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SOME GEOMETRIC PROPERTIES ON A MODEL OF UNIVERSAL TEICHMÜLLER SPACES

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Abstract

The model of the universal Teichmüller space by the derivatives of logarithm is the union of infinite disconnected components. In this paper, the fact that each component is not starlike with respect to its center is proved, and the outer radius of the space with respect to each center is obtained.

Keywords Universal Teichmüller space, Logarithmic derivative, Quasiconformal extension
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§0. Introduction

Let E_j denote the Banach space of functions ϕ which are analytic in the unit disk Δ with norm

$$\|\phi\|_j = \sup_{z\in\Delta}(1-|z|^2)^j|\phi(z)| < \infty, \quad j = 1, 2. \tag{0.1}$$

Next let $[g] = g''/g'$ for g holomorphic in Δ and $S_g = (g''/g')' - \frac{1}{2}(g''/g')^2$, the Schwarzian derivative of g, for g meromorphic in Δ, and define

$$S_1 = \{[g]|\, g \text{ conformal in } \Delta \text{ with } g(0) = g'(0) - 1 = 0,\ g(\Delta) \subset C\},$$
$$S = \{S_g | g \text{ conformal in } \Delta \text{ with } g(\Delta) \subset \overline{C}\}.$$

Let T_1 and T denote the corresponding subsets of S_1 and S for which the mapping g has, in addition, a quasiconformal extension to C. It is well-known that T is the Bers universal Teichmüller space[1], and T_1 is an alternative model of the universal Teichmüller space introduced in [2] and [3]. Astala and Gehring[2] gave a complete description of the closure of $\bar{T}_1$, and Zhuravlev[3] obtained an interesting result that T_1 is disconnected in the topology induced by the norm $\|.\|_1$. He proved $T_1 = L \cup \{\bigcup_{\theta\in[0,2\pi)} L_\theta\}$, where L and L_θ are connected components of T_1 with g bounded in Δ and $g(e^{i\theta}) = \infty$ respectively, and $L_{\theta_1} \cap L_{\theta_2} = \phi$ for $\theta_1 \neq \theta_2$ and $L \cap L_\theta = \phi$. Let $H_\theta = \frac{z}{1-e^{-i\theta}z}$, $[H_\theta] \in L_\theta$ plays an important role in the description of T_1 (see [3]), and is used as the center of the component L_θ. In this paper, we point out in chapter 1 that there exists a homeomorphism of L_θ onto T for every $\theta \in [0, 2\pi)$. In chapter 2, we prove that L_θ is not starlike with respect to $[H_\theta]$, and we also

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obtain that the distance between $[H_{\theta_1}]$ and $[H_{\theta_2}]$ is 4 for arbitrary $\theta_1 \neq \theta_2$. In chapter 3, we discuss the inner radius and outer radius of L_θ and T_1 with respect to $[H_\theta]$, and obtain some exact evaluations.

§1. The Homeomorphism of L_θ onto T

There is a natural mapping I: $T_1 \to T$ defined by the formula $I(\phi) = \phi' - \frac{1}{2}\phi^2$. By using the Cauchy integral formula, it is easy to get the estimate $\|\phi'\|_2 \leq 16\|\phi\|_1$ and $\|\phi\|_1 \leq 6$ for $\phi \in T_1$, from which it follows that

$$\|I(\phi_1) - I(\phi_2)\|_2 \leq 22\|\phi_1 - \phi_2\|_1 \tag{1.1}$$

for ϕ_1, $\phi_2 \in T_1$.

Throrem 1.1. *L_θ is homeomorphic to T for each $\theta \in [0, 2\pi)$.*

Proof. We consider I: $L_\theta \to T$. It is clear that I is surjective and injective. We can conclude from (1.1) that I is continuous. Now we need only to prove that I^{-1} is also continuous. Let $[f_1]$, $[f_2] \in L_\theta$, i.e., f_i is a *q.c.* mapping of C onto itself and f_i is conformal in Δ, $f_i(0) = f_i'(0) - 1 = 0$, $f_i(e^{i\theta}) = \infty$, $i = 1, 2$. Let μ_{f_i} be the Beltrami coefficient of f_i, and let $f = f_2 \cdot f_1^{-1}$. Then $f(\infty) = \infty$, $f(0) = 0$, $f_z(0) = 1$,

$$\mu_f(f_1(z)) = \frac{\mu_{f_2}(z) - \mu_{f_1}(z)}{1 - \bar{\mu}_{f_1}(z)\mu_{f_2}(z)} \frac{\bar{f}_{1z}}{f_{1z}}.$$

