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TOWARD A THEORY OF WEAK *I* SEQUENCES

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Abstract

The author introduces a notion of weak I sequences and characterizes such sequences by means of homological methods. This notion extends the notion of weak M-sequences and thus extends the notions of generalized Cohen-Macaulay modules and Buchsbaum modules to more general cases

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§1. Introduction

The goal of this paper is to characterize weak I sequences and M-standard ideal by means of homological methods. The notion of weak I sequence is closely related to the finiteness property of local cohomology groups with support in $\mathrm{V}(I)$. Our characterization is similar to that of regular sequences given in [7].

Let A be a commutative Noetherian local ring with the maximal ideal m and I be an ideal of A. Let M be a finite A-module of dimension d. We say that a sequence $x_1, \cdots, x_r$ contained in I is a weak I sequence with respect to M, if the inclusion

$$(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : I^n$$

holds, where $1 \leqslant i \leqslant r$, $x_0^{n_0} = 0$, n being a fixed positive integer and $n_1, \cdots, n_r$ running through all positive integers. Recall that if $I = m$ and $r = d$, then that a weak m sequence $x_1, \cdots, x_d$ with respect to M exists implies that M is a generalized Cohen-Macaulay A-module and that $x_1, \cdots, x_d$ must be a system of parameters for M. In this case, every system of parameters for M forms a weak m sequence[3]. Then it raises a natural question whether all the maximal weak I sequences have the same length. After we get a necessary and suffcient characterization of weak I sequences by means of the homology of Koszul complex ([Theorem 3.1]), we obtain a positive answer to the question ([Theorem 3.4]). At the end of the paper we consider the case that the length of the maximal weak I sequence is d and extend the notion of standard ideals in [13] to general cases.

Throughout this paper, let A be a commutative Notherian local ring with unit and m the maximal ideal of A. We always denote by I a proper ideal of A and by M a finite A-module. Let $H_I^i(\cdot)$ stand for the ith local cohomology group relative to I and $\Gamma_I(M)$ stand for $H_I^0(M)$. Finally we use Z^+ to denote the set of positive integers.

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§2. Weak *I* Sequences

In this section we will give the defintion of weak I sequences with respect to M and discuss some basic properties.

Definition 2.1. *Let I be an ideal of A and M a finite A-module. A sequence $x_1, \cdots, x_r$ contained in I is said to be a weak I sequence with respect to M, if for $1 \leq i \leq r$, n a fixed positive integer*

$$(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : I^n \tag{2.1}$$

holds for $n_1, n_2, \cdots, n_r$ running through all positive integers.

In the rest of the paper we will simply call $x_1, \cdots, x_r$ a weak I sequence if it causes no confusion. Clearly, any M-sequence contained in I is a weak I sequence. If $x_1, \cdots, x_r$ is a weak I sequence and $n_1, \cdots, n_r \in Z^+$, $x_1^{n_1}, \cdots, x_r^{n_r}$ is also a weak I sequence. If $\Gamma_I(M) \neq M$ and $M' = M/\Gamma_I(M)$, a weak I sequence $x_1, x_2, \cdots, x_r$ with respect to M is also a weak I sequence with respect to M'. In fact, if $x_1, x_2, \cdots, x_r$ satisfies (2.1), then

$$(x_1^{n_1}, \cdots, x_{n_{i-1}}^{n_{i-1}})M' : x_i^{n_i} \subseteq (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M' : I^{2n}.$$

Furthermore, if $x_1, \cdots, x_r$ is a weak I sequence, by definition $x_i, \cdots, x_r$ is a weak I sequence with respect to $M/(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M$.

The following proposition is a simple generalization of a result in [4].

Proposition 2.1 (i) *If $\Gamma_I(M) = M$, then $H_I^i(M) = 0$ for all $i > 0$.*

(ii) *If $x \in I$ and $(0 :_M x) \subseteq \Gamma_I(M)$, then we have the local cohomology long exact sequence*

$$0 \longrightarrow (0 :_M x) \longrightarrow H_I^0(M) \xrightarrow{x} H_I^0(M) \longrightarrow H_I^0(M/xM)$$
$$\longrightarrow H_I^1(M) \xrightarrow{x} H_I^1(M) \longrightarrow \cdots .$$

Lemma 2.1. *Let I be an ideal of A and M a finite A-module. Then every maximal weak I sequence has the length $r \geq 1$.*

Proof. Since A is Noetherian and M a finite A-module, the accending chain of submodules

$$(0 :_M I) \subseteq (0 :_M I^2) \subseteq (0 :_M I^3) \subseteq \cdots$$

must stop at some s ($s \in Z^+$). Hence $\Gamma_I(M) = (0 :_M I^s)$. If $\Gamma_I(M) = M$, then any element $x \in I$ is a weak I sequence. If $\Gamma_I(M) \neq M$, put $M' = M/\Gamma_I(M)$. Consider the short exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

from it we can deduce that $H_I^0(M') = 0$. Thus there exists an M'-regular element $x \in I$ such that $0 :_M x^n \subseteq (0 :_M I^s)$ for all $n > 0$.

