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THE FEYNMAN-KAC FORMULA FOR SYMMETRIC MARKOV PROCESSES**

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Abstract

Let X be an m-symmetric Markov process and M a multiplicative functional of X such that the M-subprocess of X is also m-symmetric. The author characterizes the Dirichlet form associated with the subprocess in terms of that associated with X and the bivariate Revuz measure of M.

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§1. Introduction

Let X be a Borel right Markov process on $(E, \mathcal{E})$ with semigroup (P_t) which is symmetric relative to a σ-finite measure m on E. Then P_t is self-adjoint as an operator on $L^2(m)$. Let $(a, \mathcal{D})$ denote the Dirichlet form associated with X and m. Assume that M is a decreasing multiplicative functional of X such that the M-subprocess of X is also m-symmetric. Our goal in this paper is to give a formula of Feynman-Kac type to characterize the Dirichlet form $(b, \mathcal{D}(M))$ associated with (X, M).

Our approach is that used in [2], and our tool is bivariate Revuz measures and respective Revuz formula. Some standard results on Dirichlet form are contained in §2. Our main results will be proved in §3.

§2. Preliminaries

Let $X = (X_t, P^x)$ be a Borel right process with state space $(E, \mathcal{E})$ which is assumed to be Lusinian, semigroup (P_t) and resolvent (U^q). We fix a σ-finite measure m with respect to which X is symmetric

$$(f,\ P_t g) = (P_t f,\ g)$$

for each $t > 0$ where $(\cdot, \cdot)$ is the inner product in $L^2(m)$. Note that Walsh[8] has showed that P^m a.s.

$$X_{t-} \quad \text{exists in } E \quad \text{for all} \quad t \in]0, \zeta[. \tag{2.1}$$

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The family (P_t) determines a strongly continuous semigroup of subMarkovian contractions on $L^2(m)$, which we denote by (P_t) as well. It is proved in [2] that X is also an m-special standard Markov process. Thus by [6, IV.5.1] there exists a unique Dirichlet form $(a, \mathcal{D}(X))$ associated with X such that

$$\begin{aligned} \mathcal{D}(X) &= \{u \in L^2(m) : \uparrow \lim_{p\to\infty} a^{(p)}(u,u) < \infty\}, \\ a(u,v) &= \lim_{p\to\infty} a^{(p)}(u,v) \quad \text{for } u,v \in \mathcal{D}(X), \end{aligned} \tag{2.2}$$

where $a^{(p)}(u,v) = p(u, v - pU^p v)$, called the approximating form of a.

A set B is m-polar if and only if $P^m(T_B < \infty) = 0$, and a statement is true m-quasi everywhere (m-q.e.) provided it holds off an m-polar set. It is known from [6] that any function $u \in \mathcal{D}(X)$ admits a quasi-continuous m-version. Note that by the transfer method developped in [6, VI.2], we can make free use of almost all results in [4], though we are actually in the frame of [6]. Now we are going to settle down the notations for multiplicative functionals involved in this paper.

Definition 2.1. *A real-valued, decreasing, right continuous process $M = (M_t : t \geq 0)$ is called a multiplicative functional (MF) of X if M is adapted, i.e., $M_t \in \mathcal{F}_t$ for any $t \geq 0$, and $M_{t+s}(\omega) = M_t(\omega)\cdot M_s(\theta_t\omega)$ for any $s,t \geq 0$ and $\omega \in \Omega$. In addition, an MF M is exact provided for any $t > 0$ and every sequence $t_n \downdownarrows 0$,*

$$M_{t-t_n} \circ \theta_{t_n} \to M_t \quad a.s. \text{ as } n \to \infty.$$

Let MF be the set of all exact multiplicative functionals of X. For any $M \in$ MF, denote

$$E_M := \{x \in E : P^x(M_0 = 1) = 1\}, \quad S_M := \inf\{t : \ M_t = 0\}.$$

Then by [7, §57] the M-subprocess, (X, M), is a right Markov process with state space E_M, semigroup (Q_t) and resolvent (V^q) given by

$$\begin{aligned} Q_t f(x) &:= P^x(f(X_t)M_t), \\ V^q f(x) &:= P^x \int_0^\infty e^{-qt} f(X_t) M_t dt. \end{aligned}$$