By [5], there exists a unique *q.c.* mapping $f(z,t)$ such that $f(0,t) = 0$, $f_z(0,t) = 1$, $f(\infty, t) = \infty$ and $f_{\bar{z}}(z,t) = t\mu_f(z)f_z(z,t)$ for $t \in \Delta_f = \{\ t \mid |t| < 1/\|\mu_f\|_\infty\ \}$. $f(z,t)$ is conformal for z in $f_1(\Delta)$ and $\phi(t) = f_{zz}(z,t)/f_z(z,t)\ \rho^{-1}_{f_1(\Delta)}(z)$ is conformal in Δ_f, where $\rho_{f_1(\Delta)}$ is the Poincare metric of domain $f_1(\Delta)$. For $t = 0$, we have $f(z,0) = z$, hence $\phi(0) = 0$. Osgood[6] proved that $|\phi(t)| \leq 8$.

By Schwarz Lemma, we have $|\phi(t)| \leq 8\|\mu_f\|_\infty t$ for $t \in \Delta_f$. Let $t = 1$, $f(z,1) = f(z)$. Then $|\frac{f''(z)}{f'(z)}||\rho_{f_1(\Delta)}| \leq 8\|\mu_f\|_\infty$. Hence

$$\|[f_2] - [f_1]\|_1 \leq 8\|(\mu_{f_2} - \mu_{f_1})/(1 - \bar{\mu}_{f_1}\mu_{f_2})\|_\infty. \tag{1.2}$$

Let $[\mu_{f_i}] = \{\mu_{F_i} | F_i$ is a quasiconformal mapping of C onto itself and $F_i|_\Delta = f_i|_\Delta\}$. From (1.2), we have

$$\|[f_2] - [f_1]\|_1 \leq 8 \inf_{\mu_i \in [\mu_{f_i}]} \|(\mu_2 - \mu_1)/(1 - \bar{\mu}_1\mu_2)\|_\infty. \tag{1.3}$$

Now let $\phi_0 \in T$, $\phi_n \in T$, $\phi_0 = I([f_0])$, $\phi_n = I([f_n])$ and $[f_0]$, $[f_n] \in L_\theta$, $\|\phi_n - \phi_0\|_2 \to 0$. Then $\|S_{f_n} - S_{f_0}\|_2 \to 0$.

As $f_0(\Delta)$ is a k-quasidisk for some k $(1 \leq k < \infty)$, by [7], there exists $C(k) > 0$ such that a conformal mapping f in $f_0(\Delta)$ can be quasiconformally extended to C and $\|\mu_f\|_\infty \leq C(k) \sup_{z \in f_0(\Delta)} \{|S_f| \cdot \rho^{-2}_{f_o(\Delta)}\}$ whenever

$$\sup_{z \in f_0(\Delta)} \{|S_f| \cdot \rho^{-2}_{f_0(\Delta)}\} < \frac{1}{C(k)}.$$

Now, for each $\delta > 0$, $\delta < \frac{1}{C(k)}$, there exists an n_0 such that $\|S_{f_n} - S_{f_0}\|_2 < \delta < \frac{1}{C(k)}$ for $n > n_0$, that is, $|S_{f_n \cdot f_0^{-1}}| \cdot \rho^{-2}_{f_0(\Delta)} < \delta < \frac{1}{C(k)}$. Hence

$$\|\mu_{f_n \cdot f_0^{-1}}\|_\infty \leq C(k) \sup_{z \in f_0(\Delta)} \{|S_{f_n \cdot f_o^{-1}}|\rho^{-2}_{f_0(\Delta)}\} = C(k)\|S_{f_n} - S_{f_0}\|_2.$$

From (1.3), we have

$$\|[f_n]-[f_0]\|_1 \le 8C(k)\|S_{f_n}-S_{f_0}\|_2. \tag{1.4}$$

From (1. 4), we see that I^{-1} is locally Lipschitz. Hence I^{-1} is continuous, which completes the proof of Theorem.

Remark. It is easy to see that $I: L \to T$ is not injective.

§2. Non-Starlike of Connected Component L_θ and the Distance Between Two Center Points

Krushkal[8] proved that L is not starlike with respect to zero, the center of L. In this chapter, we will prove that L_θ is not starlike with respect to $[H_\theta]$ for each $\theta \in [0, 2\pi)$. We need the following lemma.

