beta(2(1-r)^{-1})$ $(\alpha>0,\beta>0)$, then according to the definition of normal function we can take $a=\alpha-\epsilon,b=\alpha+\epsilon$, where $\epsilon>0$ is any sufficiently small number.

If $P_{s,t}$ is bounded on $L_{p,q}(\varphi)$, then from Theorem A(i) we have $t>\alpha-\epsilon,s>-(\alpha+\epsilon)$. It follows easily that $t\ge\alpha,s\ge-\alpha$. But if $t=\alpha$ or $s=-\alpha$, then $P_{s,t}$ is unbounded by (2.6) or (2.7). Hence the boundedness of $P_{s,t}$ implies $t>\alpha>-s$.

On the other hand, if $t>\alpha>-s$, then there exists an $\epsilon>0$ such that $t>\alpha+\epsilon>\alpha-\epsilon>-s$. Take $a=\alpha-\epsilon,b=\alpha+\epsilon$. Theorem A(ii) shows that $P_{s,t}$ is bounded on $L_{p,q}(\varphi)$.

(iv) We treat the two cases $1\le p<\infty$ and $p=\infty$ separately. When $1\le p<\infty$, [7,7.1.2] shows that $P_{0,t}f=f$ for $f\in H^\infty(B)$. Hence the result is an immediate consequence of Theorem A(ii), since $H^\infty(B)$ is dense in $H_{p,q}(\varphi)$ $(1\le p<\infty,\ 1\le q\le\infty)$ (see [10, Proposition 2.3]).

Now let $p=\infty$. Assume that $\varphi(r)\psi(r)=(1-r^2)^\beta,\varphi,\psi$ are both normal functions, $f\in H_{\infty,q}(\varphi),g\in H_{1,q'}(\psi)$. Then from Lebesgue dominated convergence theorem we have

$$\lim_{r\to1}\int_B\overline{g(z)}f_r(z)(1-|z|^2)^{\beta-1}dv(z)=\int_B\overline{g(z)}f(z)(1-|z|^2)^{\beta-1}dv(z).$$

For any given $w \in B$, take $g(z) = (1 - \langle z, w\rangle)^{-(n+\beta)} \in H_{1,q'}(\psi)$. Thus

$$\begin{aligned} c_{n,\beta}\int_B \frac{(1-|z|^2)^{\beta-1}}{(1-\langle w,z\rangle)^{n+\beta}} f(z)dv(z) &= \lim_{r\to 1} c_{n,\beta}\int_B \frac{(1-|z|^2)^{\beta-1}}{(1-\langle w,z\rangle)^{n+\beta}} f(rz)dv(z) \\ &= \lim_{r\to 1} f(rw) = f(w), \end{aligned}$$

that is, $P_{0,t}f = f$. This completes the proof of Theorem A.

§3. Some Applications

We now begin the proof of Theorem B.

Proof of Theorem B. Assume $m = 1$. Let $A_k f(z) = \int_0^1 \frac{\partial f}{\partial z_k}(rz)dr$. Then for any $f \in H(B), f(0) = 0$, we have $f(z) = \sum\limits_{k=1}^{n} z_k A_k f(z)$, so it remains to show that A_k is bounded on $H_{p,q}(\varphi)$ for $1 \le p, q \le \infty$ and on $H^{(0)}_{\infty,q}(\varphi)$ for $1 \le q \le \infty$.