Let $M \in$ MF. Write $m^* := m|_{E_M}$. The bivariate Revuz measure of M with respect to m is defined by

$$\nu_M(F) := \lim_{t\to 0} \frac{1}{t} P^{m^*} \int_0^t F(X_{s-}, X_s) d(-M_s), \ F \in \mathcal{E}\times\mathcal{E}, F \geq 0.$$

Denote $\rho_M := \nu_M(1\otimes\cdot)$. Note that we actually use the definition for additive functionals of M (Refer to [3] for definition of additive functionals of M, or M-additive functionals, and existence of Revuz measures) since $(1 - M_t : t > 0)$ is an additive functional of M. For the later use of definition of Revuz measures we will not repeat this technique. It is clear that ν_M does not charge any m-bi-polar set, which is a set of form $E_1 \times E_2$ with either E_1 or E_2 being m-polar.

Let sMF be the set of all $M \in$ MF such that the M-subprocess (X, M) is also symmetric with respect to m, and

$$\begin{aligned} \text{sMF}_+ &:= \{M \in \text{sMF} : \ E_M = E\}; \\ \text{sMF}_{++} &:= \{M \in \text{sMF} : \ S_M \geq \zeta, \ \text{ where } \zeta \ \text{ is the life time of } X\}. \end{aligned}$$

From §5 of [9] we know that $M \in$ sMF if and only if the measure ν_M is symmetric; that is, $\nu_M(dx,dy) = \nu_M(dy,dx)$. There certainly exists a Dirichlet form, donoted by $(b, \mathcal{D}(M))$, associated with (X, M).

Some results, which will be used throughout this paper, are listed below for handy reference.

Generalized Revuz formula: for any $f,\, g \in p\mathcal{E}$

$$(g, U_M^q f) = \nu_M(V^q g \otimes f), \tag{2.4}$$

where

$$U_M^q f(x) := P^x \int_0^\infty e^{-qt} f(X_t) d(-M_t).$$

This can be proved by the approach employed by Getoor and Sharpe[5] for proving the Revuz formula in the frame of weak duality.

Representation of bivariate Revuz measures: Let $S := S_M$ and ν_S the bivariate Revuz measure of $1_{[0,S[}$. The Stieltjes logarithm of M is defined by

$$(\,\mathrm{slog} M)_t := \int_0^t 1_{[0,S[}(s) \frac{d(-M_s)}{M_{s-}}. \tag{2.5}$$

Then $\mathrm{slog} M$ is an S-additive functional. Indeed it is easy to see that

$$(\,\mathrm{slog} M)_{t+s} = (\,\mathrm{slog} M)_t + (\,\mathrm{slog} M)_s \circ \theta_t \cdot 1_{\{t<S\}}.$$

Hence $\nu_{\,\mathrm{slog} M}$ makes sense. Let ${}^S\nu_M$ be the bivariate Revuz measure of M relative to m in the frame of the subprocess, (X,S), of X killed at S. Then it is not hard to check that

$$\nu_M = \nu_S + \nu_{\,\mathrm{slog} M} \text{ and } \nu_{\,\mathrm{slog} M} = {}^S\nu_M. \tag{2.6}$$

The readers interested in (2.4) and (2.5) may refer to [9] for details.

§3. Feynman-Kac Formula

In this section any element in $\mathcal{D}(X)$ takes its quasi-continuous m-version. Let $\mathcal{D}^+(X)$ and $\mathcal{D}^+(M)$ be the sets of those non-negative elements in $\mathcal{D}(X)$ and $\mathcal{D}(M)$, respectively, and

$$\mathcal{L}(M) = \{u \in \mathcal{D}(X) : \quad \nu_M(u \otimes u) < \infty\}.$$

Proposition 3.1. *If $M \in sMF_+$ and ν_M is finite, then $\mathcal{D}^+(M) = \mathcal{D}^+(X) \cap \mathcal{L}(M)$ and*

$$b(u,u) = a(u,u) + \nu_M(u \otimes u), \quad \text{for } u \in \mathcal{D}(M). \tag{3.1}$$

