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LIMIT CYCLES AND BIFURCATION CURVES FOR THE QUADRATIC DIFFERENTIAL SYSTEM $(\mathrm{III})_{m=0}$ HAVING THREE ANTI-SADDLES (II)**

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Abstract

As a continuation of [1], the author studies the limit cycle bifurcation around the focus S_1 other than $O(0,0)$ for the system (1) as δ varies. A conjecture on the non-existence of limit cycles around S_1, and another one on the non-coexistence of limit cycles around both O and S_1 are given, together with some numerical examples.

Keywords Quadratic differential system, Limit cycle, Bifurcation, Anti-saddle, Focus
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In [1] we have studied the bifurcation problem of the quadratic system

$$\dot{x} = -y + \delta x + lx^2 + ny^2 = P(x,y), \quad \dot{y} = x(1 + ax - y) = Q(x,y) \tag{1}$$

around the focus $O(0,0)$ under the conditions

$$m = 0, \quad -1 < l < 0, \quad b = -1, \quad n + l - 1 > 0, \quad a \leq 0, \tag{2}$$

and drawn the bifurcation diagram in the (a, δ) plane. Now, we will study (1) and (2)[1)] in the neighbourhood of the other anti-saddle $S_1(x_1, y_1)$ lying on $y = 1 + ax$, where $x_1 > 0, y_1 > 0$. The results got by us are not so satisfactory as in [1]; nevertheless, they are very interesting, as we can see below.

Before discussing the bifurcation phenomena around S_1 we prove first the following

Theorem 1.[2)] *If*

$$a < 0, \quad n > 1, \quad n + l > 0, \quad na^2 + l < 0, \quad a^2 n < (n-1)(l+n)^2, \tag{3}$$

then[3)]

$$a^2 - 4(n-1)(1-l) < 0. \tag{4}$$

Proof. Assume on the contrary, $a^2 - 4(n-1)(1-l) \geq 0$. Then we have

$$a^2/4(1-l) \geq n - 1 > a^2 n/(l+n)^2;$$

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1) The condition $n + l - 1 > 0$ is now relaxed into $n + l > 0$.
2) This result was proved by Prof. Ye Weiyin.
3) Similarly, we can prove: If $a^2 n > (n-1)(l+n)^2$, $n > 1$, $n + l < -1$, then $a^2 - 4(n-1)(1-l) > 0$.

therefore

$$(l+n)^2 > 4n(1-l) > 4, \quad \text{so } l+n > 2, \quad n > 2-l > 2. \tag{5}$$

Actually, we can prove by induction that

$$l+n > 2^{(2^{k+1}-1)/2^k} \quad (\to 4)$$

for any natural number k, so we have $n > 4$.

On the other hand, from $na^2 < -l < n$ we get

$$1/4 > a^2/4 \geq (n-1)(1-l) > n-1,$$

i.e., $n < 5/4$; this contradicts (5).

In order to see whether limit cycle (LC, for abbreviation) can appear around S_1, we study first the condition for S_1 to be on the line of divergence:

$$x = \delta/(1-2l). \tag{6}$$

The y-coordinates of the intersection points of (6) with $P(x,y)=0$ satisfy the equation

$$n(1-2l)^2y^2 - (1-2l)^2y + \delta^2(1-l) = 0, \tag{7}$$

which gives

$$y'_{1,2} = [1-2l \pm \sqrt{(1-2l)^2 - 4n(1-l)\delta^2}]/2n(1-2l). \tag{8}$$

So y'_1 and y'_2 are both positive when

$$0 < \delta < (1-2l)/2\sqrt{n(1-l)} = \delta_1, \tag{9}$$

but (6) and $P(x,y)=0$ have no intersection point when $\delta > \delta_1$.