Lemma 2.1.[9] *There exists an isolated point in S in the topology induced by $\|\cdot\|_2$.*

The proof of Lemma 2.1 is based on the existence of rigid simply connected domains $D \subset \bar{C}$ having the property that any univalent meromorphic function h on D with sufficiently small norm $\sup\limits_D\{\rho_D^{-2}|S_h|\}$ must be a linear fractional transformation (see [9]).

Theorem 2.1. *L_θ is not starlike with respect to $[H_\theta]$ for each $\theta \in [0, 2\pi)$.*

Proof. Let S_{f_0} be an isolated point in S and $f_0(0) = f_0'(0) - 1 = 0$. Assume, contrary to the assertion of the theorem, that for every $[f] \in L_\theta$, $t[f] + (1-t)[H_\theta] \in L_\theta$ holds for $0 \le t \le 1$.

Let $f_r = \frac{1}{r} f_0(rz)$, $\quad G_r(w) = \frac{w}{1-(f_r(e^{i\theta}))^{-1}w}$ $\quad (0 < r < 1)$. Then $G_r \cdot f_r$ has a quasiconformal extension to C. Denote it by F_r. Then $F_r(0) = F_r'(0) - 1 = 0$, $\quad \lim\limits_{z\to e^{i\theta}} F_r(z) = \infty$. Hence $[F_r] \in L_\theta$. By assumption, $t[F_r] + (1-t)[H_\theta] \in L_\theta$, so there exists an $f_{r,t}$ for each $r, t \in (0, 1)$ such that $[f_{r,t}] = t[F_r] + (1-t)[H_\theta] \in L_\theta$. We can choose $r_n \to 1$ such that $F_{r_n} \to F_0$ locally uniformly in Δ, where F_0 is conformal in Δ, $f_{r_n}(e^{i\theta}) \to w_0$ (w_0 may be ∞ but nonzero) and $f_{r_n,t} \to f_{0,t}$, where $f_{0,t}$ is conformal in Δ. So $F_0 = G_0 \cdot f_0$, $\quad G_0 = \frac{w}{1-w_0^{-1}w}$ and $[f_{0,t}] = t[F_0] + (1-t)[H_\theta] \in S_1$. But

$$\|[f_{0,t}]-[F_0]\|_1 = \|t[F_0]+(1-t)[H_\theta]-[F_0]\|_1 = (1-t)\|[F_0]-[H_\theta]\|_1 \le 12(1-t) \to 0$$

as $t \to 1$, hence

$$\|S_{f_{0,t}} - S_{F_0}\|_2 \le 22\|[f_{0,t}]-[F_0]\|_1 \le 264(1-t) \to 0,$$

where $S_{f_{0,t}} \in S$ and $S_{f_0} = S_{G_0\cdot f_0} = S_{F_0}$. Hence S_{f_0} is not isolated in S, which is a contradiction to Lemma 2.1. This completes the proof of Theorem 2.1.

Now we give the estimation on the distance between the centers of different components.

Theorem 2.2. $\|[H_\theta]\|_1 = 4, \quad \|[H_{\theta_1}]-[H_{\theta_2}]\|_1 = 4 \quad$ *for* $\theta_1 \ne \theta_2, \theta_1, \theta_2 \in [0, 2\pi)$.

Proof. By computation we know that $[H_\theta] = \frac{2e^{-i\theta}}{1-e^{-i\theta}z}$, hence $\|[H_\theta]\|_1 = 4$ for $\theta \in [0, 2\pi)$.

Without loss of generality, we assume $\theta_0 = \theta_1 - \theta_2 > 0$, so

$$\begin{aligned}\|[H_{\theta_1}]-[H_{\theta_2}]\|_1 &= 2\sup_{z\in\Delta}\Big\{\Big|\frac{1-e^{-i\theta_0}}{(1-z)(1-e^{-i\theta_0}z)}\Big|(1-|z|^2)\Big\}\\ &= 4\Big|\sin\frac{\theta_0}{2}\Big|\sup_{z\in\Delta}\Big\{\frac{1-|z|^2}{|1-z||1-e^{-i\theta_0}z|}\Big\}.\end{aligned}$$