Proof. Let $u \in L^2(m)$ and $u \geq 0$. By Dynkin's formula[7, §56]:

$$U^p = V^p + U_M^p U^p,\ p \geq 0$$

and using the approximating form, we have

$$b^{(p)}(u,u) = p(u, u - pV^p u) = a^{(p)}(u,u) + p^2(u, U_M^p U^p u).$$

Then using the Revuz formula (2.4), we have

$$(u, U_M^p U^p u)_m = \nu_M(V^p u \otimes U^p u).$$

Bring them together, we have

$$b^{(p)}(u,u) = a^{(p)}(u,u) + \nu_M(pV^p u \otimes pU^p u). \tag{3.2}$$

Immediately from (3.2), it follows that $\mathcal{D}^+(M) \subset \mathcal{D}^+(X)$.

(1) If $u \in \mathcal{D}(M)$, by [4, Theorem 3.1.4], there is a sequence $p_k \uparrow \infty$ such that $p_k U^{p_k} u$ converges to u m-q.e. Since $E = E_M$, we can pick (the same) p_k such that $p_k V^{p_k} u \to u$ q.e. m. Since ν_M does not charge m-bi-polar sets, $p_k V^{p_k} u \otimes p_k U^{p_k} u$ converges to $u \otimes u$ a.e. ν_M. Again invoking Fatou's Lemma, we have

$$\infty > \lim_k \nu_M(p_k V^{p_k} u \otimes p_k U^{p_k} u) \geq \nu_M(u \otimes u).$$

Therefore $\mathcal{D}^+(M) \subset \mathcal{D}^+(X) \cap \mathcal{L}(M)$.

(2) If $u \in \mathcal{D}(X) \cap \mathcal{L}(M)$ and u is bounded, there is a p_k such that $p_k U^{p_k} u \to u$ m-q.e. Since u is bounded and ν_M is finite, the dominated convergence theorem gives

$$\begin{aligned}\sup_p b^{(p)}(u,u) &= \lim_k b^{(p_k)}(u,u)\\ &\leq \lim_k [a^{(p_k)}(u,u) + \nu_M(p_k U^{p_k} u \otimes p_k U^{p_k} u)]\\ &= a(u,u) + \nu_M(u \otimes u) < \infty,\end{aligned}$$

i.e., $u \in \mathcal{D}^+(M)$. Then pick a subsequence p_k such that $p_k V^{p_k} u \to u$ m-q.e., and (3.2) becomes (3.1) as $k \uparrow \infty$ by the dominated convergence theorem again.

(3) If $u \in \mathcal{D}^+(X) \cap \mathcal{L}(M)$, define $u_n = u \wedge n$. Then $u_n \in \mathcal{D}^+(M)$ and

$$b(u_n, u_n) = a(u_n, u_n) + \nu_M(u_n \otimes u_n). \tag{3.3}$$

Clearly

$$\sup_n b(u_n, u_n) \leq a(u,u) + \nu_M(u \otimes u) < \infty$$

and by [6, (I.2.12)] $u \in \mathcal{D}^+(M)$. Then by the monotone convergence theorem and [4, Theorem 1.4.2], we have (3.1).

In the rest of this section, we aim to remove two auxilliary conditions in Proposition 3.1: the positivity of u and the finiteness of ν_M.

Now we introduce the well-known Beurling-Deny's formula of Dirichlet forms. Since $(a, \mathcal{D}(X))$ is quasi-regular, by [6, VI.2.5] the Fukushima's decomposition still holds: for any $u \in \mathcal{D}(X)$, $u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}$ where $M^{[u]}$ is a martingale additive functional of finite energy and $N^{[u]}$ is a continuous additive functional of zero-energy. Hence Beurling-Deny's formula (see [6] or [4]) follows:

$$a(u,u) = a^c(u,u) + \frac{1}{2}\int\int (u(x) - u(y))^2 \nu^a(dx, dy) + k^a(u^2), \tag{3.4}$$

where a^{sc} is the strongly continuous part, ν^a the jumping measure of a, k^a the killing measures of a.