The line (6) intersects $y = 1+ax$ at $(\delta/(1-2l),\ (1-2l+a\delta)/(1-2l))$. In order that this point lies on $P(x,y)=0, \delta$ must satisfy the equation

$$[na^2+1-l]\delta^2 - a(1-2l)(1-2n)\delta + (n-1)(1-2l)^2 = 0. \tag{10}$$

The discriminant of (10) is

$$(1-2l)^2[a^2 - 4(n-1)(1-l)]. \tag{11}$$

From Theorem 1 we see that (11) is negative under condition (3), so S_1 can never be on (6), i.e., S_1 is always a stable node or focus, because $\mathrm{div}|_{S_1} < 0$ for $\delta = 0$ and therefore for all δ.

On the other hand, (3) implies that l_5 goes to S_1, but l_1 always remains in the half plane $1+ax-y>0$, and to the left of l_5.[1)] We conjecture that under conditions (3), there is no LC around S_1 when $\delta > 0$ (see Conjecture 1 after Example 2).

Assume $a^2 - 4(n-1)(1-l) > 0$. Then (10) has real roots. Denote them by $0 < \delta_2 < \delta_3 (< \delta_1)$. Therefore, under conditions $a<0,\ n>1,\ n+l>0$ at least one of the inequalities $na^2+l \geq 0$ and $a^2n \geq (n-1)(l+n)^2$ exists. This means either a saddle S'_2 appears to the right of S_1 on $y=1+ax$ or l_1 lies to the right of L.[2)] In the latter case since both l_1 and l_3 are not completely lying in a half plane $1+ax-y>0$ or $1+ax-y<0$, they need not both turn clockwise or counter-clock-wise as δ increases. Numerical examples show that l_1 and l_3 can coincide twise and LC can appear around S_1 when δ lies in two different intervals.

[1)] For the meaning of the l'_is, see Fig.1, or Fig.4 in [1].

[2)] For the meaning of the L, see Fig.3, or Fig.5 in [1].

Example 1. Take $n = 9/8, a = -2, l = -5$. Then the two roots of (10) are

$$\delta_2 = 11/14, \quad \delta_3 = 11/6. \tag{12}$$

Now

$$\delta_1 = 11/3\sqrt{3} > \delta_3, \quad a = -2 > -\sqrt{-l/n} = -\sqrt{40/3},$$
$$a^2 n - (n-1)(l+n)^2 = 1343/512 > 0, \quad n\delta_2^2 + l = 1089/1568 - 5 < 0.$$

But here $n + l < 0, a^2 - 4(n-1)(1-l) > 0$ can be proved directly. So l_1 lies on the upper side of L when $\delta = 0$.

For $\delta = \delta_2$, the system

$$\dot{x} = -y + 11x/14 - 5x^2 + 9y^2/8, \quad \dot{y} = x(1 - 2x - y) \tag{13}$$

has critical points $O, N, S_1(1/14, 6/7)$ and $S_2(-7/2, 8)$. The line $P_x + Q_y = 11/14 - 11x = 0$ passes through S_1, so S_1 is a weak focus.

Transforming the origin to S_1, we get

$$\dot{u} = u/14 + 13v/14 - 5u^2 + 9v^2/8, \quad \dot{v} = -u/7 - v/14 - 2u^2 - uv. \tag{14}$$

Then the transformation

$$u = 7(\eta + 5\xi)/5, \quad v = -14\eta/5, \quad dt/d\tau = 14/5$$

or

$$\xi = (2u + v)/14, \quad \eta = -5v/14, \quad dt/d\tau = 14/5$$

changes (14) into

$$\frac{d\xi}{d\tau} = -\eta - 77\eta^2/125 - 1078\xi\eta/25 - 588\xi^2/5, \quad \frac{d\eta}{d\tau} = \xi + 98\xi\eta/5 + 98\xi^2. \tag{15}$$

So, S_1 is an unstable focus, since the first focal value at S_1 is

$$1078(588/5 + 77/125)/25 - 98(98/5 - 1176/5) > 0.$$

For $\delta = \delta_3$, the system

$$\dot{x} = -y + 11x/6 - 5x^2 + 9y^2/8, \quad \dot{y} = x(1 - 2x - y) \tag{16}$$

has critical points $O(0,0), N(0, 8/9), S_1(1/6, 2/3)$ and $S_2(-3/2, 4)$. The line $P_x + Q_y = 11/6 - 11x = 0$ passes through S_1, so S_1 is again a weak focus. Transformaing the origin to S_1, we get

$$\dot{u} = u/6 + v/2 - 5u^2 + 9v^2/8, \quad \dot{v} = -u/3 - v/6 - 2u^2 - uv. \tag{17}$$