Given $f \in H_{p,q}(\varphi)$, for $t > b$ we have

$$f(z) = c_{n,t}\int_B \frac{(1-|w|^2)^{t-1}}{(1-\langle z,w\rangle)^{n+t}} f(w)dv(w) \tag{3.1}$$

by Theorem A(iv). Differentiate (3.1) under the integral and then substitute the result into the integral formula of A_k. We have

$$\begin{aligned} A_k f(z) &= (n+t)c_{n,t}\int_0^1 dr \int_B \frac{\overline{w_k}(1-|w|^2)^{t-1} f(w)}{(1-r\langle z,w\rangle)^{n+t+1}} dv(w) \\ &= (n+t)c_{n,t}\int_B \frac{\overline{w_k}(1-|w|^2)^{t-1} f(w)}{(1-\langle z,w\rangle)^{n+t}} \frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle} dv(w). \end{aligned}$$

Take $t \in N, t > b$. Then $\frac{1-(1-\langle z,w\rangle)^{n+t}}{\langle z,w\rangle}$ is a polynomial of $\langle z, w\rangle$, and so is bounded on B. Therefore we have

$$|A_k f(z)| \le C\int_B \frac{(1-|w|^2)^{t-1}|f(w)|}{|1-\langle z,w\rangle|^{n+t}} dv(w) = C\tilde{P}_{0,t}f(z).$$

From (2.9), we get

$$||A_k f||_{p,q,\varphi} \le C||\tilde{P}_{0,t}f||_{p,q,\varphi} \le C||f||_{p,q,\varphi}.$$

We see that A_k is bounded on $H_{p,q}(\varphi)$ $(1 \le p, q \le \infty)$. Using Theorem A(ii) gives the proof for $H^{(0)}_{\infty,q}(\varphi)$ $(1 \le q \le \infty)$.

For m in general case, the proof is the same as that of Theorem 5 in [13]. This proves Theorem B.

Now we apply Theorem A to prove Theorem D.

Proof of Theorem D. Let $f \in H_{p,q}(\varphi)$ and $g \in H_{p',q'}(\phi)$. Holder's inequality implies that

$$\begin{aligned} |(f,g)| &\le 2n\int_0^1 r^{2n-1}(1-r^2)^{-1}\varphi(r)\psi(r)M_q(r,f)M_{q'}(r,g)dr \\ &\le 2n||f||_{p,q,\varphi}||g||_{p',q',\psi}. \end{aligned}$$

This shows that every $g \in H_{p',q'}(\psi)$ defines a bounded linear functional T_g, by the formula $T_g(f) = (f, g)$ on $H_{p,q}(\varphi), 1 \le p \le \infty, 1 \le q \le \infty$, and $||T_g|| \le C||g||_{p',q',\psi}$.

Conversely, when $1 < p < \infty, 1 \le q < \infty$, let $T \in (H_{p,q}(\varphi))^*$, by Hahn-Banach Theorem $T \in (L_{p,q}(\varphi))^*$. Thus from Lemma 2.4, there exists $G \in L_{p',q'}(\varphi^{\frac{p}{p'}})$ such that for any

$f \in L_{p,q}(\varphi)$,

$$T(f) = \frac{1}{2n}\int_B f(z)\overline{G(z)}(1-|z|)^{-1}\varphi^p(|z|)dv(z),$$

and $||T|| = ||G||_{p',q',\varphi^{\frac{p}{p'}}}$.

We define $g = P_{0,\beta}\tilde{G}$, where $\tilde{G}(z) = \frac{1}{2n}(1+|z|)\frac{\varphi(|z|)^{\frac{p}{p'}}}{\psi(|z|)}G(z)$. Then Theorem A shows that

$$||g||_{p',q',\psi} \le C||\tilde{G}||_{p',q',\psi} \le C||G||_{p',q',\varphi^{\frac{p}{p'}}} = C||T||,$$

so $g \in H_{p',q'}(\psi)$. It is easy to verify that $T(f) = (f,g)$ for any $f \in H_{p,q}(\varphi)$.

When $p = 1, 1 \le q < \infty$, using the similar method as in the proof of Theorem A(ii) gives the proof. This completes the proof of Theorem D.

Before proving Theorem C, we first prove

Lemma 3.1. *There exists a constant C such that for any $f \in H_{p,q}(\varphi)$ $(0 < p, q \le \infty)$ we have*

(i) $\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right| \le C||f||_{p,q,\varphi}$,

(ii) $||\sum\limits_{|\alpha|\le m-1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)z^\alpha||_{p,q,\varphi} \le C||f||_{p,q,\varphi}$.