Denote $a^c(u,u) := a^{sc}(u,u) + k^a(u^2)$ and $u_d(x,y) := u(x) - u(y)$ for any $u \in \mathcal{E}$.

Proposition 3.2. *If $M \in sMF_+$ and ν_M is finite, then*

$$\begin{aligned}\mathcal{D}(M) &= \mathcal{D}(X) \cap L^2(\rho_M),\\ b(u,u) &= a(u,u) + \nu_M(u \otimes u)\\ &= a(u,u) - \frac{1}{2}\nu_M(u_d^2) + \rho_M(u^2),\end{aligned} \tag{3.5}$$

for any $u \in \mathcal{D}(M)$.

Proof. Let D be the diagonal of $E \times E$. Since each jump of M is less than or equal to 1, $1_{D^c} \cdot \nu_M \le \nu^a$. If $u \in \mathcal{D}(M)$ and $u \ge 0$, then by Proposition 3.1 $u \in \mathcal{D}(X)$, $\nu_M(u \otimes u) < \infty$. But it is easy to check that

$$\nu_M(u \otimes u) = -\frac{1}{2}\nu_M(u_d^2) + \rho_M(u^2) \tag{3.6}$$

and

$$\nu_M(u_d^2) \le \nu^a(u_d^2) \le a(u,u) < \infty.$$

Thus we have $\rho_M(u^2) < \infty$, i.e.,

$$\mathcal{D}^+(M) \subset \mathcal{D}^+(X) \cap L^2(\rho_M).$$

Conversely, if $u \ge 0$ and $u \in \mathcal{D}(X) \cap L^2(\rho_M)$, it follows from (3.6) that $\nu_M(u \otimes u) < \infty$. Then by Proposition 3.1

$$\mathcal{D}^+(X) \cap L_M^2(\rho_M) \subset \mathcal{D}^+(M).$$

In general $u = u^+ - u^-$. Since $u \longrightarrow u^+$ and $u \longrightarrow u^-$ are contractions which operate both sides of (3.5), the result follows immediately.

We will now consider two basic types of multiplicative functionals for which we can remove the finiteness condition on ν_M in Proposition 3.2.

Proposition 3.3. *If $M \in sMF_{++}$, then we have*

$$\begin{aligned} \mathcal{D}(M) &= \mathcal{D}(X) \cap L^2(\rho_M), \\ b(u,u) &= a(u,u) + \nu_M(u \otimes u) \\ &= a(u,u) - \frac{1}{2}\nu_M(u_d^2) + \rho_M(u^2) \end{aligned} \tag{3.7}$$

for any $u \in \mathcal{D}(M)$,

Proof. Since ν_M is symmetric, we can find a sequence of symmetric functions $\{g_n\} \subset \mathcal{E} \times \mathcal{E}$ satisfying that $0 < g_n < 1$, $g_n \uparrow 1$ and $g_n \cdot \nu_M$ is finite. Due to the work of Fitzsimmons[2] it suffices to show Proposition 3.3 when M is purely discontinuous. Then

$$M_t = \prod_{s \le t} [1 - \Phi(X_{s-}, X_s)],$$

where $\Phi \in p\mathcal{E} \times \mathcal{E}$, $\Phi < 1$ and Φ vanishes on D. It is clear that

$$(\text{ slog} M)_t = \sum_{s \le t} \Phi(X_{s-}, X_s) \quad \text{and} \quad \nu_M = \Phi \cdot \nu^a.$$

Define

$$M_t^n = \prod_{s \le t} [1 - g_n \cdot \Phi(X_{s-}, X_s)].$$

It is easy to see that $M_t^n \downarrow M_t$ a.s. for each t, $M^n \in \mathrm{MF}_{++}$, $\nu_{M^n} = g_n \cdot \nu_M$ which is finite and symmetric, and thus $M^n \in \mathrm{sMF}_{++}$. We can apply Proposition 3.2 to $(b^n, \mathcal{D}(M^n))$.

Now let (V_n^p) be the resolvent of (X, M^n) and $(b^n, \mathcal{D}(M^n))$ the Dirichlet form associated with (X, M^n). Clearly $V_n^p f(x) \downarrow V^p f(x)$ if $f \ge 0$ and $V_1^p f(x) < \infty$. We need only to show that $\mathcal{D}(X) \cap \mathcal{L}(M) \subset \mathcal{D}(M)$.