Then the transformation

$$u = 3(\eta + \sqrt{5}\xi)/\sqrt{5}, \quad v = -6\eta/\sqrt{5}, \quad \frac{dt}{d\tau} = 6/\sqrt{5},$$

or

$$\xi = (2u + v)/6, \quad \eta = -\sqrt{5}v/6, \quad \frac{dt}{d\tau} = 6/\sqrt{5}$$

changes (17) into

$$\frac{d\xi}{d\tau} = -\eta - 9\eta^2/5\sqrt{5} - 198\xi\eta/5 - 108\xi^2/\sqrt{5}, \quad \frac{d\eta}{d\tau} = \xi + 18\xi\eta/\sqrt{5} + 18\xi^2. \tag{18}$$

So, the first focal value at S_1 is

$$-198(-9/5\sqrt{5} - 108/\sqrt{5})/5 - 18(18/\sqrt{5} - 216/\sqrt{5}) > 0,$$

S_1 is again an unstable weak focus.

Now, from (6) of [1] we get

$$\frac{dF(x_i)}{d\delta} = x_i + [(na^2 + l)2x_i + (\delta + 2na - a)]\frac{\partial x_i}{\partial \delta} = 0,$$

which gives

$$\frac{\partial x_1}{\partial \delta} = -x_1 \Big/ \frac{\partial F}{\partial x}\Big|_{x_1} > 0. \tag{19}$$

Hence

$$\frac{\partial \mathrm{div}}{\partial \delta}\Big|_{S_1'} = 1 + (2l-1)\frac{\partial x_1}{\partial \delta} = 1 + (1-2l)x_1 \Big/ \frac{\partial F}{\partial x}\Big|_{x_1}. \tag{20}$$

When $\delta = 11/14$, we have $x_1 = 1/14$,

$$\frac{\partial x_1}{\partial \delta} = \frac{-1/14}{-1/14 + 11/14 - 9/2 + 2} = \frac{1}{25}, \quad \frac{\partial \mathrm{div}}{\partial \delta}\Big|_{S_1'} = 1 - \frac{11}{25} > 0.$$

This mwans that $\mathrm{div}|_{S_1'}$ increases as δ increases from $11/14$. So, for $\delta \geq 11/14$, S_1' is an unstable focus. Therefore, when $\delta < 11/14$ but increases to $11/14$, there is an unstable LC Γ_1 contracting to S_1'. When $\delta = 11/6$, we have $x_1 = 1/6$.

$$\frac{\partial x_1}{\partial \delta} = \frac{-1/6}{-1/6 + 11/6 - 9/2 + 2} = \frac{1}{5}, \quad \frac{\partial \mathrm{div}}{\partial \delta}\Big|_{S_1} = 1 - \frac{11}{5} < 0.$$

This means that $\mathrm{div}|_{S_1}$ decreases as δ increases from $11/6$. So, for $\delta < 11/6$ but $|\delta - 11/6| << 1$, S_1 is an unstable focus. As δ increases from $11/6$, S_1 becomes stable, an unstable LC Γ appears around S_1. Where comes the unstable LC when $\delta < 11/14$ but $|\delta - 11/14| << 1$? Let us see the global phase-portrait of (13) shown in Fig.1, which was drawn by computer.[1)]

Fig.1

From the phase-portrait in the first quadrant we see apparently that the unstable LC contracting to S_1 when δ increases to $11/14$ must come from the appearance of a separatrix

[1)] The author thanks Prof. J. C. Artes very much for drawing this figure and the explanation of the facts shown in the next paragraph.

loop passing through N at a certain value δ_4 of δ. Numerical calculation in the computer shows that $\delta_4 \in [0.6, 0.7]$. Notice that $\mathrm{div}|_N = \delta$, so this separatrix loop must be inner unstable. Moreover, the unstable LC generated from S_1 when δ increases from $11/6$ also finally becomes again another separatrix loop passing through N at a certain value δ_5 of δ, where $\delta_5 \in [1.9, 2]$. For $\delta \in [11/14, 11/6]$, no LC exists around S_1; this is also proved only by computer.