Letting $z = re^{i\theta}, r \to 1$, we have $\|[H_{\theta_1}] - [H_{\theta_2}]\|_1 \geq 4$. Now we prove

$$\left|\sin\frac{\theta_0}{2}\right|(1-|z|^2) \leq |1-z||1-e^{-i\theta_0}z|$$

for $z \in \Delta$. Let

$$F(r,\theta) = (1+r^2-2r\cos\theta)(1+r^2-2r\cos(\theta-\theta_0)) - (1-r^2)^2\sin^2\frac{\theta_0}{2}, \tag{2.1}$$

$0 \leq r \leq 1, 0 \leq \theta \leq 2\pi$. It is easy to see that

$$\begin{aligned} F_\theta(r,\theta) &= 2r\sin\theta(1+r^2-2r\cos(\theta-\theta_0)) + 2r\sin(\theta-\theta_0)(1+r^2-2r\cos\theta) \\ &= 4r\sin\Big(\theta-\frac{\theta_0}{2}\Big)\Big[(1+r^2)\cos\frac{\theta_0}{2} - 2r\cos\Big(\theta-\frac{\theta_0}{2}\Big)\Big]. \end{aligned}$$

Let $F_\theta(r,\theta) = 0$. Then $r = 0$, or $\sin(\theta - \frac{\theta_0}{2}) = 0$, or

$$(1+r^2)\cos\frac{\theta_0}{2} - 2r\cos\Big(\theta-\frac{\theta_0}{2}\Big) = 0. \tag{2.2}$$

If $r = 0$, then $F(0,\theta) \geq 0$. If $\sin(\theta - \frac{\theta_0}{2}) = 0$, then $\theta = \frac{\theta_0}{2}$ or $\theta = \pi + \frac{\theta_0}{2}$. By computation we have $F(r, \frac{\theta_0}{2}) \geq 0$ and $F(r, \frac{\theta_0}{2} + \pi) \geq 0$.

It is clear that when $\theta_0 = \pi$,

$$F(r,\theta) = (1+r^2-2r\cos\theta)(1+r^2+2r\cos\theta) - (1-r^2)^2 = 4r^2\sin^2\theta \geq 0.$$

When $\theta_0 \neq \pi$, it follows from (2.2) that

$$1+r^2-2r\cos\theta = \frac{2r\sin\theta\sin\frac{\theta_0}{2}}{\cos\frac{\theta_0}{2}}, \tag{2.3}$$

$$1+r^2-2r\cos(\theta-\theta_0) = \frac{-2r\sin(\theta-\theta_0)\sin\frac{\theta_0}{2}}{\cos\frac{\theta_0}{2}}. \tag{2.4}$$

Substituting (2.3) and (2.4) into (2.1), we have

$$\begin{aligned} F(r,\theta) &= \tan^2\frac{\theta_0}{2}\Big[-4r^2\sin\theta\sin(\theta-\theta_0) - (1-r^2)^2\cos^2\frac{\theta_0}{2}\Big] \\ &= \tan^2\frac{\theta_0}{2}\Big[-4r^2\sin\theta\sin(\theta-\theta_0) - (1+r^2)^2\cos^2\frac{\theta_0}{2} + 4r^2\cos^2\frac{\theta_0}{2}\Big] \\ &= 2r^2\tan^2\frac{\theta_0}{2}\Big[\cos(2\theta-\theta_0) - \cos\theta_0 - 2\cos^2\Big(\theta-\frac{\theta_0}{2}\Big) + 2\cos^2\frac{\theta_0}{2}\Big] = 0. \end{aligned}$$

So we conclude that $F(r,\theta) \geq 0$ for all $r \in [0,1]$ and $\theta \in [0,2\pi]$, which implies $\|[H_{\theta_1}] - [H_{\theta_2}]\|_1 = 4$.

§3. Inner and Outer Radius of T_1 with Respect to $[H_\theta]$

We define the inner radius $\tau(\Delta,\theta)$ of T_1 with respect to $[H_\theta]$ to be the supremum of the constants b with the following properties: if f is analytic in Δ with $f(0) = f'(0) - 1 = 0$ and $\|[f] - [H_\theta]\|_1 \leq b$, then f is injective. The outer radius $O(\Delta,\theta)$ of T_1 with respect to $[H_\theta]$ is defined to be the supremum of $\|[f] - [H_\theta]\|_1$ for $[f] \in T_1$.

Theorem 3.1. $O(\Delta,\theta) = 6$ *for* $\theta \in [0,2\pi)$.

