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# ASYMPTOTIC HODGE THEORY IN SEVERAL VARIABLES: THE FLAT CASE

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**Abstract**

In the flat case, the answer to the problem posed by Steenbrink and Zucker is given.

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## §1. Introduction

We consider the degeneration of Hodge structure in the geometric case. By the latter, we mean a diagram

$$\begin{array}{ccc} X^* = X - Y & \to & X \\ f' \downarrow & & \downarrow f \\ S^* = S - T & \to & S \end{array}$$

where $X$ is a complex Kähler manifold, $S$ is a complex manifod, $f$ is a proper surjective holomorphic morphism, $T$ and $Y(= f^{-1}(T))$ are divisors with normal crossings, and $f' = f|_{X^*}$ is proper and smooth.

Then $\mathbf{V} = R^m f'_* \mathbf{C}$ is a local system on $S^*$ underlying a variation of Hodge structure of weight $m$. The associated holomorphic vector bundle $\mathcal{V} = \mathcal{O}_{S^*} \otimes_{\mathbf{C}} \mathbf{V} \cong \mathbf{R}^m f'_* \Omega^{\cdot}_{X^*/S^*}$ has the decreasing Hodge filtration, given by $\mathcal{F}^p \cong \mathbf{R}^m f'_* F^p \Omega^{\cdot}_{X^*/S^*}$, where $F$ denotes the usual truncation from below. Steenbrink[6] and Clemens[1] studied the de Rham theoretic realization of the limit mixed Hodge structure of Schmid[5], defind for abstract variations of Hodge structure, in the geometric case when $\dim_{\mathbf{C}} S = 1$. Part of their theory gives:

(i) The "canonical extension" of $\mathcal{V}$ (see [2, p.91]) is given by $\widetilde{\mathcal{V}} \cong \mathbf{R}^m f_* \Omega^{\cdot}_{X/S}(\log Y)$, with Hodge filtration $\widetilde{\mathcal{F}}^p \cong \mathbf{R}^m f_* F^p \Omega^{\cdot}_{X/S}(\log Y)$; the fiber $\widetilde{\mathcal{V}}(0)$ of $\widetilde{\mathcal{V}}$ at 0 is given by $\widetilde{\mathcal{V}}(0) = \mathbf{H}^m(Y, \Omega^{\cdot}_{X/S}(\log Y)) \otimes \mathcal{O}_Y)$, whose filtration $\{\mathcal{F}^p(0)\}$ is induced by $F$.

(ii) Assume for simplicity that $Y$ is reduced. Steenbrink constructed an $F$-filtered "resolution" $A^{\cdot\cdot}$ of $\Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Y$ that admits a second (useful) filtration and underlies a cohomological mixed Hodge complex (in the sense of [4, §8]). This is shown to induce the limit mixed Hodge structure.

Steenbrink and Zucker posed the following (see [7, p.495])

**Problem.** Carry out the analogue of de Rham theoretic realization of the limit mixed Hodge structure in the geometric case when the dimension of $S$ is greater than one.

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**Assumption.** Towards finding an answer to the above problem, we make the hypothesis that $f$ is flat over $S$. The mapping $f : X \to S$ then separates variables in local coordinates. We assume, only for simplicity of notation, that $\dim_{\mathbf{C}} S = 2$. The local data of this geometric case (at a singular point of $T$) is given by $S = \Delta^2$, $T = T_1 \cup T_2$, where $T_i = \{(t_1, t_2) \in \Delta^2 | t_i = 0\} (i = 1, 2)$, $Y = f^{-1}(T) = Y_1 \cup Y_2$, $Y_i = f^{-1}(T_i)$ $(i = 1, 2)$ and $Z = Y_1 \cap Y_2 = f^{-1}(0, 0)$. In a coordinate system $(x, y) = (x_1, \cdots, x_m; y_1, \cdots, y_n)$ on a neighborhood of $Q \in Z$, the mapping $f$ can be written as $f(x, y) = \Big(\prod_{i=1}^{m} x_i^{a_i}, \prod_{j=1}^{n} y_j^{b_j}\Big)$ with $a_i, b_j \in \mathbf{Z}$ and $a_i, b_j \geq 0$.