The change of phase-portraits around S_1 is shown in Figs. 2.

$\delta = 0$ $\delta = \delta_4$ $\delta_4 < \delta < \frac{11}{14}$ $\delta = \frac{11}{14}$ $\delta = \frac{11}{6}$

$\frac{11}{6} < \delta < \delta_5$ $\delta = \delta_5$ $\delta > \delta_5$ $\delta = 2.045 > \delta_5$ $\delta > 2.045$

Fig.2

Example 2. Consider the system

$$\dot{x} = -y + \delta x - x^2 + 3y^2/2, \quad \dot{y} = x(1 - x/3 - y), \tag{21}$$

here $na^2 + l < 0,\ n + l > 0,\ n + l - 1 < 0,\ a^2 n - (n-1)(l+n)^2 > 0,\ a^2 - 4(n-1)(1-l) < 0$. The equation of critical points at infinity: $9k^3 + 2 = 0$ has only one real root $k \doteq -0.61$. So, $S_1(x_1, y_1)$ is always stable, while $S_2(x_2, y_2)$ is always unstable for all δ. The phase-portrait of the system for $\delta = 0$ is shown in Fig.3. Although $n + l - 1 < 0,\ l_2$ still goes into $O(0, 0)$.

Fig.3

Conjecture 1. System (21), or more general, system (1) has no LC around S_1 under conditions (2) and

$$na^2 < -l, \quad a^2 - 4(n-1)(1-l) < 0. \tag{22}$$

Example 3. Consider the system

$$\dot{x} = -y + \delta x - x^2 + 13y^2, \quad \dot{y} = x(1 - 10x - y). \tag{23}$$

Here

$$n = 13,\ l = -1,\ a = -10,\ na^2 + l > 0,\ na^2 < (n-1)(l+n)^2,$$
$$a^2 - 4(n-1)(1-l) > 0, \quad a > -\sqrt{n(n-1)},$$

S_1 and S_2 are both on the right side of the y-axis, $l_1(l_2)$ lies on the lower side of $L(L')$, and so goes around O when $\delta = 0$.

Equation (10), which is now

$$1302\delta^2 - 750\delta + 108 = 0$$

has two real roots

$$\delta_2 = 2/7 \doteq 0.2857, \quad \delta_3 = 9/31 \doteq 0.2903.$$

For $\delta = \delta_2$, (23) becomes

$$\dot{x} = -y + 2x/7 - x^2 + 13y^2, \quad \dot{y} = x(1 - 10x - y). \tag{24}$$

$\mathrm{div}(P, Q) = 0$ is $2/7 - 3x = 0$, it passes through $S_1(2/21, 1/21)$ (see Fig.4). Since the normal form of (24) at S_1 is

$$\frac{d\xi}{d\tau} = -\eta + 441[7794\eta^2 - 36\sqrt{6}\xi\eta - 288\xi^2]/5760,$$
$$\frac{d\eta}{d\tau} = \xi + 441(144\xi^2 + 12\sqrt{6}\xi\eta)/5760,$$

S_1 is a stable weak focus of (24), $S_2(42/433, 13/433)$ is a saddle point. When $\delta > 2/7$, S_1 becomes unstable, a stable LC Γ_1 appears around S_1.

Fig.4 Fig.5

For $\delta = \delta_3$, (23) becomes

$$\dot{x} = -y + 9x/31 - x^2 + 13y^2, \quad \dot{y} = x(1 - 10x - y), \tag{25}$$

div$= 9/31 - 3x = 0$ passes through $S_2(3/31, 1/31)$, S_2 is a weak saddle, while $S_1(124/1299, 59/1299)$ is an unstable focus.

