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FINITE GROUPS WHOSE AUTOMORPHISM GROUP HAS ORDER CUBEFREE**

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Abstract

Let G denote a finite group. It is shown that if $|\mathrm{Aut}(G)|$ is cubefree then G possesses the Sylow tower property

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§1. Introduction and Results

In this paper G always denotes a finite group and $\mathrm{Aut}(G)$ the automorphism group of G. H. K. Iyer[6] showed that for any given positive integer number n, the number of solutions G of $|\mathrm{Aut}(G)| = n$ is at most finite. Some special cases have been studied. Mashale[8] and Curran[3] showed that for each odd prime p the equation $|\mathrm{Aut}(G)| = p^s$ $(1 \leq s \leq 5)$ has no solution. For any $n \geq 6$, A. Caranti and C. M. Scopploa[1] gave some examples G such that $|\mathrm{Aut}(G)| = p^n$ for some odd prime p. The cases of $n = pqr, p^3q$ and p^2q^2 were investigated in [2], [12] and [13] respectively, where p, q and r are distinct primes. In the present paper we shall prove the following

Theorem. *If $|\mathrm{Aut}(G)|$ is cubefree, then G possesses the Sylow tower property.*

Corollary. *If $|\mathrm{Aut}(G)|$ is cubefree and odd, then $|G| \leq 2$.*

Our result is a contribution for classifying groups G such that $|\mathrm{Aut}(G)|$ is cubefree. We also note that a group whose order is cubefree need not be solvable.

The corollary is a middle result of the theorem, its proof will occur on the second step of the proof of the theorem.

A central automorphism of G is an automorphism which induces the identity automorphism of $G/Z(G)$. We denote the central automorphism group of G by $\mathrm{Cent}(G)$. If H and K are groups, $[H]K$ denotes a semi-direct product of H by K. C_n denotes the cyclic group of order n. All further unexplained notation and terminology are standard.

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§2. Preliminaries

In order to prove our theorem, we need the following lemmas, some of which are well known.

Lemma 2.1.[8] $\mathrm{Cent}(G) = C_{\mathrm{Aut}(G)}(\mathrm{Inn}(G))$.

We take the primary decomposition of G/G' and $Z(G)$ as

$$G/G' = G_{p_1}/G' \times G_{p_2}/G' \times \cdots \times G_{p_r}/G',$$
$$Z(G) = Z_{p_1} \times Z_{p_2} \times \cdots \times Z_{p_r},$$

where $p_i (i = 1, 2, \cdots, r)$ are the prime factors of $|G|$.

Lemma 2.2.[10] *Let G be a finite group with no non-trivial abelian direct factor. Then*

$$|\mathrm{Cent}(G)| = \prod_{i=1}^{r} \prod_{j=1}^{k_i} |Z_{p_i,j}|^{r_{ij}},$$

where $Z_{p_i,j}$ is the subgroup of Z_{p_i} of elements whose order divides p^j, $p_i^{k_i}$ is the exponent of G_{p_i}/G' and in the decomposition of G_{p_i}/G' there occur precisely r_{ij} direct factors of order p_i^j.

Lemma 2.3. *Let p be an odd prime and let G be a p-group such that $|G/Z(G)| = p^2$. Then $G = [A]B$, where A is an abelian group and B is a cyclic group. In particular, G has an automorphism of order* 2.

Proof. If $G = [A]B$, then $ab \mapsto a^{-1}b$, $\forall a \in A, b \in B$ is an automorphism of G of order 2. We now show that $G = [A]B$. Let P be a minimal nonabelian subgroup of G. Then $G = Z(G)P$ and P' is of order p by Miller-Mereno's Theorem[9] and hence G' is a group of order p. Write $[y, x] = c$ for $x, y \in G$. We have $c^p = 1$ and

$$\begin{aligned}(x^{-1}y)^p &= \overbrace{x^{-1}y \cdot x^{-1}y \cdots x^{-1}y}^{p} \\ &= y^x y^{x^2} \cdots y^{x^p} x^{-p} \\ &= ycyc^2 \cdots yc^p x^{-p} \\ &= y^p c^{p(p+1)/2} x^{-p} \\ &= y^p x^{-p} \\ &= x^{-p} y^p. \end{aligned} \tag{2.1}$$