Now, we consider the converse to the part (i) of Proposition 2.1.

Proposition 2.2. *Let I be an ideal of the local ring A and M a finite A-module. If there exists a positive integer n such that $I^n H_I^i(M) = 0$ for all $i \geq 0$, then $M = \Gamma_I(M)$.*

Proof. We use induction on the dimension of M. For $\dim M = 0$, the result is trivial. Now suppose the statement holds for those A-modules with dimension less than $\dim M$. If $M \neq \Gamma_I(M)$, then we put $M' = M/\Gamma_I(M)$. Clearly $\Gamma_I(M') \neq M'$. Consider the short exact sequence

$$0 \longrightarrow H_I^0(M) \longrightarrow M \longrightarrow M' \longrightarrow 0,$$

this implies $H_I^i(M') \simeq H_I^i(M)$ for all $i \geq 1$. Since $H_I^0(M') = 0$, there exists an M'-regular element $x \in I$. For a positive integer s such that $s > 2n$, consider the following short exact sequence

$$0 \longrightarrow M' \xrightarrow{x^s} M' \longrightarrow M'/x^sM' \longrightarrow 0.$$

We have the long exact sequence

$$0 \longrightarrow H_I^0(M'/x^sM') \longrightarrow H_I^1(M') \xrightarrow{x^s} H_I^1(M') \longrightarrow \cdots .$$

From this we have $I^{2n}H_I^i(M'/x^sM') = 0$ for all $i \geq 0$. Since

$$\dim M \geq \dim M' > \dim (M'/x^sM'),$$

by the induction hypothesis, we have

$$\Gamma_I(M'/x^sM') = M'/x^sM', \ \text{ i.e. } \ I^{2n}M' \subseteq x^sM'.$$

So $I^{2n}M' \subseteq x^{s-2n}I^{2n}M'$. By Nakayama lemma, we have $I^{2n}M' = 0$. Hence $\Gamma_I(M) = M$, this is a contradiction.

Proposition 2.3. *Let I be an ideal of A and M a finite A-module. If there exists a weak I sequence $x_1, \cdots, x_s$ such that*

$$(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : I^r,$$

where $1 \leq i \leq s$, r is a fixed positive integer and $n_1, \cdots, n_s$ run through all positive integers. Then there exists an integer $k \in Z^+$ which depends only on r such that $I^kH_I^i(M) = 0$ for $i < s$.

Proof. We use induction on the dimension d of M. For $d = 0$, the result is trivial. Suppose the conclusion holds for those A-modules M_1 with $\dim M_1 < d$. If $\Gamma_I(M) = M$, the result follows from Proposition 2.1. Now, If $\Gamma_I(M) \neq M$, put $M' = M/\Gamma_I(M)$, then x_1 is M'-regular. For $n \in Z^+$, consider the following long exact sequence

$$\cdots \longrightarrow H_I^0(M'/x_1^nM') \longrightarrow H_I^1(M') \xrightarrow{x_1^n} H_I^1(M') \longrightarrow H_I^1(M'/x_1^nM') \longrightarrow \cdots .$$

Since $x_2, \cdots, x_s$ is a weak I sequence with respect to M'/x_1^nM', the integer r in the theorem may be selected such that r does not change for each A-module $M'/x_1^nM'(n \in Z^+)$. On the other hand, $\dim M \geq \dim M' > \dim(M'/x_1^nM')$. Hence we can use our induction hypothesis to assert that there exists an integer $k > 0$ such that

$$I^kH_I^i(M'/x_1^nM') = 0 \text{ for all } i < s-1, \ \text{ and all } n \in Z^+.$$

Now, for any $a \in H_I^i(M')$ $(i < s)$, we can choose $n \in Z^+$ such that $x_1^na = 0$. Hence it can be seen easily from the long exact sequence that $I^ka = 0$. This implies $I^kH_I^i(M) = 0$ for $i < s$.

One can prove immediately the following by Proposition 2.2 and Proposition 2.3.

Corollary 2.1. *Let I be an ideal of A and M a finite A-module. If $\Gamma_I(M) \neq M$, then any maximal weak I sequence has length $r \leq \dim M$.*

By means of Proposition 2.1 and by induction on r, we have

Propostion 2.4. *Suppose $I^kH_I^i(M) = 0$, for $i < r$. Let $x_1, \cdots, x_r$ be a weak I sequence. Then*

$$(x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : x_i^{n_i} \subseteq (x_1^{n_1}, \cdots, x_{i-1}^{n_{i-1}})M : I^{2^rk} \qquad (2.2)$$

where $1 \le i \le r$, *and* $n_1, \cdots, n_r$ *run through all positive integers.*

In the following, we use $H_p(\underline{x}_j, M)$ to denote the p-th Koszul homology group of a module M with respect to a sequence $x_1, \cdots, x_n$. For properties of Koszul homology one can refer to [1].