## §2. Main Results

We assume throughout that $f$ is flat over $S$. Our main results are Theorems 2.1 and 2.2 below.

**Theorem 2.1.** *The canonical extension of $\mathcal{V}$ is given by $\widetilde{\mathcal{V}} \cong \mathbf{R}^m f_* \Omega^{\cdot}_{X/S}(\log Y)$ and the fiber $\widetilde{\mathcal{V}}(0,0)$ of $\widetilde{\mathcal{V}}$ at $(0,0)$ is given by $\widetilde{\mathcal{V}}(0,0) \cong \mathbf{H}^m(Z, \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z)$.*

The limit mixed Hodge structure lives naturally on the fiber $\widetilde{\mathcal{V}}(0,0)$, which is the reason why we are interested in the expression for $\widetilde{\mathcal{V}}(0,0)$ (see §1).

In order to put a mixed Hodge structure on $\widetilde{\mathcal{V}}(0,0)$, we replace the complex $\Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z$ by $B^{\cdot}$, a bigraded complex with terms

$$B^{p,q} = \bigoplus_{q'+q''=q} \Omega_X^{p+q+2}(\log Y)/(W'_{q'} + W''_{q''})\Omega_X^{p+q+2}(\log Y) \ (p, q \geq 0),$$

where $W'$ and $W''$ are the weight filtrations with respect to $Y_1$ and $Y_2$ respectively, given by

$$W'_{q'}\Omega_X^r(\log Y) = \Omega_X^{q'}(\log Y) \wedge \Omega_X^{r-q'}(\log Y_2), \quad W''_{q''}\Omega_X^r(\log Y) = \Omega_X^{q''}(\log Y) \wedge \Omega_X^{r-q''}(\log Y_1),$$

and differentials

$$d : B^{p,q} \to B^{p+1,q} \text{ induced by } d \text{ in } \Omega^{\cdot}_X(\log Y),$$

$$\Theta : B^{p,q} \to B^{p,q+1} \text{ induced by } \wedge(\theta_1 + \theta_2), \text{ where } \theta_i = f^* dt_i / t_i \ (i = 1, 2).$$

One places filtrations $M$ and $F$ on $B^{\cdot}$ as follows:

(i) $M_k B^{p,q} = \bigoplus_{q'+q''=q} W_{k+2q+2}\Omega_X^{p+q+2}(\log Y)/(W'_{q'} + W''_{q''})\Omega_X^{p+q+2}(\log Y)$,

where $W_l \Omega_X^r(\log Y) = \Omega_X^l(\log Y) \wedge \Omega_X^{r-l}$.

(ii) $F^r B^{p,q} = \begin{cases} B^{p,q} & \text{if } p \geq r, \\ 0 & \text{if } p < r. \end{cases}$

**Theorem 2.2.** *Assume for simplicity that $Z$ is reduced. Then*

(i) *$B^{\cdot}$ is an $F$-filtered resolution of $\Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z$,*

(ii) *$B^{\cdot}$ (with a suitable rational structure) is a cohomological mixed Hodge complex that induces the limit mixed Hodge structure on $\mathbf{H}^m(Z, \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z)$.*

This gives the solution of the above problem in the flat case. The underlying theme of the proof is to use the local separation of variables to reduce to the local results in [6].

**Remark.** The objects in Theorem 2.2 make sense even without the flatness assumption.

## §3. Proof of Theorem 2.1.

The canonical extension is characterized by certain properties, so it is enough to check that the following conditions hold:

(i) $\mathbf{R}^m f_*\Omega^{\cdot}_{X/S}(\log Y)$ is locally free on $S$.