Next, we have

$$G/Z(G) = \langle aZ(G)\rangle \times \langle bZ(G)\rangle.$$

If $b^p = 1$, choose $A = \langle a\rangle Z(G)$. Then A is an abelian group and $G = [A]\langle b\rangle$ as desired. We therefore may assume

$$x^p \neq 1, \quad \forall x \in G - Z(G). \tag{2.2}$$

We now claim that we can suitably choose a and b such that $Z(G)$ possesses the following form

$$Z(G) = \langle a^p\rangle \times \langle b^p\rangle \times Z. \tag{2.3}$$

To see this, let $Z(G) = \langle z_1\rangle \times \langle z_2\rangle \times \cdots \times \langle z_r\rangle$ with $|z_1| \le |z_2| \le \cdots \le |z_r|$. Then

$a^p = \prod_{i=1}^{r} z_i^{m_i}$ for suitable integers m_i. If each m_i is divided by p, then $\widehat{a} = a\prod_{i=1}^{r} z_i^{-m_i/p}$ is of order p and $\widehat{a} \notin Z(G)$. This contradicts (2.2) and hence we may assume that $p \dagger m_l$ for some $l \in \{1, 2, \cdots, r\}$ such that $p|m_j$ $\forall j \in \{l+1, \cdots, r\}$. Put $\widehat{a} = a \prod_{i=l+1}^{r} z_i^{-m_i/p}$. Then $|\widehat{a}^p| = |z_l|$. By replacing a by $\widehat{a}$, we may assume

$$\langle a^p \rangle = \langle z_l \rangle \quad \text{for some} \quad l. \tag{2.4}$$

Now we have $b^p = a^{np} \prod_{i \neq l} z_i^{n_i}$ for suitable integers n and n_i. Put $\widehat{b} = a^{-n}b$. Then $\widehat{b}^p = a^{-np}b^p = \prod_{i \neq l} z_i^{n_i}$ by (2.1). As above, we may replace b by $\widehat{b}$ and have

$$\langle b^p \rangle = \langle z_k \rangle \quad \text{for some} \quad k \neq l. \tag{2.5}$$

Thus (2.3) is proved.

We let $|a| = p^u$ and $|b| = p^v$, $2 \leq u \leq v$. Since $[a, b]^p = 1$ and $[a, b] \in Z(G)$, by (2.3) we may assume that

$$[a, b] = a^{sp^{u-1}} b^{rp^{v-1}} z \quad \text{for some} \quad z \in Z,$$

where s and r are suitable integers. If $p|s$, then $[a, b] \in \langle b^p \rangle \times Z$ and so $G = [\langle b, Z \rangle]\langle a \rangle$ and the proof has been completed. Hence we may assume $p \dagger rs$. Choose $\widehat{a} = a^s b^{rp^{v-u}}$. We have $\widehat{a}^{p^u} = a^{sp^u} b^{rp^v} = 1$. By replacing a by $\widehat{a}$, (2.3) becomes $Z(G) = \langle \widehat{a}^p \rangle \times \langle b^p \rangle \times Z$ and $[\widehat{a}, b] = \widehat{a}^{p^{u-1}} z$ and hence $G = [\langle \widehat{a}, Z \rangle]\langle b \rangle$ as desired. Thus the lemma is proved.

Lemma 2.4. *If the finite group $G = Z(G)K$ and α is an automorphism of subgroup K such that $\alpha_{Z(K)} = 1$, where $\alpha_{Z(K)}$ is the restriction of α to $Z(K)$, then G has an automorphism σ such that the restriction $\sigma_K = \alpha$ and $\sigma_{Z(G)} = 1$.*

Proof. See [9, Lemma 1].

Lemma 2.5. *Let $p > 3$ be an odd prime and let K be a subgroup of $GL(2, p)$ of odd order. If $p \dagger |K|$, then K is either cyclic or abelian with* $\exp(K)|p - 1$.