Proposition 2.5. *Let* A, I, M *be as in Proposition* 2.3 *and* $I = (y_1, y_2, \cdots, y_n)$. *If* $x_1, \cdots, x_s$ *is a weak* I *sequence such that*

$$(x_1^{r_1}, \cdots, x_{i-1}^{r_{i-1}})M : x_i^{r_i} \subseteq (x_1^{r_1}, \cdots, x_{i-1}^{r_{i-1}})M : I^r,$$

where r *is a fixed positive integer and* $r_1, \cdots, r_s$ *run through all positive integers, then there exists an integer* $k \in Z^+$ *which depends only on* r *such that* $I^k H_i(\underline{y}_j^{r_j}, M) = 0$, *for all* $i > n - s$ *and* $r_1, \cdots, r_s$ *running through all positive integers.*

Proof. If $M = \Gamma_I(M)$, the result is trivial. Suppose $M \ne \Gamma_I(M)$. Put $M' = M/\Gamma_I(M)$. We apply induction on the dimension d. Suppose the conclusion holds for those A-modules with dimension less than d. Consider the following short exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow M' \longrightarrow 0.$$

For any $r_1, \cdots, r_n \in Z^+$, we have

$$\cdots \longrightarrow H_i(\underline{y}_j^{r_j}, \Gamma_I(M) \longrightarrow H_i(\underline{y}_j^{r_j}, M) \longrightarrow H_i(\underline{y}_j^{r_j}, M') \longrightarrow \cdots .$$

It can be seen easily from the definition of Koszul complex that

$$I^r H_i(\underline{y}_j^{r_j}, \Gamma_I(M)) = 0, \text{ for all } i \ge 0.$$

Hence in order to prove that there exists $k \in Z^+$ such that $I^k H_i(\underline{y}_j^{r_j}, M) = 0$, for $i > n - s$, it suffices to prove that $I^k H_i(\underline{y}_j^{r_j}, M') = 0$, for $i > n - s$. Since x_1 is M'-regular, we have the short exact sequence for each $n_1 \in Z^+$,

$$0 \longrightarrow M' \xrightarrow{x_1^{n_1}} M' \longrightarrow M'/x_1^{n_1}M' \longrightarrow 0.$$

As $x_2, x_3, \cdots, x_s$ is a weak I sequence with respect to $M'/x_1^{n_1}M'$ for each n_1, and the integer r in the theorem may be chosen independent of the choice of n_1, by the induction hypothesis we have an integer $k \in Z^+$ such that

$$I^k H_i(\underline{y}_j^{r_j}, M'/x_1^{n_1}M') = 0, \text{ for all } i > n - s + 1 \text{ and } n_1 \in Z^+.$$

Now, consider the following exact sequence

$$\cdots \longrightarrow H_i(\underline{y}_j^{r_j}, M'/x_1^{n_1}M') \longrightarrow H_{i-1}(\underline{y}_j^{r_j}, M') \overset{(-1)^{i-1}x_1^{n_1}}{\longrightarrow} H_{i-1}(\underline{y}_j^{r_j}, M') \to .$$

For any fixed $r_1, r_2, \cdots, r_n$, $H_{i-1}(\underline{y}_j^{r_j}, M')$ is annihilated by $x_1^{n_1}$ for n_1 large enough, i.e. $H_i(\underline{y}_j^{r_j}, M/x_1^{n_1}M') \to H_{i-1}(\underline{y}_j^{r_j}, M')$ is surjective for large n_1. Hence, for $i > n - s$, we have

$$I^k H_i(\underline{y}_j^{r_j}, M') = 0.$$

Noting the arbitrary choices of $r_1. \cdots, r_n$, we have

$$I^k H_i(\underline{y}_j^{r_j}, M') = 0, \quad \text{for all } r_1, \cdots, r_n \in Z^+.$$

§3. Characterizations

In the section 2, we have proved that, if $\Gamma_I(M) \ne M$, then the length of any maximal weak I sequence must be finite. In this section we obtain a necessary and sufficient condition for a sequence to be a weak I sequence and give explicitly the length of a (hence all) maximal weak I sequence by means of Koszul homology groups and local cohomology groups.

Theorem 3.1. *Let I be an ideal of A and M a finite A-module such that $M \neq 0$. Let $x_1, \cdots, x_n$ be a sequence contained in I. Then the following conditions are equivalent:*

(i) *$x_1, \cdots, x_n$ is a weak I sequence;*

(ii) *There exists a $k > 0$ such that $I^k H_1(\underline{x}_i^{r_i}, M) = 0$, for $r_1, \cdots, r_n$ running through all positive integers.*

Proof. We use induction on n.