(ii) The Gauss-Manin connection $\nabla$ extends to have logarithmic poles on $T$:

$$\nabla: \ \mathbf{R}^m f_*\Omega^{\cdot}_{X/S}(\log Y) \to \Omega^1_S(\log T) \otimes \mathbf{R}^m f_*\Omega^{\cdot}_{X/S}(\log Y).$$

(iii) The eigenvalues $\lambda$ of the residues of $\nabla$ on $T_i (i = 1, 2)$ satisfy $0 \le \mathrm{Re}\lambda < 1$.

We prove (i) by showing that for any $m \in \mathbf{Z}$, the function

$$h^m(t_1, t_2) = \dim_{\mathbf{C}} \mathbf{H}^m(X_{(t_1,t_2)}, \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_{X_{(t_1,t_2)}})$$

is constant on $S$, where $X_{(t_1,t_2)} = f^{-1}(t_1, t_2)$ for $(t_1, t_2) \in \Delta^2$. The only serious point is to verify that $h^m(t_1, t_2)$ does not jump when $(t_1, t_2)$ becomes $(0, 0) \in \Delta^2$.

Let $\tilde{\Delta}^{*2}$ be the universal covering of $\Delta^{*2}$. Denote $\widetilde{X}^* = X \times_{\Delta^2} \tilde{\Delta}^{*2}$ and let $k : \widetilde{X}^* \to X$ be the projection, $i : Z \to X$ the injection. We introduce an intermediate complex

$$L^{\cdot} = \bigoplus_{\alpha_1,\alpha_2} L^{\cdot}_{\alpha_1,\alpha_2} \ (L^{\cdot}_{\alpha_1,\alpha_2} = i^{-1} t_1^{-\alpha_1} t_2^{-\alpha_2} \Omega^{\cdot}_X(\log Y)[\log t_1, \log t_2], \ \alpha_1, \alpha_2 \in \mathbf{Q} \cap [0, 1])$$

to establish a quasi-isomorphism between $\Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z$ and $i^{-1}k_*\Omega^{\cdot}_{\widetilde{X}^*}$ :

$$i^{-1}k_*\Omega^{\cdot}_{\widetilde{X}^*} \overset{\phi}{\hookleftarrow} L^{\cdot} \overset{\psi}{\to} \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z,$$

where the natural map $\phi$ is injective, $\psi$ is defined by sending

$$\omega = t_1^{-\alpha_1} t_2^{-\alpha_2} \sum_{i=0}^{r} \sum_{j=0}^{s} \omega_{i,j} (\log t_1)^i (\log t_2)^j,$$

a section of $L^{\cdot}_{\alpha_1,\alpha_2}$, to the image of $\omega_{0,0}$ under the natural map

$$i^{-1}\Omega^{\cdot}_X(\log Y) \to \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z.$$

Both $\phi$ and $\psi$ are quasi-isomorphisms, because they induce the isomorphisms of cohomology sheaves:

$$\mathcal{H}^q(i^{-1}k_*\Omega^{\cdot}_{\widetilde{X}^*}) \underset{\mathcal{H}^q(\phi)}{\overset{\cong}{\longleftarrow}} \mathcal{H}^q(L^{\cdot}) \underset{\mathcal{H}^q(\psi)}{\overset{\cong}{\longrightarrow}} \mathcal{H}^q(\Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z).$$

This can be checked by calculating the stalks of these cohomology sheaves at a point $Q \in Z$, as in [6, §2]. In fact, they have the same representatives.

Therefore we have

$$\begin{aligned}\mathbf{H}^q(Z, \Omega^{\cdot}_{X/S}(\log Y) \otimes \mathcal{O}_Z) &\cong \mathbf{H}^q(Z, i^{-1}k_*\Omega^{\cdot}_{\widetilde{X}^*}) \cong \mathbf{H}^q(X, k_*\Omega^{\cdot}_{\widetilde{X}^*}) \\ &\cong H^q(\widetilde{X}^*, \mathbf{C}) \cong H^q(X_{(t_1,t_2)}, \mathbf{C})\end{aligned}$$

for $(t_1, t_2) \in \Delta^{*2}$.