Proof. Set $K_0 = K \cap SL(2, p)$. By checking the subgroup table of $SL(2, p)$ (see [5, p.213, Dickson's Theorem]), we see that K_0 is cyclic and $|K_0|$ is a divisor of $p - 1$ or $p + 1$. Next, by [5, p.178, Theorem 7.3], $GL(2, p)$ contains a cyclic subgroup B of order $p^2 - 1$, $B \cap SL(2, p) \cong C_{p+1}$ and $GL(2, p) = SL(2, p)B$. So

$$K/K_0 \cong C_n \quad \text{for some} \quad n|p - 1.$$

Suppose that $|K_0|\,|p + 1$. Let r be a prime divisor of $|K_0|$, and $R_0 \in Syl_r(K_0)$. Since every Sylow subgroup of odd order of B is a Sylow subgroup of $GL(2, p)$ too, we may assume that $R_0 \leq B$. Choose $1 \neq u \in R_0$ and write $G = GL(2, p)$. Then by [5, p.187, Theorem 7.3], $C_G(u) \leq B$ and $|N_G(\langle u \rangle) : C_G(u)| = 2$. Also, obviously $K \leq N_G(\langle u \rangle)$ and K is of odd order. We conclude that $K \leq B$ and hence K is cyclic because B is cyclic.

Suppose that $|K_0| \dagger p+1$. Then $|K_0|\,|p-1$. In this case $|K_0|$ is a divisor of $(p-1)^2$. Let π denote the set of the odd prime divisors of $p - 1$ and let $p - 1 = 2^a m$, $2 \dagger m$. It is clear that $GL(2, p)$ contains a π-Hall subgroup $H \cong C_m \times C_m$. Hence the π-subgroup K is contained by conjugate in H. So K is abelian and $\exp(K)|p - 1$. The proof is now complete.

Lemma 2.6. *$PSL(2, p^n)$ and $SL(2, p^n)$ have an outer automorphism of order 2, where p is an odd prime.*

Proof. The automorphism group of $PSL(2,p^n)$ is a semi-direct product of $PGL(2,p^n)$ by C_n and $|PGL(2,p^n):PSL(2,p^n)|=2$. Therefore we conclude that the lemma is true for $PSL(2,p^n)$.

Set $q=p^n, G=GL(2,q)$ and let F_q denote a field with q elements. Write $q-1=2^m s$, $m\geq 1$ and $2\dagger s$. Choose $\lambda\in F_q$ such that $|\lambda|=2^m$. Then

$$\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\in GL(2,q)\backslash SL(2,q)$$

induces an automorphism φ of $SL(2,q)$ of order 2^m. If φ is an inner automorphism, then there is

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in SL(2,q)$$

with $ad-bc=1$ such that

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\lambda & 0\\ 0 & 1\end{pmatrix}\in C_G(SL(2,q)).$$

From this we have $\lambda=d^2$ and so $\lambda^{2^{m-1}}=1$, contradicting $|\lambda|=2^m$. Thus we conclude that $SL(2,p^n)$ has an outer automorphism of order 2.

Lemma 2.7. *Let H be a finite nonsolvable group. If $|H|$ is cubefree, then $H\cong PSL(2,p^n)$ $\times N$, where $p^n\equiv 3,5 (\mathrm{mod}\, 8)$ and N is solvable.*

Proof. Let N be the largest solvable normal subgroup of H and let K/N be a chief factor of H. We have

$$K/N\cong PSL(2,p^n)\quad p^n\equiv 3,5 \pmod 8)$$

and K has the derived series:

$$K=K^{(0)}\geq K^{(1)}\geq\cdots\geq K^{(l)}.$$

Put $M=K^{(l)}$. Then $K=MN$ and $M'=M$. We claim that $D=M\cap N=1$. Suppose not. Obviously, $D=\Phi(M)$ and so D is nilpotent. Let $R>1$ be a Sylow r- subgroup of D with r a prime dividing $|D|$. By hypothesis we see that $2p\dagger|D|$ and $|R|=r$ or r^2. If $D\leq Z(M)$, then $M\cong PSL(2,p^n)$ (see [4, p.302]) and hence $D=1$ as desired. We thus may assume that $R\not\leq Z(M)$. We have $PSL(2,p^n)\cong M/D=M/C_M(R)\cong$ some subgroup of $GL(2,r)$. But as well-known, $GL(2,r)$ cannot contain any nonabelian simple group. Thus the claim holds. Hence

$$K=M\times N\quad\text{and}\quad M\cong PSL(2,p^n).$$

Finally, we show that $H=K$. We have $H/N=H/C_H(M)\cong$ some subgroup of $\mathrm{Aut}(M)=PGL(2,p^n)C_n,\ n\leq 2$. Also, by hypothesis, $2^3\dagger|H|$, we conclude that $H/N\cong PSL(2,p^n)$ and hence $H=K$. The proof of Lemma 2.7 is now complete.