(i)$\Rightarrow$(ii) For $n = 1$, we have $H_1(x_1^{n_1}, M) = (0 :_M x_1^{n_1})(n_1 \in Z^+)$, so the assertion holds. For $n > 1$, we have the exact sequence

$$\begin{gathered}\cdots \longrightarrow H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) \longrightarrow H_1(x_1^{r_1}, \cdots, x_n^{r_n}, M) \longrightarrow \\ M/(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M \stackrel{x_n^{r_n}}{\longrightarrow} M/(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M \longrightarrow \cdots .\end{gathered} \tag{3.1}$$

By the induction hypothesis, there exists an integer k' such that $I^{k'} H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) = 0$ for all $r_1, \cdots, r_{n-1} \in Z^+$. On the other hand $x_1, \cdots, x_n$ is a weak I sequence, we have $k'' > 0$ such that $(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M : x_n^{r_n} \subseteq (x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M : I^{k''}$ for all $r_1, \cdots, r_n \in Z^+$. So in the above sequence, $I^{k''} \ker(x_n^{r_n}) = 0$ and it implies

$$I^{k'+k''} H_1(x_1^{r_1}, \cdots, x_n^{r_n}, M) = 0$$

for any $r_1, \cdots, r_n \in Z^+$.

(ii)$\Rightarrow$(i) For $1 \le i \le n$, set $M_i = M/(x_1, \cdots, x_i)M$. Then $M_i \neq 0$. By the hypothesis and by Proposition 2.5 we have

$$\cdots \to H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) \stackrel{-x_n^{r_n}}{\to} H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) \to H_1(x_1^{r_1}, \cdots, x_n^{r_n}, M) \to \cdots,$$

where $r_1, \cdots, r_{n-1}$ are arbitrary positive integers, and $r_n > k$. Hence

$$x_n^{r_n} H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M \supseteq I^k H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M).$$

This implies

$$x_n^{r_n - k} I_n^k H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) \supseteq I^k H_1(x_z^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M).$$

But quite generally $H_1(\underline{x}, M)$ is a finite A-module. By Nakayama Lemma we have

$$I^k H_1(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}}, M) = 0$$

for all $r_1, \cdots, r_{n-1} \in Z^+$. Thus by the induction hypothesis, $x_1, \cdots, x_{n-1}$ is a weak I sequence. Now consider the exact sequence in (3.1). We can see that

$$(x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M : x_n^{r_n} \subseteq (x_1^{r_1}, \cdots, x_{n-1}^{r_{n-1}})M : I^k$$

for all $r_1, \cdots, r_n \in Z^+$. Hence $x_1, \cdots, x_n$ is a weak I sequence.

Corollary 3.1. *Let $x_1, \cdots, x_n$ be a weak I sequence. Then $x_{i_1}, \cdots, x_{i_n}$ is a weak I sequence, where $x_{i_1}, \cdots, x_{i_n}$ is a permutation of $x_1, \cdots, x_n$.*

Now, we prove a lemma which will play an important role in the proof of Theorem 3.2.

Lemma 3.1. *Let A be a Notherian local ring and I an ideal of I. Let M be a finite A-module and $x_1, \cdots, x_r$ be a weak I sequence. If there exists an integer s such that $I^s H_I^0(M/(x_1^n, \cdots, x_r^{n_r})M) = 0$, for all $n_1, \cdots, n_r \in Z^+$, then there exists an element $x_{r+1} \in I$ such that $x_1, \cdots, x_{r+1}$, is a weak I sequence.*

Proof. It suffices to construct an element x_{r+1} such that for any $n_1, \cdots, n_r, n_{r+1} \in Z^+$,

$$(x_1^{n_1}, \cdots, x_r^{n_r})M : x_{r+1}^{n_{r+1}} \subseteq (x_1^{n_1}, \cdots, x_r^{n_r})M : I^s. \tag{3.2}$$

We first put $N = \sum_{i=1}^{r} n_i$ and choose an element $x_{r+1} \in I$ such that (3.2) holds for $N = r$, this is possible because $I^s H_I^0(M/(x_1, \cdots, x_r)M = 0$. Now we use induction on N to prove that x_{r+1} satisfies (3.2) for all $N \geq r$. Suppose the conclusion holds for those N' with $r \leq N' < N$. Due to Corollary 3.1, without loss of generality we may assume $n_1 > 1$. For any $a \in (x_1^{n_1}, \cdots, x_r^{n_r})M : x_{r+1}^{n_{r+1}}$, we may express

$$x_{r+1}^{n_{r+1}} a = x_1^{n_1} a_1 + a_1', \tag{3.3}$$

where $a_1 \in M$, $a_1' \in (x_2^{n_2}, \cdots, x_r^{n_r})M$. By the induction hypothesis, for any $y \in I^s$, we may write

$$ya = x_1^{n_1 - 1} a_2 + a_2', \tag{3.4}$$

where $a_2 \in M$, $a_2' \in (x_2^{n_2}, \cdots, x_r^{n_r})M$. From (3.3) and (3.4), we assert that

$$x_1^{n_1-1}(yx_1 a - x_{r+1}^{n_{r+1}} a_2) \in (x_2^{n_2}, \cdots, x_r^{n_r})M.$$