Given $u \in p\mathcal{D}(X) \cap \mathcal{L}(M) \subset L^2(m)$, we have $V_1^p u \le U^p u < \infty$ a.e. m. Since

$$\mathcal{D}(M^n) = \mathcal{D}(X) \cap \mathcal{L}(M^n) \quad \text{for any } n,$$

it follows that

$$\begin{aligned} b(u,u) &= \sup_p b^{(p)}(u,u) \\ &= \sup_n \sup_p (u, p(u - pV_n^p u)) \\ &= \sup_n b^n(u,u) \\ &= \sup_n (a(u,u) + \nu_{M^n}(u \otimes u)) \\ &= \sup_n (a(u,u) + g_n \cdot \nu_M(u \otimes u)) \\ &= a(u,u) + \nu_M(u \otimes u) < \infty. \end{aligned}$$

That completes the proof.

Remark. Actually Proposition 3.3 was proved in [10] without revoking the generalized Revuz formula (2.4), but a bit more indirectly.

Proposition 3.4. *Let $T \in sMF_+$ and $(b, \mathcal{D}(T))$ be the Dirichlet form associated with the subprocess (X,T). Then*

$$\begin{aligned} \mathcal{D}(T) &= \mathcal{D}(X) \cap L^2(\rho_T); \\ b(u,u) &= a(u,u) + \nu_T(u \otimes u) \\ &= a(u,u) - \frac{1}{2}\nu_T(u_d^2) + \rho_T(u^2), \quad u \in \mathcal{D}(T). \end{aligned} \tag{3.8}$$

Proof. By representation of terminal times, there exists $L \in \mathcal{E} \times \mathcal{E}$, which is disjoint from D, such that

$$T = J_L := \inf\{t > 0 : \ (X_{t-}, X_t) \in L\}.$$

Then by [9] $\nu_T = 1_L \cdot \nu^a$. For $n \geq 1$ let

$$G_n := \Big\{(x,y) \in E \times E : \ d(x,y) > \frac{1}{n}\Big\},$$

where d is a metric on E compatible with the topology. Since ν^a is symmetric, σ-finite and carried by $E \times E - D$, we can choose a sequence of sets $\{E_n\} \subset \mathcal{E} \times \mathcal{E}$ such that

(1) $E_n \subset E \times E - D$ and $E_n \uparrow E \times E - D$;

(2) E_n is symmetric;

(3) $\nu^a(E_n) < \infty$.

Define

$$T_n := \inf\{t > 0 : \ (X_{t-}, X_t) \in L \cap G_n \cap E_n\}.$$

It is easy to check $T_n \downarrow T$. But we need a stronger convergence in the sense of the following lemma.

Lemma 3.1. *For P^m-a.s. ω, there exists an integer $n = n(\omega)$ such that $T_k = T$ for $k \geq n$.*

Proof. [9, I.3.2] tells us that P^m-a.s. $(X_{T-}, X_T) \in L$. For such an ω, $X_{T-} \neq X_T$, and since $G_n \cap E_n \uparrow E \times E - D$, we can find $n = n(\omega)$ large enough such that $(X_{T-}, X_T) \in G_n \cap E_n$. Hence $(X_{T-}, X_T) \in L \cap G_n \cap E_n$, or $T_n = T$.

Back to the proof of Proposition 3.4, let V_n^p be the resolvent of (X, T_n). By this strong convergence of $\{T_n\}$, we have

$$V_n^p f(x) = P^x \int_0^\infty e^{-pt} f(X_t) 1_{\{t<T_n\}} dt$$
$$\downarrow P^x \int_0^\infty e^{-pt} f(X_t) 1_{\{t<T\}} dt = V^p f(x)$$

for m-a.e. x. Since G_n and E_n are symmetric and

$$\nu_{T_n}(1) = \nu^a(L \cap G_n \cap E_n) < \infty,$$

$T_n \in \mathrm{sMF}_+$ and (3.8) holds for T_n. On the other hand,

$$\nu_{T_n} = 1_{L\cap G_n \cap E_n} \cdot \nu^a \uparrow 1_L \cdot \nu^a = \nu_T.$$