Proof. (i) Let $t > \frac{n}{q} + b$. For any $f \in H_{p,q}(\varphi)$, the equality (3.1) gives

$$(1-|z|^2)^{|\alpha|}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z) = C(1-|z|^2)^{|\alpha|}\int_B \frac{(1-|w|^2)^{t-1}\overline{w}^\alpha f(w)dv(w)}{(1-\langle z,w\rangle)^{n+t+|\alpha|}}. \tag{3.2}$$

Here $C = (n+t)(n+t+1)\cdots(n+t+|\alpha|-1)c_{n,t}$. Take $z = 0$ in (3.2) to get

$$\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right| \le C\int_0^1 (1-r^2)^{t-1}M_1(r,f)dr. \tag{3.3}$$

By the inequality $\varphi(r) \ge C(1-r)^b$, and Proposition 2.1 in [10],

$$M_1(r,f) \le C\varphi^{-1}(r)(1-r)^{-\frac{n}{q}}||f||_{p,q,\varphi} \le C(1-r)^{-b-\frac{n}{q}}||f||_{p,q,\varphi}.$$

The desired inequality follows from (3.3) and the above inequality.

(ii) is a direct corollary of (i). This completes the proof of Lemma 3.1.

Proof of Theorem C. (i) For any $f \in H_{p,q}(\varphi)$ $(1 \le p, q \le \infty)$, from (3.2) we have

$$(1-|z|^2)^m\frac{\partial^m f}{\partial z^\alpha}(z) = (n+t)\cdots(n+t+m-1)P_{m,t}S_\alpha f(z), \tag{3.4}$$

where $t > b, |\alpha| = m, S_\alpha f(z) = \overline{z}^\alpha f(z)$.

Since S_α and $P_{m,t}(t > b)$ are bounded on $L_{p,q}(\varphi)$ by Theorem A(ii), we have

$$\left\|(1-|z|^2)^m\frac{\partial^m f}{\partial z^\alpha}(z)\right\|_{p,q,\varphi} \le C||P_{m,t}S_\alpha f(z)||_{p,q,\varphi} \le C||f||_{p,q,\varphi}.$$

This proves that $f \in H_{p,q}(\varphi)$ implies $(1-|z|^2)^{|\alpha|}\frac{\partial^{|\alpha|}f}{\partial z^\alpha} \in L_{p,q}(\varphi)$.

Conversely we prove that $f \in H_{p,q}(\varphi)$ if $f \in H(B)$ and $(1-|z|^2)^{|\alpha|}\frac{\partial^{|\alpha|}f}{\partial z^\alpha} \in L_{p,q}(\varphi)$. We treat the two cases $1 \le p < \infty, 1 \le q \le \infty$ and $p = \infty, 1 \le q \le \infty$ separately.

Case 1. $1 \le p < \infty, 1 \le q \le \infty$.

For $f \in H_{p,q}(\varphi)$, denote

$$||f||_{m,p,q,\varphi} = \sum_{|\alpha|\le m-1}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\right| + \sum_{|\alpha|=m}||(1-|z|^2)^m\frac{\partial^m f}{\partial z^\alpha}(z)||_{p,q,\varphi}.$$

We claim that $||\cdot||_{p,q,\varphi}$ and $||\cdot||_{m,p,q,\varphi}$ are equivalent norms on $H_{p,q}(\varphi)$.