By Corollary 3.1, $x_2, \cdots, x_r, x_1$ is also a weak I sequence. Hence we can find an integer $s' \in Z^+$ such that for any $y' \in I^{s'}$, $y'(yx_1 a - x_{r+1}^{n_{r+1}} a_2) \in (x_2^{n_2}, \cdots, x_r^{n_r})M$. This implies $x_{r+1}^{n_{r+1}} y' a_2 \in (x_1, x_2^{n_2}, \cdots, x_r^{n_r})M$. By the induction hypothesis, we have for any $y'' \in I^s$, $y' y'' a_2 \in (x_1, x_2^{n_2}, \cdots, x_r^{n_r})M$. Hence, from (3.4), we obtain $yy'y''a \in (x_1^{n_1}, \cdots, x_r^{n_r})M$. By the arbitrary choices of y, y' and y'', we see that $I^{2s+s'} a \in (x_1^{n_1}, \cdots, x_r^{n_r})M$. But by the assumption, $I^s H_I^0(M/(x_1^{n_1}, \cdots, x_r^{n_r})M) = 0$. Therefore $a \in (x_1^{n_1}, \cdots, x_r^{n_r})M : I^s$ and this proves the lemma.

Theorem 3.2. *Let I be an ideal of A and M a finite A-module such that $\Gamma_I(M) \neq M$. Set $r = \inf_i \{i \mid$ for some $s > 0,\ I^s H_I^i(M) \neq 0\}$. Then every maximal weak I sequence in I has the same length r.*

Proof. Let $x_1, \cdots, x_s$ be a maximal weak I sequence in I. We argue by induction on s.

For $s = 1$, if $r \neq 1$, then there exists a positive integer k such that $I^k H_I^1(M) = 0$. By Proposition 2.1 (ii), we have the long exact sequence

$$0 \longrightarrow (0 :_M x_1^{n_1}) \longrightarrow H_I^0(M) \xrightarrow{x_1^{n_1}} H_I^0(M) \longrightarrow H_i^0(M/x_1^{n_1}M) \longrightarrow H_i^1(M) \longrightarrow \cdots.$$

Hence $I^{2k} H_I^0(M/x_1^{n_1}M) = 0$ for all $n_1 > 0$. By Theorem 3.1, we have a contradiction. Thus $r = 1$.

For $s > 1$, according to Proposition 2.3, we have $s \leq r$. If $s \neq r$, then there exists an integer $k > 0$ such that $I^k H_I^i(M) = 0$, for $i \leq s$. Using Proposition 2.1 (ii) s times, we can choose an integer k' (cf. $k' = 2^s k$) such that $I^{k'} H_I^0(M/(x_1^{n_1}, \cdots, x_s^{n_s})M) = 0$, where $n_1, \cdots, n_s$ run through all positive integers. By Lemma 3.1, we can construct an element $x_{s+1} \in I$ such that $x_1, \cdots, x_s, x_{s+1}$ is a weak I sequence. This contradicts the choices of $x_1, \cdots, x_s$. So $s = r$.

Write wdepth$(I, M) = r$. We call r the weak I-depth of M. If $M = \Gamma_I(M)$, the weak I-depth is by convention ∞. We make a remark here. For $i < r$, $H_I^i(M)$ is a Noetherian A-module, i.e., $H_I^i(M)$ is fintely generated. In fact, letting $x_1, \cdots, x_r$ be a weak I sequence, without loss of generality, we may assume depth$_I(M) \geq 1$. If $r > 1$, we have an integer k

as in the proof of Proposition 2.3, such that $I^kH_I^i(M/x_1^nM)=0$, for all $i<r-1$. Now, choose k large enough such that $I^kH_I^i(M)=0$, for $i<r$. Clearly, we have the short exact sequences

$$0\to H_I^i(M)\to H_I^i(M/x_1^kM)\to H_I^{i+1}(M)\to 0,\ \text{for } i<r-1.$$

So by induction on the dimension of M, $H_I^i(M)$ is finitely generated.

Corollary 3.2. *Let $I=(y_1,\cdots,y_n)$ be an ideal of A and M a finite A-module with $\Gamma_I(M)\neq M$. Set*

$$r'=\sup\{i\mid \textit{for any } s\in Z^+,\ \textit{there exist } r_1,\cdots,r_n\in Z^+ \textit{such that } I^sH_i(\underline{y}_j^{r_j},M)\neq 0\}.$$

Then $n-r'=\mathrm{wdepth}(I,M)$.

Proof. By Proposition 2.5 and Theorem 3.2, $n-r'\geq \mathrm{wdepth}(I,M)$. If $n-r'\neq \mathrm{wdepth}(I,M)$, then from

$$\lim_{r\to\infty}H_{n-i}(\underline{y}_i^r,M)=H_I^i(M),$$

there is an integer k such that $I^kH_I^r(M)=0$ ($r=\mathrm{wdepth}(I,M)$), a contradiction.