(ii) is automatic, since the Gauss-Manin connection can be constructed as the connecting homomorphism in the long exact sequence of hypercohomology, associated to the exact sequence of complexes:

$$0 \to f^*\Omega^1_S(\log T) \otimes \Omega^{\cdot}_{X/S}(\log Y)[-1] \to \Omega^{\cdot}_X(\log Y) \to \Omega^{\cdot}_{X/S}(\log Y) \to 0.$$

Finally, applying Steenbrink's result (see [6, (2.20)]), we see that (iii) is true.

## §4. Proof of Theorem 2.2

The rational structure that underlies $B^{\cdot}$ can be constructed as a complex of $\mathbf{Q}$-vector spaces $C^{\cdot}$ (see [7, §5]), as follows. For any space $V$, let $\mathbf{C}^{\cdot}(V)$ denote the complex of sheaves of germs of rational-valued singular cochains on $V$. Hence $\mathbf{H}^m(V, \mathbf{C}^\cdot(V)) \cong H^m(V, \mathbf{Q})$. If $h : V \to X$ is a morphism, put $K^\cdot(V) = i^{-1}h_* C^\cdot(V)$. For $j = 1, 2$, let $pr_j : \Delta^2 = \Delta \times \Delta \to \Delta$ be the projection from the product $\Delta \times \Delta$ onto its $j$th factor $\Delta$. We have the composition $f_j = pr_j \circ f : X \overset{f}{\to} \Delta^2 \overset{pr_j}{\to} \Delta$, and the composition $\pi_j = pr_j \circ \pi : \tilde{\Delta}^{*2} \overset{\pi}{\to} \Delta^2 \overset{pr_j}{\to} \Delta$, where $\pi$ is the covering map, given by $(u_1, u_2) \to (e^{2\pi\sqrt{-1}u_1}, e^{2\pi\sqrt{-1}u_2})$. Denote $X_j^* = X - Y_j$, and let $\widetilde{X}_j^* = X \times_\Delta \tilde{\Delta}^{*2}, k_j$ the projection:

$$\begin{array}{ccc} \widetilde{X}_j^* & \overset{k_j}{\to} & X \\ \downarrow & & \downarrow f_j \\ \tilde{\Delta}^{*2} & \overset{\pi_j}{\to} & \Delta \end{array}$$

The monodromy transformation $T_1$, induced by the automorphism $(x, y; u_1, u_2) \mapsto (x, y; u_1 - 1, u_2)$ of $\widetilde{X}_1^*$, lifts to an automorphism of $K^\cdot(\widetilde{X}_1^*)$. Let $'B^\cdot = \bigcup_{m \geq 0} \ker\{(T_1 - I)^{m+1} : K^\cdot(\widetilde{X}_1^*) \to K^\cdot(\widetilde{X}_1^*)\}$. If $K^\cdot$ is a complex of $\mathbf{Q}$-vector spaces and $r \in \mathbf{Z}$, put $K^\cdot(r) = (2\pi\sqrt{-1})^r K^\cdot$. Let $\rho('B)^\cdot$ denote the mapping cone of the morphism $\delta_1 = -\frac{1}{2\pi\sqrt{-1}} \log T_1 :' B^\cdot \to' B^\cdot(-1)$, i.e., $\rho('B)^p =' B^p \oplus' B^{p-1}(-1)$ and $d_1 : \rho('B)^p \to \rho('B)^{p+1}$ is given by $d_1(x', y') = (d_1 x', -dy' + \delta_1 x'), \theta_1 : \rho('B)^\cdot \to \rho('B)^\cdot(1)$ (see [1]) is given by $\theta_1(x', y') = (0, x')$.

Starting out with the complex $K^\cdot(\widetilde{X}_2^*)$ and its automorphism $T_2$, the monodromy transformation induced by the automorphism $(x, y; u_1, u_2) \to (x, y; u_1, u_2 - 1)$ of $\widetilde{X}_2^*$, we obtain the mapping cone $\rho(''B)^\cdot$ with two maps $d_2, \theta_2$ by performing the same construction.