Lemma 2.8. *Let K be a minimal nonnilpotent group such that $K/\Phi(K)\cong A_4$, the alternating group of degree 4. Then K has an outer automorphism of order 2.*

Proof. Let P be a Sylow 2-subgroup of K and Q a Sylow 3-subgroup of K. Then $P=\langle x,y\rangle, |x|=|y|=2$ if $\Phi(P)=1$ and $|x|=|y|=4$ if $\Phi(P)>1$, and $Q=\langle w\rangle, |w|=3^m$ for some m. Also, $\Phi(P)=P'=Z(P)$ is of order 2 if $\Phi(P)>1$. By suitably choosing x and y, K has the defining relation:

$$x^w=y,\quad y^w=xyz,\quad z\in P',$$

$|z| = 1$ if $P' = 1$ and $|z| = 2$ if $P' > 1$. Define the map σ by

$$\sigma = \begin{pmatrix} x & y & z & w \\ yz & xz & z & w^{-1} \end{pmatrix}.$$

Then

$$\begin{aligned} [\sigma(x), \sigma(y)] &= [yz, xz] = [y, x] = z^{-1} = z, \\ \sigma(x)^{\sigma(w)} &= (yz)^{w^{-1}} = y^{w^{-1}} z = xz = \sigma(y), \\ \sigma(y)^{\sigma(w)} &= (xz)^{w^{-1}} = y(y^{w^{-1}})^{-1} = yx^{-1} \\ &= yzxzx^{-2} = \sigma(x)\sigma(y)z. \end{aligned}$$

So σ is an automorphism of K and has order 2. Obviously σ cannot be an inner automorphism. Thus the lemma holds.

Lemma 2.9. *Suppose that* $|\mathrm{Aut}(G)|$ *is cubefree and* $G = [A]B$*,* A *is a nontrivial abelian subgroup of odd order. Then* G *is* 2*-nilpotent.*

Proof. If G is not 2-nilpotent, then $|G/Z(G)|_2 = 4$ and so G has no outer automorphism of order 2. On the other hand, $\alpha : ab \to a^{-1}b, \quad \forall a \in A, \quad b \in B$ is an automorphism of G of order 2. So, there is $g \in G$ such that

$$(ab)^g = a^{-1}b, \quad \forall a \in A, \quad b \in B.$$

Put $g = uv$, where u is 2-element and v is 2'-element, and $|v| = m$. Then $(ab)^{u^m} = (ab)^{g^m} = a^{-1}b \ \ \forall a \in A, b \in B$. So we may assume that g is 2-element and $g \in B$. Clearly $g \in Z(B)$ but $g \notin Z(G)$. Hence we have

$$|B/Z(B)|_2 < |B/Z(G) \cap B|_2 \le 4.$$

This implies that B is 2-nilpotent and so is G. This is a contradiction and thus the lemma is proved.

§3. Proof of the Theorem

Suppose that the theorem is false and let G be a counterexample. We prove the theorem in 3 steps.

(a) Solvability of G.

Suppose that G is nonsolvable and put $Z = Z(G)$. By Lemma 2.7 we have

$$G/Z = M/Z \times N/Z,$$

where N is the largest solvable normal subgroup of G, $M/Z = PSL(2, p^n)$, $1 \le n \le 2$ and $p^n \equiv 3, 5 (\mathrm{mod}\, 8)$. Let $M^{(r)}$ be the terminal member of the derived series of M. We have $M = M^{(r)}Z$ and $M^{(r)} \lhd G$. Hence $M^{(r)}/M^{(r)} \cap Z \cong PSL(2, p^n)$. By checking the Shur multipliers of $PSL(2, p^n)$ (see [4, p. 302]) we see

$$M^{(r)} \cong PSL(2, p^n) \ \text{ or } \ SL(2, p^n).$$

If $M^{(r)} \cong PSL(2, p^n)$, then $G \cong PSL(2, p^n) \times N$. By Lemma 2.6 G has an outer automorphism of order 2. This is a contradiction.

Suppose that $M^{(r)} = SL(2, p^n)$. We let S be a Sylow 2-subgroup of Z and so $N = S \times N_1$ where N_1 is a 2'-group because $|N/Z|$ must be an odd number. We have

$$G = SL(2, p^n)S \times N_1.$$

If $|S| > 2$, then $2\big||G : G'|$ because $Z(SL(2, p^n))$ is of order 2. Thus G has a central automorphism of order 2 by Lemma 2.2. This is impossible. Hence $|S| = 2$ and so $G = SL(2, p^n) \times N_1$. By Lemma 2.6 G has an outer automorphism of order 2, again a contradiction. Thus (a) is proved.