Corollary 3.3. *Let $I=(y_1,\cdots,y_n)$ be an ideal of A and M a finite A-module such that $\Gamma_I(M)\neq M$. Then the following conditions are equivalent:*

(i) *$y_1,\cdots,y_n$ is a weak I sequence;*

(ii) $\mathrm{wdepth}(I,M)=n$.

§4. The Case wdepth(I,M)=dim$M$

In this section we consider the case $\mathrm{wdepth}(I,M)=\mathrm{dim}M$. It is known that if $\mathrm{wdepth}(m,M)=\mathrm{dim}M$, then a sequence $x_1,x_2,\cdots,x_d$ is a weak m sequence if and only if $x_1,x_2,\cdots,x_d$ is a system of parameters of M. Now we extend this result to our case.

Theorem 4.1. *Let I be an ideal of A and M a finite A-module of dimension d with $\Gamma_I(M)\neq M$, and* $\mathrm{wdepth}(I,M)=d$. *Let $x_1,x_2,\cdots,x_d$ be a sequence contained in I. Then the following conditions are equivalent:*

(i) *$x_1,x_2,\cdots,x_d$ is a weak I sequence;*

(ii) *there exists a positve integer n such that $I^nM\subseteq(x_1,x_2,\cdots,x_d)M$.*

Proof. (i)$\Longrightarrow$(ii) We use induction on d. For $d=1$, put $M'=M/\Gamma_I(M)$. Then x_1 is M'-regular and $\dim M'/x_1M'=0$. So $\Gamma_I(M'/x_1M')=M'/x_1M'$. This implies $I^nM'\subseteq x_1M'$ for some $n\in Z^+$, namely $I^nM\subseteq x_1M+\Gamma_I(M)$. For n large enough, we have $I^nM\subseteq x_1M$. Suppose the conclusion holds for those A-modules with dimension less than d ($d>1$). Put $M'=M/\Gamma_I(M)$. Then $\Gamma(M')\neq M'$ and $\dim M'=d$ (because of wdepth $(I,M')=d$). Now consider the short exact sequence for each $s\in Z^+$

$$0\longrightarrow M'\xrightarrow{x_1^s}M'\longrightarrow M'/x_1^sM'\longrightarrow 0.$$

We have the long exact sequence

$$0\longrightarrow H_I^0(M'/x_1^sM')\longrightarrow H_I^1(M')\xrightarrow{x_1^s}H_I^1(M')\longrightarrow\cdots.$$

Since $d>1$, we can choose $k\in Z^+$ such that $I^kH_I^1(M')=0$. Hence $I^kH_I^0(M'/x_1^sM')=0$ for all $s\in Z^+$. Now we claim that $\Gamma_I(M'/x_1^{k+1}M')\neq M'/x_1^{k+1}M'$. Otherwise, $I^kM'\subseteq x^{k+1}M'$. This implies $I^kM'\subseteq I^kx_1M'$. By Nakayama lemma, $I^kM'=0$, a contradiction. So from $\Gamma_I(M'/x_1^{k+1}M')\neq M'/x_1^{k+1}M$, we assert that $\Gamma_I(M'/x_1M')\neq M'/x_1M'$,

$\dim M'/x_1M' = d-1$ and wdepth $(M'/x_1M') = d-1$. By the induction hypothesis, we can choose a positive integer n such that $I^nM' \subseteq (x_1, \cdots, x_d)M'$. For n large enough, we have $I^nM \subseteq (x_1, \cdots, x_d)M$.

(ii)$\Longrightarrow$(i) Clealy, $\text{rad}((x_1, \cdots, x_d) + \text{ann}M) = \text{rad}(I + \text{ann}M)$. So by the characterization of local cohomology via Koszul cohomology, we have $H^i_{I'}(M) \simeq H^i_I(M)$ for all $i \geq 0$, where $I' = (x_1, \cdots, x_d)$. From this we can assert that $x_1, x_2, \cdots, x_d$ is a weak I sequence by Corollary 3.3.

Now, we prove a result which states that I has a generator consisting of weak I sequences.

Lemma 4.1. *If* wdepth$(I, M) = d$, *then there exists* $n \in Z^+$ *such that for every weak* I *sequence* $x_1, x_2, \cdots, x_d$

$$\dim M/(x_1^{n_1}, \cdots, x_{d-1}^{n_{d-1}})M : I^n = 1,$$

where $n_1, \cdots, n_{d-1}$ *run through all positive integers.*

Proof. We use induction on the dimension d. For $d = 1$, the result is obvious. Suppose the result holds for those A-module with dimension $d' < d$. For $d > 1$, put $M' = M/\Gamma_I(M)$. As $x_2, \cdots, x_d$ is a weak I-sequence with respect to $M'/x_1^{n_1}M'$, and $\dim M'/x_1^{n_1}M' = d-1$ (see the proof of Theorem 4.1), by the induction hypothesis, we assert that

$$\dim M'/(x_2^{n_1}, \cdots, x_{d-1}^{n_{d-1}})M' : I^n) = 1$$

for n satisfying Proposition 2.4. This implies that $\dim M/((x_1^{n_1}, \cdots, x_{d-1}^{n_{d-1}})M : I^{n'}) = 1$, for a fixed large n'.