Then we put $R^\cdot = \rho('B)^\cdot \otimes_{\mathbf{Q}} \rho(''B)^\cdot$. The two morphisms $d, \Theta$ on $R^m$ are given by $d = d_1 \otimes 1 + (-1)^m(1 \otimes d_2)$ and $\Theta = \theta_1 \otimes 1 + (-1)^m(1 \otimes \theta_2)$ respectively.

Let $\tau_r K^\cdot$ be the canonical filtration of a complex $K^\cdot$, given by

$$\tau_r K^p = \begin{cases} K^p & \text{if } p < r, \\ \ker d & \text{if } p = r, \\ 0 & \text{if } p > r. \end{cases}$$

For a tensor product $K^\cdot \otimes_{\mathbf{Q}} L^\cdot$ of complexes, we have partial canonical filtrations

$$\tau_r'(K^\cdot \otimes_{\mathbf{Q}} L^\cdot) = (\tau_r K^\cdot) \otimes_{\mathbf{Q}} L^\cdot \quad \text{and} \quad \tau_r''(K^\cdot \otimes_{\mathbf{Q}} L^\cdot) = K^\cdot \otimes_{\mathbf{Q}} (\tau_r L^\cdot).$$

Note that

$$Gr_l^\tau(K^\cdot \otimes_{\mathbf{Q}} L^\cdot) \cong \mathcal{H}^l(K^\cdot \otimes_{\mathbf{Q}} L^\cdot)[-l] = \bigoplus_{l'+l''=l} \mathcal{H}^{l'}(K^\cdot) \otimes_{\mathbf{Q}} \mathcal{H}^{l''}(L^\cdot)[-l]$$
$$\cong \bigoplus_{l'+l''=l} Gr_{l'}^{\tau'}(K^\cdot) \otimes_{\mathbf{Q}} Gr_{l''}^{\tau''}(L^\cdot).$$

We finally define the double complex $C^\cdot$ as

$$C^{\cdot,q} = \bigoplus_{q'+q''=q} R^\cdot(q+2)[q+2]/(\tau_{q'}' + \tau_{q''}'')\{R^\cdot(q+2)[q+2]\} \ (q \geq 0)$$

with differentials

$$d : C^{p,q} \to C^{p+1,q} \text{ induced by } d \text{ in } R^\cdot, \quad \Theta : C^{p,q} \to C^{p,q+1} \text{ induced by } \Theta \text{ in } R^\cdot.$$

Then $C^\cdot \otimes_{\mathbf{Q}} \mathbf{C}$ is quasi-isomorphic to $B^\cdot$. The filtration $M$ on $C^\cdot$ is given by

$$M_k C^{\cdot,q} = \bigoplus_{q'+q''=q} \tau_{k+2q+2}\{R^\cdot(q+2)[q+2]\}/(\tau_{q'}' + \tau_{q''}'')\{R^\cdot(q+2)[q+2]\}.$$

In the complex $Gr_k^M C^\cdot, \Theta$ induces the zero map. We have

$$Gr_k^M C^\cdot \cong \bigoplus_{q\geq 0,-k} Gr_k^M C^{\cdot,q}[-q]$$
$$= \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} Gr^{\tau}_{k+2q+2}\Big\{\frac{R^\cdot(q+2)[q+2]}{(\tau'_{q'}+\tau''_{q''})(R^\cdot(q+2)[q+2])}\Big\}[-q]$$
$$\cong \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} \bigoplus_{\substack{k'+k''=k\\ k'\geq -q'\ k''\geq -q''}} \mathbf{Q}_{\widetilde{Y}_1^{(k'+2q'+1)}\cap \widetilde{Y}_2^{(k''+2q''+1)}}(-k-q)[-k-2q].$$