Proof. Let D_θ be the image of Δ under H_θ. Then for $[f] \in S_1$,

$$\|[f] - [H_\theta]\|_1 = \sup_{z\in D_\theta}\{|[f\cdot H_\theta^{-1}]|\rho_{D_\theta}^{-1}\},$$

where ρ_{D_θ} is the Poincare metric of D_θ and $\rho_{D_\theta}(H_\theta(z))|H'_\theta(z)| = \frac{1}{1-|z|^2}$.

Let $f_1 = f \cdot H_\theta^{-1}$. Then f_1 is conformal in D_θ. For each $w \in D_\theta$, let $z \in \Delta$ such that $H_\theta(z) = \frac{z}{1-e^{-i\theta}z} = w$. Next define

$$g(\eta) = H_\theta\Big(\frac{\eta + z}{1+\bar{z}\eta}\Big) = \frac{\eta + z}{1 - e^{-i\theta}z + (\bar{z} - e^{-i\theta})\eta}.$$

Then $g(\eta) : \Delta \to D_\theta$ is conformal and $g(0) = H_\theta(z) = w$,

$$\rho_{D_\theta}(w) = \frac{1}{|g'(0)|}. \tag{3.1}$$

By computation and [10], we have $[f_1 \cdot g] = ([f_1] \cdot g)g' + [g]$, $\quad |[f_1]||g'(0)| \le |[f_1 \cdot g](0)| + |[g](0)| \le 4 + |[g](0)|$ and

$$|[g](0)| = \Big|\frac{g''(0)}{g'(0)}\Big| = 2\Big|\frac{\bar{z} - e^{-i\theta}}{1 - e^{-i\theta}z}\Big| = 2.$$

Hence $|[f_1](w)||g'(0)| \le 6$, i.e., $|[f_1](w)| \le 6\rho_{D_\theta}(w)$. Then $\|[f] - [H_\theta]\|_1 \le 6$ for $[f] \in S_1$.

Let $f_r(z) = z - \frac{1}{2}\, e^{-i\theta}\, rz^2$ $(0 < r < 1)$, f_r can be quasiconformally extended to C and

$$\|[f_r] - [H_\theta]\|_1 = \sup_{z\in\Delta}\Big\{\frac{|2 + r - 3rz|}{|1 - z||1 - rz|}(1 - |z|^2)\Big\} \ge \frac{(2 + r - 3r^{n+1})(1 + r^n)}{1 - r^{n+1}}.$$

Then $\lim\limits_{r\to1} \|[f_r] - [H_\theta]\|_1 \ge \frac{6n+4}{n+1}$ for all $n \in N$, hence

$$O(\Delta, \theta) = \sup_{[f]\in T_1} \|[f] - [H_\theta]\|_1 = 6.$$

Zhuravlev's result[3,Theorem 3] implies $\tau(\Delta, \theta) \ge 1$. Now we have

Theorem 3.2. $\tau(\Delta, \theta) = 1$ *for* $\theta \in [0,\, 2\pi)$.

Proof. By [11], for $\varepsilon > 0$, there exists an f_ε, which is not injective in $H = \{z|\mathrm{Re}z > 0\}$, and $2\mathrm{Re}\Big\{z\Big|\frac{f''_\varepsilon}{f'_\varepsilon}\Big|\Big\} \le 1+\varepsilon$. Let $f^*_\varepsilon = f_\varepsilon \cdot \phi \cdot H_\theta$, where $\phi : D_\theta \to H$ with $\phi = e^{-i\theta}\,\eta + \frac{1}{2}$. Then

$$(f^*_\varepsilon)' = f_\varepsilon e^{-i\theta} H'_\theta, \quad (f^*_\varepsilon)'' = e^{-2i\theta} f''_\varepsilon H'^2_\theta + e^{-i\theta} f'_\varepsilon H''_\theta.$$

Hence $[f^*_\varepsilon] = e^{-i\theta}[f_\varepsilon]H'_\theta + [H_\theta]$,

$$\begin{aligned}\|[f^*_\varepsilon] - [H_\theta]\|_1 &= \sup_{z\in\Delta}\{|[f_\varepsilon]||H'_\theta|(1 - |z|^2)\}\\ &= \sup_{w\in H}\Big\{\Big|\frac{f''_\varepsilon}{f'_\varepsilon}\Big|\,\Big|w + \frac{1}{2}\Big|^2\Big(1 - \Big|\frac{w + \frac{1}{2}}{w - \frac{1}{2}}\Big|^2\Big)\Big\}\\ &= 2\sup_{w\in H}\Big\{\mathrm{Re}w\Big|\frac{f''_\varepsilon(w)}{f'_\varepsilon(w)}\Big|\Big\} \le 1 + \varepsilon.\end{aligned}$$

Thus $\tau(\Delta, \theta) \le 1$ can be deduced from the fact that f^*_ε is not injective in Δ. Since $\tau(\Delta, \theta) \ge 1$, we know that $\tau(\Delta, \theta) = 1$. This completes the proof of Theorem 3.2.