In fact, by Theorem D, $(H_{p,q}(\varphi))^* = H_{p',q'}(\psi)$, the pairing is given by

$$(f,g) = \int_B f(z)\overline{g(z)}(1-|z|^2)^{\beta-1}dv(z).$$

For any $g \in H_{p',q'}(\psi)$, by Theorem B and Lemma 3.1(ii), we can write

$$g(z) = g_0(z) + \sum_{|\alpha|=m} z^\alpha g_\alpha(z),$$

where $g_0(z) = \sum\limits_{|\alpha|\le m-1} \frac{1}{\alpha!}\frac{\partial^{|\alpha|}g}{\partial z^\alpha}(0)z^\alpha$, $||g_0||_{p',q',\psi} \le C||g||_{p',q',\psi}, ||g_\alpha||_{p',q',\psi} \le C||g||_{p',q',\psi}$. Thus we have

$$(f,g) = (f_0,g_0) + \sum_{|\alpha|=m}(S_\alpha f, g_\alpha) = (f_0,g_0) + \sum_{|\alpha|=m}(S_\alpha f, P_{0,m+\beta}g_\alpha),$$

where $f_0(z) = \sum\limits_{|\alpha|\le m-1}\frac{1}{\alpha!}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)z^\alpha$. A direct computation gives

$$(S_\alpha f, P_{0,m+\beta}g_\alpha) = \frac{c_{n,m+\beta}}{c_{n,\beta}}(P_{m,\beta}S_\alpha f, g_\alpha) = C(T_\alpha f, g_\alpha),$$

where $T_\alpha f(z) = (1-|z|^2)^\alpha \frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z)$, and the last step uses the equality (3.4). Now we have

$$(f,g) = (f_0,g_0) + C\sum_{|\alpha|=m}(T_\alpha f, g_\alpha). \tag{3.5}$$

By Lemma 3.1, we get

$$|(f_0,g_0)| \le ||f_0||_{p,q,,\varphi}||g_0||_{p',q',\psi} \le C\sum_{|\alpha|\le m-1}\left|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}\right| \, ||g||_{p',q',\psi},$$

$$|(T_\alpha f, g_\alpha)| \le C||T_\alpha f||_{p,q,\varphi}||g_\alpha||_{p',q',\psi} \le C||T_\alpha f||_{p,q,\varphi}||g||_{p',q',\psi}.$$

From (3.5) and the above inequality, we have

$$\begin{aligned}||f||_{p,q,\varphi} &\le C\sup\{|(f,g)| : g\in H_{p',q'}(\psi), ||g||_{p',q',\psi}=1\}\\ &\le C\Big\{\sum_{|\alpha|\le m-1}\Big|\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(0)\Big| + \sum_{|\alpha|=m}||(1-|z|^2)^m\frac{\partial^m f}{\partial z^\alpha}(z)||_{p,q,\varphi}\Big\}.\end{aligned}$$

Together with the above results we have proved the claim.

Next by the claim we prove that $f\in H(B), ||f||_{m,p,q,\varphi} \le \infty$ implies $f \in L_{p,q}(\varphi)$.

Let $f_r(z) = f(rz)$. Then $f_r \in H_{p,q}(\varphi)$. Using the claim and the monotonicity of $M_q(r, \frac{\partial^m f}{\partial z^\alpha})$ with respect to r yields $||f_r||_{p,q,\varphi} \le C||f_r||_{m,p,q,\varphi} \le C||f||_{m,p,q,\varphi}$. Letting $r\to 1$, we obtain $||f||_{p,q,\varphi} \le C||f||_{m,p,q,\varphi}$. This proves that $f\in H_{p,q}(\varphi)$ if $||f||_{m,p,q,\varphi} < \infty$ and $f \in H(B)$.

Case 2. $p=\infty, 1\le q\le\infty$.