In the complex $Gr_k^M B^\cdot$, we have

$$Gr_k^M B^\cdot = \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} \bigoplus_{\substack{k'+k''=k\\ k'\geq -q'\ k''\geq -q''}} Gr^{W'}_{k'+2q'+1} Gr^{W''}_{k''+2q''+1}\Omega^\cdot_X(\log Y)(\text{see [2]})$$
$$\cong \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} \bigoplus_{\substack{k'+k''=k\\ k'\geq -q'\ k''\geq -q''}} \Omega^\cdot_{\widetilde{Y}_1^{(k'+2q'+1)}\cap \widetilde{Y}_2^{(k''+2q''+1)}}[-k-2q],$$

$$F^p Gr_k^M B^\cdot \cong \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} \bigoplus_{\substack{k'+k''=k\\ k'\geq -q'\ k''\geq -q''}} F^{p+q+2}\Omega^\cdot_{\widetilde{Y}_1^{(k'+2q'+1)}\cap \widetilde{Y}_2^{(k''+2q''+1)}}[-k-2q].$$

Thus

$$\mathbf{H}^n(X, F^p Gr_k^M B^\cdot) \cong \bigoplus_{q\geq 0,-k} \bigoplus_{q'+q''=q} \bigoplus_{\substack{k'+k''=k\\ k'\geq -q'\ k''\geq -q''}} F^p[H^{n-k-2q}(\widetilde{Y}_1^{(k'+2q'+1)}$$
$$\bigcap \widetilde{Y}_2^{(k''+2q''+1)}, \mathbf{C})\langle -k-q\rangle],$$

where $\langle -r\rangle$ indicates a Tate twist. So $Gr_k^M B^\cdot$ is a direct sum of cohomological Hodge complexes of weight $-k-2q-2(-k-q)=k$, as is required for applying Deligne's Theorem (see [4, (8.1.9)]): $B^\cdot$ and the quasi-isomorphisms with its rational structure comprise a cohomological mixed Hodge complex that induces a mixed Hodge structure on its hypercohomology.

To put a mixed Hodge structure on $\mathbf{H}^m(Z, \Omega^\cdot_{X/S}(\log Y)\otimes \mathcal{O}_Z)$, we verify that $B^\cdot$ is actually an $F$-filtered resolution of $\Omega^\cdot_{X/S}(\log Y)\otimes \mathcal{O}_Z$, thus reducing to the preceding. This can be done by showing that

$$0 \longrightarrow \Omega^p_{X/S}(\log Y)\otimes \mathcal{O}_Z \overset{\wedge\theta_1\wedge\theta_2}{\longrightarrow} B^{p,0} \overset{\wedge\Theta}{\longrightarrow} B^{p,1} \overset{\wedge\Theta}{\longrightarrow} B^{p,2} \longrightarrow \cdots \tag{4.1}$$

is exact.

For this, we place another weight filtration $W$ on $B^\cdot$:

$$W_l B^{p,q} = \bigoplus_{q'+q''=q} W_{l+q+2}\Omega_X^{p+q+2}(\log Y)/(W'_{q'}+W''_{q''})\Omega_X^{p+q+2}(\log Y)\ (0\leq l\leq p).$$

The weight filtration $W_l$ on $\Omega^\cdot_{X/S}(\log Y)\otimes \mathcal{O}_Z$ is the image of $W_l\Omega^\cdot_X(\log Y)$.

We show that the complex

$$0\to Gr_l^W(\Omega^p_{X/S}(\log Y)\otimes \mathcal{O}_Z \to Gr_l^W B^{p,0}\to Gr_l^W B^{p,1}\to Gr_l^W B^{p,2}\to\cdots \tag{4.2}$$

is exact for all $l$. Via Poincaré residues, the complex (4.2) is isomorphic to the complex

$$0\longrightarrow \frac{\Omega_X^r}{I(Z)\Omega_X^r(\log Y)} \overset{\tilde\theta_1\otimes\tilde\theta_2}{\longrightarrow} E^0 \overset{\widetilde{\Theta}}{\longrightarrow} E^1 \overset{\widetilde{\Theta}}{\longrightarrow} E^2 \longrightarrow\cdots, \tag{4.3}$$