(b) $|G| = 2^n 3^m$.

Suppose that (b) is not true. Since we are assuming that G has no Sylow tower, we see that G cannot be 2-nilpolent because $|G/Z(G)|$ is cubefree. Thus G contains a subgroup K such that $K/Z(K) \cong A_4$. So

(b_1) G has no outer automorphism of order 2.

Let p denote the largest prime divisor of $|G|$ and let P be a Sylow p-subgroup of G. Then $P \lhd G$ and G has a p-complement H. So $G = H[P]$. We may assume that $K \le H$. If $H \lhd G$, then by Lemma 2.3 G has an automorphism α of order 2 with $\alpha_H = 1$. Obviously α cannot be an inner automorphism. This contradicts (b_1). Hence H acts nontrivially on P by conjugation and P is nonabelian by Lemma 2.9.

We claim that H acts trivially on each H-invariant abelian subgroup of P. Suppose not. Let A be an H-invariant abelian subgroup of P and $[A, H] > 1$. We have $A = C_A(H) \times [A, H]$. Also, by hypothesis $|P : Z(P)| = p^2$ and $Z(P) = Z(G) \cap P \le C_P(H)$, so $[A, H]$ has order p and by Mashke's Theorem (see [7, 12.3]) we see

$$P/Z(P) = A/Z(P) \times B/Z(P),$$

where $Z(P) \le B$ and B is some H-invariant subgroup. Thus $G = (H[A, H])[B]$. This contradicts Lemma 2.9 and hence the claim holds.

We now choose a subgroup P_0 of P satisfying conditions: P_0 is nonabelian, H acts nontrivially on P_0 and P_0 has the smallest possible order. Put

$$\begin{aligned} G_0 &= HP_0 \quad \text{and} \quad \text{so} \\ G &= Z(G)G_0. \end{aligned} \tag{3.1}$$

(b_2) $|P_0| = p^3$ and $\exp(P_0) = p$.

Indeed, by choice of P_0 we see that H acts trivially on each H-invariant proper subgroup of P_0. It follows by Hall-Higman's Theorem (see [7, 7.25]) that $P_0' = \Phi(P_0) = Z(P_0)$ and $\exp(P_0) = p$. On the other hand, P_0 contains a minimal nonabelian subgroup, say N. By Miller-Mereno's Theorem[9], $N' = \langle c \rangle, c^p = 1$. Thus $P_0' = \langle c \rangle$ since $P_0 = Z(P_0)N$. Therefore we conclude $|P_0| = p^3$.

(b_3) $H/C_H(P_0)$ is cyclic.

By above $P_0' = \Phi(P_0) = Z(P_0) = \langle c \rangle$ with $c^p = 1$. It follows by a theorem of Hall that we have $C_H(P_0) = C_H(P_0/\langle c \rangle)$. So $C_H(P_0)$ is isomorphic to some subgroup of $\mathrm{Aut}(P/\langle c \rangle) = GL(2, p)$.

We first consider the case when $H/C_H(P_0)$ has odd order. Suppose that $H/C_H(P_0)$ is noncyclic. By Lemma 2.5 $H/C_H(P_0)$ is an abelian group with exponent a divisor of $p - 1$. It follows that $P_0/\langle c \rangle$ has an H-invariant subgroup of order p, say $A/\langle c \rangle$. Obviously A is an abelian group of order p^2. Put $\widehat{A} = AZ(P)$. Then $\widehat{A}$ is also abelian and H- invariant. It follows by Mashke's theorem that we have $G = (H[\widehat{A}, H])[B]$ where $B > Z(P)$ is abelian. This contradicts Lemma 2.9.

Next suppose that $H/C_H(P_0)$ has even order. In this case $\overline{K} = KC_H(P_0)/C_H(P_0) \cong A_4$ or a 2-group. But $GL(2,p)$ cannot contain a subgroup which is isomorphic to A_4. Thus the only possibility of $\overline{K}$ is a 2-group and hence $C_H(P_0)$ contains a Sylow 3-subgroup of K and $K \cap C_H(P_0) \lhd K$. This is impossible because $K/Z(K) \cong A_4$. This proves (b$_3$).