Therefore for $u \in p\mathcal{D}(X) \cap \mathcal{L}(T)$,

$$\begin{aligned} b(u,u) &= \sup_p b^{(p)}(u,u) \\ &= \sup_n \sup_p (u, p(u - pV_n^p u)) \\ &= \sup_n (a(u,u) + \nu_{T_n}(u \otimes u)) \\ &= \sup_n (a(u,u) + \nu_T(u \otimes u)) < \infty; \end{aligned}$$

i.e., $u \in \mathcal{D}(T)$.

Here is our final version of the Feynman-Kac formula for Dirichlet forms.

Theorem 3.1. *Let $M \in sMF$ and $(b, \mathcal{D}(M))$ be the Dirichlet form associated with the subprocess (X, M). Then*

$$\begin{aligned} \mathcal{D}(M) &= \mathcal{D}_{E_M}(X) \cap L^2(\rho_M); \\ b(u,u) &= a(u,u) + \nu_M(u \otimes u) \\ &= a^{sc}(u,u) + \frac{1}{2}(\nu^a - \nu_M)(u_d^2) + (k^a + \rho_M)(u^2), \quad u \in \mathcal{D}(M), \end{aligned} \tag{3.9}$$

where $\mathcal{D}_{E_M}(X) := \{u \in \mathcal{D}(X) : \ u = 0 \quad q.e. \ on \ E_M^c\}$.

Remark. (3.9) actually gives the Beurling-Deny's decomposition of b. It says that $b^{sc} = a^{sc}$, $\nu^b = \nu^a - \nu_M|_{D^c}$ and $k^b = k^a + \rho_M$.

Proof. The whole proof will be accomplished in three steps.

(1) Killing X by $R := T_{E_M^c}$.

Let $(b', \mathcal{D}(R))$ be the Dirichlet form associated with (X, R). It follows from the discussion in [6, IV. Lemma 5.6] that

$$\mathcal{D}(R) = \mathcal{D}_{E_M}(X) \quad \text{and } b' = a \ \text{ on } \mathcal{D}_{E_M}(X).$$

(2) Killing (X, R) by $S := S_M$.

It follows from [1] that (X, S) is m-symmetric. Let $(b'', \mathcal{D}(S))$ be the Dirichlet form associated with (X, S). Start from the subprocess (X, R), $S > 0$ a.s.; i.e., P^{m^*}-a.e. $S > 0$. Thus using Proposition 3.4 we have

$$\mathcal{D}(S) = \mathcal{D}(R) \cap L^2({}^R\rho_S) = \mathcal{D}_{E_M}(X) \cap L^2(\rho_S)$$

and for any $u \in \mathcal{D}(S)$,

$$b''(u,u) = b'(u,u) + {}^R\nu_S(u \otimes u) = a(u,u) + \nu_S(u \otimes u),$$

since $u \otimes u$ is supported by $E_M \times E_M$.

(3) Killing (X, S) by M.

Clearly the resulting Dirichlet form is $(b, \mathcal{D}(M))$ and M does not vanish under the subprocess (X, S). Hence we can apply Proposition 3.3 and obtain

$$\mathcal{D}(M) = \mathcal{D}(S) \cap L^2({}^S\rho_M) = \mathcal{D}_{E_M}(X) \cap L^2(\rho_S) \cap L^2(\rho_{\,\mathrm{slog}M}).$$

Since $\rho_M = \rho_S + \rho_{\,\mathrm{slog}M}$ and $\nu_M = \nu_S + \nu_{\,\mathrm{slog}M}$,

$$\mathcal{D}(M) = \mathcal{D}_{E_M}(X) \cap L^2(\rho_M)$$

and for any $u \in \mathcal{D}(M)$,

$$\begin{aligned} b(u,u) &= b''(u,u) + {}^S\nu_M(u \otimes u) \\ &= a(u,u) + \nu_S(u \otimes u) + \nu_{\,\mathrm{slog}M}(u \otimes u) \\ &= a(u,u) + \nu_M(u \otimes u). \end{aligned}$$

That completes the proof.

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