where $E^k = \bigoplus_{k'+k''=k} \Omega^r_{\widetilde{Y}_1^{(k'+1)} \cap \widetilde{Y}_2^{(k''+1)}}$ $(k = 0, 1, \cdots)$; $\tilde{\theta}_1, \tilde{\theta}_2$ and $\widetilde{\Theta}$ are induced by $\theta_1, \theta_2$ and $\Theta$. The stalk of the complex (4.3) at $Q \in Z$ is isomorphic to the complex

$$0 \longrightarrow \frac{\mathbf{C}\{\mathbf{x}, y\}}{(x_1 \cdots x_m; y_1 \cdots y_n)} \xrightarrow{\delta_1 \otimes \delta_2} K^0 \xrightarrow{\delta} K^1 \xrightarrow{\delta} K^2 \xrightarrow{\delta} \cdots , \tag{4.4}$$

where

$$K^k = \bigoplus_{k'+k''=k} \prod_{\substack{1 \le i'_1 < \cdots < i'_{k'+1} \le m \\ 1 \le i''_1 < \cdots < i''_{k''+1} \le n}} \frac{\mathbf{C}\{x, y\}}{(x_{i'_1}, \cdots, x_{i'_{k+1}}; y_{i''_1}, \cdots, y_{i''_{k+1}})},$$

$\delta_1, \delta_2$ and $\delta_3$ are combinatorial objects induced by $\tilde{\theta}_1, \tilde{\theta}_2$ and $\widetilde{\Theta}$.

Instead of the complex (4.4), we look at the complex

$$0 \to \frac{\mathbf{C}[x, y]}{(x_1 \cdots x_m; y_1 \cdots y_n)} \overset{\delta_1 \otimes \delta_2}{\to} L^0 \overset{\delta}{\to} L^1 \overset{\delta}{\to} L^2 \to \cdots , \tag{4.5}$$

where

$$L^k = \bigoplus_{k'+k''=k} \prod_{\substack{1 \le i'_1 < \cdots < i'_{k'+1} \le m \\ 1 \le i''_1 < \cdots < i''_{k''+1} \le n}} \frac{\mathbf{C}[x, y]}{(x_{i'_1}, \cdots, x_{i'_{k+1}}; y_{i''_1}, \cdots, y_{i''_{k+1}})} \ (k = 0, 1, 2, \cdots).$$

The complex (4.5) can be thought of the tensor product of two exact complexes

$$0 \longrightarrow \frac{\mathbf{C}[x]}{(x_1 \cdots x_m)} \xrightarrow{\delta_1} {}'L^0 \xrightarrow{\delta_1} {}'L^1 \xrightarrow{\delta_1} {}'L^2 \longrightarrow \cdots ,$$

$$0 \longrightarrow \frac{\mathbf{C}[y]}{(y_1 \cdots y_n)} \xrightarrow{\delta_2} {}''L^0 \xrightarrow{\delta_2} {}''L^1 \xrightarrow{\delta_2} {}''L^2 \longrightarrow \cdots ,$$

where

$${}'L^{k'} = \prod_{1 \le i'_1 < \cdots < i'_{k'+1} \le m} \frac{\mathbf{C}[x]}{(x_{i'_1}, \cdots, x_{i'_{k'+1}})} \ (k' = 0, 1, 2, \cdots),$$

$${}''L^{k''} = \prod_{1 \le i''_1 < \cdots < i''_{k''+1} \le n} \frac{\mathbf{C}[y]}{(y_{i''_1}, \cdots, x_{i''_{k''+1}})} \ (k'' = 0, 1, 2, \cdots).$$

By the Künneth formula, we obtain the exactness of the complex (4.5). Since $\mathbf{C}\{\mathbf{x}, y\}$ is flat over $\mathbf{C}[x, y]$, the complex (4.4) is exact. Therefore the complex (4.1) is exact as desired.

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