(b$_4$) G_0 has an outer automorphism γ of order 2 such that the restriction $\gamma_{Z(G_0)} = 1$.

Put $C = C_H(P_0)$ and let uC be a generator of H/C. Then u defines an automorphism σ_u of P_0 by

$$\sigma_u : x \mapsto u^{-1}xu, \quad \forall x \in P_0.$$

We may assume

$$\sigma_u = \begin{pmatrix} a & b & c \\ b & a^r b^s c^t & c \end{pmatrix}, \quad r \not\equiv 0 \pmod p,$$

where a, b are generators of P_0 and $[a,b] = c$. Choose k such that $kr \equiv t (\operatorname{mod} p)$. Then

$$\sigma_{ub^{-k}} = \begin{pmatrix} a & b & c \\ b & a^{-1}b^s & c \end{pmatrix}.$$

We may replace u by ub^{-k}. Further choose m such that $2m \equiv s (\operatorname{mod} p)$ and replace u by ub^{-m} again. Then we get

$$\sigma_u = \begin{pmatrix} a & b & c \\ b & a^{-1}b^{2m}c^m & c \end{pmatrix}. \tag{3.2}$$

On the other hand, it is clear that P_0 has an automorphism α of order 2:

$$\alpha = \begin{pmatrix} a & b & c \\ a^{-1} & b^{-1} & c \end{pmatrix}$$

and it is easy to check that $\alpha\sigma_u = \sigma_u\alpha$, that is, $(u^{-1}xu)^\alpha = u^{-1}x^\alpha u \ \ \forall x \in P_0$. Thus we obtain

$$(h^{-1}xh)^\alpha = h^{-1}x^\alpha h, \ \forall h \in H, \quad x \in P_0. \tag{3.3}$$

Using (3.3) we can define an automorphism γ of G_0 with order 2 as follows:

$$(hx)^\gamma = hx^\alpha, \quad \forall h \in H, \quad x \in P_0.$$

Obviously $\gamma_{Z(G_0)} = 1$ and γ must be an outer automorphism (see the proof of Lemma 2.9). This proves (b$_4$).

Now by Lemma 2.4 and $G = Z(G)G_0$ (see (3.1)) γ is extendible to G. Thus G has an outer automorphism of order 2. This contradicts (b$_1$) and hence (b) is proved.

(c) G cannot exist.

As in (b), G contains a minimal nonnilpotent subgroup K such that $K/Z(K) \cong A_4$. By Lemma 2.8 we see that $G \neq K$, namely $K < G$.

(c$_1$) First let $|G/Z(G)| = 2^2 \cdot 3$. In this case

$$G = Z(G)K \text{ with } \ K < G.$$

If $2\big||G : K|$, then by Lemma 2.2 $|\mathrm{Cent}(G)|_2 > 1$. But $Z(G/Z(G)) = 1$. Lemma 2.1 implies that G has an outer automorphism of order 2. This is a contradiction.

If $3\big||G : K|$, as in above, $|\mathrm{Cent}(G)|_3 > 1$. But $|\mathrm{Cent}(G)|_3 \leq 3$. So $|\mathrm{Cent}(G)|_3 = 3$. Applying Lemma 2.2 we see that G has a cyclic Sylow 3-subgroup. This yields $G = K$, again a contradiction.

(c$_2$) Next let $|G:Z(G)|=2^2\cdot 3^2$.

In this case $|G:Z(G)K|=3$. Let P be a Sylow 2-subgroup of G and Q a Sylow 3-subgroup of G and put $M=Z(G)K$. We have $G/\mathrm{Core}(M)\cong$ a subgroup of S_3. It follows that $M\lhd G$ because M cannot contain a subgroup of index 2. Thus $P\lhd G$ and $3\big||G:G'|$. If $3\big||Z(G)|$, we have $|\mathrm{Cent}(G)|_3=3$. It follows that Lemma 2.2 implies that G/G' has a cyclic Sylow 3-subgroup, and hence G has a cyclic Sylow 3-subgroup. Thus $G=Z(G)K$, a contradiction.

If $3\dagger|Z(G)|$, then $|Q|=3^2$ and G has no central automorphism of order 3. On the other hand, since A_4 has no outer automorphism of order 3, we have $G/Z(G)\cong A_4\times C_3$. Thus Lemma 2.1 implies that G has a central automorphism of order 3. This is a contradiction and hence the proof of the theorem is completed.

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