Proposition 4.1. *If* wdepth$(I, M) = d$, *then there exists* $B = \{y_1, \cdots, y_n\} \subseteq I$ *such that every* d*-element of* B *forms a weak* I *sequence and* $I = (y_1, \cdots, y_n)$.

Proof. Since wdepth $(I, M) = d$, we have a weak I sequence $y_1, \cdots, y_d$. Let $B' \supseteq \{y_1, \cdots, y_d\}$ be a maximal subset of I such that every d-element of B' forms a d-sequence and $(B') \subseteq I$. As A is Notherian, we can choose a finite subset $B \subseteq B'$, $B = \{y_1, \cdots, y_d, y_{d+1}, \cdots, y_n\}$ such that $(B') = (B)$. By Proposition 2.4 and Lemma 4.1, we have $k \in Z^+$, such that for every d-element $y_{i_1}, \cdots, y_{i_d}$ of B,

$$(y_{i_1}, \cdots, y_{i_{d-1}})M : y_{i_d} \subseteq (y_{i_1}, \cdots, y_{i_{d-1}})M : I^k,$$

and $\dim M/(y_{i-1}, \cdots, y_{i_{d-1}})M : I^k = 1$. This implies that y_{i_d} is $M/((y_{i_1}, \cdots, y_{i_{d-1}})M : I^k)$-regular. If $(B) \neq I$, let $P_1, \cdots, P_s$ be the non-embedded associated primes of all submodules $(y_{i_1}, \cdots, y_{i_{d-1}})M : I^k$, where $y_{i_1}, \cdots, y_d$ is an arbitrary $(d-1)$-element of B. Choose $y \in I \setminus (B)$. If $y \notin P_i$ for all $i = 1, \cdots, s$, then we set $y_{n+1} = y$. If $y \in P_i$ for $i = 1, \cdots, t$ and $y \notin P_i$ for $i = t+1, \cdots, s$, $1 \leq t \leq s$, we first choose an element $y' \in ((B) \cap P_{t+1} \cdots \cap P_s) \setminus P_1 \cup \cdots \cup P_t$, which is possible because $B \not\subseteq P_i$ for $1 \leq i \leq t$. Otherwise $B \subseteq P_i$, P_i is a non-embedded associated prime ideal of some submodule $(y_{i_1}, \cdots, y_{i_d-1})M : I^k$. It means there exists $b \in M/((y_{i_1}, \cdots, y_{i_{d-1}})M : I^k)$, $b \neq 0$ and ann $b = P_i$, so $y_{i_d}b = 0$. This implies $b = 0$, a contradiction. Set $y_{n+1} = y' + y$. Clearly, in either case, we have $y_{n+1} \notin P_1, P_2, \cdots, P_s$. From this we can see that y_{n+1} is $(M/(y_{i_1}, \cdots, y_{i_{d-1}})M : I^k)$-regular for any $\{i_1, \cdots, i_{d-1}\} \subseteq \{1, \cdots, n\}$. Hence, for any fixed $i_1, \cdots, i_{d-1}$, we have an integer t such that $I^tM \subseteq (y_{i_1}, \cdots, y_{i_{d-1}}, y_{n+1})M$. By Theorem 4.1, $y_{i_1}, \cdots, y_{i_{d-1}}, y_{n+1}$ is a weak I sequence.

Since $(y_1, \cdots, y_n, y_{n+1}) \nsubseteq (y_1, \cdots, y_n)$, this contradicts the choice of B'. So $I = (y_1, \cdots, y_n)$ and the proof is complete.

§5. Standard Ideals

In this section, we extend the notion of standard ideals in [13] to more general cases and discuss some basic facts about it.

Definition 5.1. *We say that I is an M-standard ideal if* $\mathrm{wdepth}(I, M) = d$ *and every maximal weak I sequence with respect to M $x_1, \cdots, x_d$ forms an I-weak sequence, i.e. the equality $(x_1, \cdots, x_{i-1})M : x_i = (x_1, \cdots, x_{i-1}) : I$ holds for i $(1 \le i \le d)$.*

Recall that a sequence $x_1, \cdots, x_r \in I$ is said to be a d-sequence if

(i) $x_iM \not\subset x_1M + \cdots + x_{i-1}M + x_{i+1}M + \cdots + x_rM$,

(ii) $(x_1, \cdots, x_{i-1})M : x_ix_k = (x_1, \cdots, x_{i-1})M : x_i$

for $1 \le i \le r$, $i \le k \le r$.

For the properties of d-sequences, one can refer to [13] and so on. The following result is an extension to the results in [13].