From [10], we know that $(H^{(0)}_{\infty,q}(\varphi))^* = H_{1,q'}(\psi)$. Hence the same proof as in case 1 shows that $||\cdot||_{\infty,q,\varphi}$ and $||\cdot||_{m,\infty,q,\varphi}$ are equivalent norms on $H^{(0)}_{\infty,q}(\varphi)$. Therefore

$$||f_r||_{\infty,q,\varphi} \le C||f_r||_{m,p,q,\varphi} \le C||f||_{m,\infty,q,\varphi}.$$

That is, $\varphi(\rho)M_q(r\rho, f) \le C||f||_{m,\infty,q,\varphi}$, $0\le\rho<1$. Letting $r\to 1$, we get $\varphi(\rho)M_q(\rho,f) \le C||f||_{m,\infty,q,\varphi}$, thus $f\in H_{\infty,q}(\varphi)$. This proves Theorem C(i).

(ii) By Theorem A(ii) and (3.4), we have $(1-|z|^2)^{|\alpha|}\frac{\partial^{|\alpha|}f}{\partial z^\alpha}(z) \in L^{(0)}_{\infty,q}(\varphi)$ if $f\in H^{(0)}_{\infty,q}(\varphi)$ $(1\le q\le\infty)$.

Conversely, if $f \in H(B), (1-|z|^2)^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(z) \in L^{(0)}_{\infty,q}(\varphi)$ $(|\alpha| = m)$, then by Theorem C(i) we get $f \in H_{\infty,q}(\varphi)$. Hence by Proposition 2.4 in [10], in order to prove $f \in H^{(0)}_{\infty,q}(\varphi)$ we need only to prove that $\lim\limits_{\sigma\to 1} ||f - f_\sigma||_{\infty,q,\varphi} = 0$. Since $||\centerdot||_{\infty,q,\varphi}$ and $||\centerdot||_{m,\infty,q,\varphi}$ are equivalent norms on $H^{(0)}_{\infty,q}(\varphi)$, and

$$\frac{\partial^{|\alpha|} f_\sigma}{\partial z^\alpha}(rz) = \sigma^{|\alpha|} \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(r\sigma z), \tag{3.6}$$

we have

$$\begin{aligned} ||f - f_\sigma||_{\infty,q,\varphi} &\le C||f - f_\sigma||_{m,\infty,q,\varphi} \\ &\le C\Big\{ \sum_{|\alpha|\le m-1} (1-\sigma^{|\alpha|}) \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(0) + \sum_{|\alpha|=m} ||T_\alpha(f-f_\sigma)||_{\infty,q,\varphi} \Big\}. \end{aligned}$$

Therefore it remains to prove $\lim\limits_{\sigma\to 1} ||T_\alpha(f - f_\sigma)||_{\infty,q,\varphi} = 0$.

Recall that $T_\alpha f \in L^{(0)}_{\infty,q}(\varphi)$. By (3.6) and the monotonicity of means M_q, there exists $0 < \rho_0 < 1$ such that for $\rho > \rho_0$,

$$\varphi(\rho) M_q(\rho, T_\alpha(f - f_\rho)) \le C\varphi(\rho) M_q(\rho, T_\alpha f) < \epsilon.$$

On the other hand, if $0 < \rho < \rho_0$, since $f \in H(B)$, we have

$$\begin{aligned} \varphi(\rho) M_q(\rho, T_\alpha(f - f_\sigma)) &\le \varphi(\rho)(1-\rho^2)^{|\alpha|} M_\infty\Big(\rho, \frac{\partial^{|\alpha|}}{\partial z^\alpha}(f - f_\sigma)\Big) \\ &\le C \max_{z\in B} \Big| \frac{\partial^{|\alpha|} f}{\partial z^\alpha}(\rho z) - \frac{\partial^{|\alpha|} f_\sigma}{\partial z^\alpha}(\rho z)\Big| < \epsilon \quad (\sigma > \sigma_0). \end{aligned}$$

Namely $\sup\limits_{0<\rho<1} \varphi(\rho) M_q(\rho, T_\alpha(f - f_\sigma)) < \epsilon, (\sigma > \sigma_0)$. That is,

$$\lim_{r\to 1} ||T_\alpha(f - f_\sigma)||_{\infty,q,\varphi} = 0.$$

This proves Theorem C.

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