For Koebe function $K(z) = \frac{z}{(1-z)^2}$, we know that $[K_r] = [\frac{1}{r}K(rz)] \in L$. By some simple computation, we have $\lim\limits_{r\to1} \|[K_r]\|_1 \ge 6$.

From [3] and the above discussion, we know that the inner and outer radius of L (with respect to zero) are 1 and 6 respectively. From our proof of Theorem3.2, we also know that the inner radius of L_θ with respect to $[H_\theta]$ is 1. Whether the outer radius of L_θ with repect to $[H_\theta]$ is also 6 is still open.

From [2], we know $[K(z)] \in \partial T_1$ and $\|[K]\|_1 = 6$. Though $[K_r] \in L$ and $K_r(z) \to K(z)$ locally uniformly in Δ, we point out $[K] \notin \partial L$. Actually, we have the following

Theorem 3.3. $[K] \in \partial L_0$, $[K] \notin \partial L \bigcup \Big(\bigcup\limits_{\theta\ne0} \partial L_\theta\Big)$.

Proof. Let $\Delta_r = \{z : |z-1| < r \cap \Delta\ \}$ for sufficiently small $r > 0$ and $[g_n] \in T_1$ such that

$$\|[g_n] - [K]\|_1 \to 0 \quad (n \to \infty).$$

Now we prove $[g_n] \in L_0$ for $n > n_0$.

If it is not the case, then there exists $n_k \to \infty$ such that $[g_{n_k}] \notin L_0$, i,e., $g_{n_k}(1) \neq \infty$.

For a given $\varepsilon > 0$, there exists an n_0 such that

$$\Big|\frac{K''(z)}{K'(z)} - \frac{g_n''(z)}{g_n'(z)}\Big| < \frac{\varepsilon}{1-|z|^2}$$

holds for $n > n_0$. Using the well-known relation

$$t\frac{\partial}{\partial t}\log|g_n'(z)| = \mathrm{Re}\Big(z\frac{g_n''(z)}{g_n'(z)}\Big), \quad z = te^{i\theta},$$

we obtain

$$t\frac{\partial}{\partial t}\log\Big(\frac{|K'(z)|}{|g_n'(z)|}\Big) \leq \varepsilon t(1-t^2)^{-1}.$$

Dividing this inequality by t and integrating both sides of the expression obtained with respect to t from 0 to $|z|$, we have $|K'(z)| < \left(\frac{1+|z|}{1-|z|}\right)^{\frac{\varepsilon}{2}}|g_n'(z)|$.

By Cauchy inequality, we have

$$\Big(\iint_{\Delta_r}|K'(z)|dxdy\Big)^2 \leq \iint_{\Delta_r}\Big(\frac{1+|z|}{1-|z|}\Big)^{\varepsilon}dxdy\iint_{\Delta_r}|g_{n_k}'(z)|^2dxdy.$$

Since $g_{n_k}(1) \neq \infty$, we know that $m(g_{n_k}(\Delta_r)) < \infty$ for sufficiently small r, where m denotes the planar Lebesgue measure. Hence for $\varepsilon < 2$, we have $\iint_{\Delta_r}|K'(z)|dxdy < \infty$. This is a contradiction to $\iint_{\Delta_r}|K'(z)|dxdy = \infty$. Hence $[K] \in \partial L_0$ and $[K] \notin \partial L \cup \big(\bigcup\limits_{\theta\neq 0} L_\theta\big)$. This completes the proof of Theorem 3.3.

From our proof, we also have $\mathrm{dist}([K], L_\theta) \geq 2$ for $\theta \neq 0$ and $\mathrm{dist}([K], L) \geq 2$.

Remark. Let $K_\theta = \frac{z}{(1-e^{-i\theta}z)^2}$. We can have $[K_\theta] \in \partial L_\theta$, $\mathrm{dist}([K_\theta], L_{\theta'}) \geq 2$, for $\theta \neq \theta'$ and $\mathrm{dist}([K_\theta], L) \geq 2$.

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