Theorem 5.1. *If* $\mathrm{wdepth}(I, M) = d$*, then the following conditions are equivalent:*

(i) *I is M-standard;*

(ii) *every I-weak sequence $x_1, \cdots, x_d$ forms a d-sequence.*

Proof. (i)$\Rightarrow$ (ii) Let $x_1, \cdots, x_d$ be an I-weak sequence, since every permutation of $x_1, \cdots, x_d$ is also an I-weak sequence. In order to prove $x_iM \not\subset x_1M + \cdots + x_{i-1}M + x_{i+1}M + \cdots + x_rM$, it suffices to prove $x_rM \not\subset x_1M + \cdots + x_{r-1}M$. As $x_1, \cdots, x_{r-1}, x_r$ is a weak I sequence, according to Lemma 4.1, $\dim M/(x_1, \cdots, x_{d-1})M : I^n = 1$ for n large enough. If $x_rM \subseteq (x_1, \cdots, x_{r-1})M$, we have $(x_1, \cdots, x_{d-1})M : I^n = M$; this is a contrdiction. For any i,k $(1 \le i \le d,\ i \le k \le d)$, as I is M-standard, we have

$$(x_1, \cdots, x_{i-1})M : x_i = (x_1, \cdots, x_{i-1})M : x_i^2,$$
$$(x_1, \cdots, x_{i-1})M : x_k = (x_1, \cdots, x_{i-1})M : I.$$

As $x_1, \cdots, x_{i-1}, x_k$ and $x_1, \cdots, x_{i-1}, x_i^2$ are I-weak sequences, assume $c \in (x_1, \cdots, x_{i-1})M : x_ix_k$, that is, $x_ix_kc \in (x_1, \cdots, x_{i-1})M$. This implies $x_ic \in (x_1, \cdots, x_{i-1})M : I$. It shows that $x_i^2c \in (x_1, \cdots, x_{i-1})M$. Hence $c \in (x_i, \cdots, x_{i-1})M : x_i$, that is,

$$(x_1, \cdots, x_{i-1})M : x_ix_k = (x_1, \cdots, x_{i-1})M : x_i.$$

(ii)$\Rightarrow$(i) By the proof of Proposition 4.1, every weak I-sequence $x_1 \cdots, x_d$ can be extended to $B = \{x_1, \cdots, x_d, \cdots, x_n\}$ such that every d-element subset of B forms a weak I-sequence and $I = (x_1, \cdots, x_n)$. For each i $(1 \le i \le d)$, since $x_1, x_2, \cdots, x_{i-1}, x_k$ $(i \le k \le n)$ is a weak I-sequence, by the hypothesis it is a d-sequence. We have

$$(x_1, \cdots, x_{i-1})M : x_k^s = (x_1, \cdots, x_{i-1})M : x_k$$

for any $s \in Z^+$, it means that

$$(x_1, \cdots, x_{i-1})M : x_k \supseteq \bigcup_{n=1}^{\infty} (x_1, \cdots, x_{i-1})M : I^n.$$

On the other hand, $x_1, \cdots, x_{i-1}, x_k$ is a weak I sequence, and we have

$$(x_1, \cdots, x_{i-1})M : x_k \subseteq \bigcup_{n=1}^{\infty} (x_1, \cdots, x_{i-1})M : I^n.$$

Hence we have

$$(x_1, \cdots, x_{i-1})M : x_k = \bigcup_{n=1}^{\infty} (x_1, \cdots, x_{i-1})M : I^n,$$

so

$$(x_1, \cdots, x_{i-1})M : x_i = \bigcap_{x \in B} (x_1, \cdots, x_{i-1})M : x = (x_1, \cdots, x_{i-1})M : I.$$

Because of Theorem 5.1, many results concerning m-primary standard ideals in [8, 13] can be extended to our cases. We quote two important results here, which can be proved word by word as that in [8, 13]. We will omit the proof of them.

Theorem 5.2.[8,Theorem 1] *Let M be a finite A module of dimesion d and I an ideal of A such that $\Gamma_I(M) \neq M$. If the canonical maps ϕ_i: $\mathrm{Ext}\,^i_A(A/I, M) \to H^i_I(M)$ are surjective for all $i \neq d$, then I is M-standard.*

Let $G_q(M) = \underset{n \geq 0}{\oplus} q^n M / q^{n+1} M$ be the associated module of M relative to an ideal $q = (x_1, \cdots, x_d) \subseteq I$ of A where I is an M-standard ideal and $x_1, \cdots, x_d$ is an I-weak sequence. It is well known that[7] the dimension of $G_q(M)$ as $G_q(A)$-module is the same as that of M. Then we have

Theorem 5.3.[13,Theorem 5.4] *Let I be an M-standard ideal and $x_1, \cdots, x_d$ be an I-weak sequence. Then*

(i) $H^i_P(G_q(M)) = H^i_I(M)(i)$ *for* $i = 0, \cdots, d-1$,

(ii) $[H^d_P(G_q(M))]_n = 0$ *for* $n > -d$,

where $H^i_I(M)$ is considered as a graded module concentrated in degree 0 (the integer in the round brackets denotes the shifting degree) and $P = I/q \oplus q/q^2 \oplus \cdots$.

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