The tangent line of the separatrix l_1 of (24) at N is $y - 1/13 \doteq -1.143x$, which intersects the x-axis at $M(0.0674, 0)$. M lies on the left side of div$= 2/7 - 3x = 0$. So LC around O already disappears when $\delta \geq \delta_2$. Therefore, LC cannot appear both around O and S_1 at the same time. Notice that when

$$\delta = \delta_1 = (1-2l)/2\sqrt{n(1-l)} = 3/2\sqrt{26} = 0.2924 > 9/31, \tag{26}$$

$S_1 = S_2 =$ saddle-node, LC around S_1 must disappear.

Conjecture 2. System (23), or more general, system (1) under the conditions

$$-l < na^2 < (n-1)(l+n)^2, \quad a^2 - 4(n-1)(1-l) > 0$$

cannot have LC around O and S_1 at the same time. Moreover, if LC around S_1 exists, it must be unique.

Finally, let us prove the interesting

Theorem 2. *For the quadratic system*

$$\dot{x} = -y + lx^2 + mxy + ny^2, \quad \dot{y} = x(1 + ax - y), \quad n > 1,\ a < 0,\ l < 0, \tag{27}$$

*it is impossible that the phase-portrait shown in Fig.*5 *apperas, where $O(S_1)$ is a stable (unstable) weak focus, and*

$$\mathrm{div}(P,Q)|_N > 0, \quad (\mathrm{div}(P,Q)|_{S_2} < 0).$$

Proof. The condition for div$= (2l-1)x + my = 0$ to pass through S_1 is

$$(1-l)m^2 + (1-2l)am + (n-1)(1-2l)^2 = 0, \tag{28}$$

which gives

$$m = (2l-1)[a + \sqrt{a^2 - 4(n-1)(1-l)}]/2(1-l) > 0. \tag{29}$$

(Here we only take the positive sign before the square root, for the negative sign case, the proof is the same.) The condition for O to be a stable weak focus is

$$\begin{aligned} W_1 &= m(l+n) - a(2l-1) \\ &= (2l-1)[(l+n)(a + \sqrt{a^2 - 4(n-1)(1-l)}) - 2(1-l)a]/2(1-l) < 0, \end{aligned} \tag{30}$$

i.e.,

$$(l+n)(a + \sqrt{a^2 - 4(n-1)(1-l)}) > 2(1-l)a,$$

or

$$a(n+3l-2) > -(l+n)\sqrt{a^2 - 4(n-1)(1-l)}. \tag{31}$$

(i) Assume $n + 3l - 2 > 0$. Then $n + l > 0$, and (31) means that the absolute value of $a(n+3l-2)$ is less than that of $(n+l)\sqrt{a^2 - 4(n-1)(1-l}$. The condsition for l_1 lying below L (tangent of l_1 at N) is

$$\Sigma = a^2 n + am(l+n) - (n-1)(l+n)^2 < 0,$$

i.e.,

$$a^2 n + a(l+n)(2l-1)[a + \sqrt{a^2 - 4(n-1)(1-l)}]/2(1-l) - (n-1)(l+n)^2 < 0. \tag{32}$$

Taking the square of (31) and using (32) we get

$$a(2l-1)[2a(1-l) - a(l+n) - (l+n)\sqrt{a^2 - 4(n-1)(1-l)}]/2(1-l) > 0,$$

or

$$a(2-3l-n) > (l+n)\sqrt{a^2-4(n-1)(1-l)}; \tag{33}$$

this contradicts (31).

(ii) Assume $n+l>0$, but $n+3l-2<0$. Then (31) holds naturally, but (33) does not hold, a contradiction to $\Sigma<0$.

(iii) Evidently, when $n+l=0$, we have $\Sigma>0$.

(iv) Assume now $n+l<0$. Then surely we have $W_1<0$, so O is always stable. We have already proved that $\Sigma>0$ when $l+n\geq o$; moreover, if $0<-(l+n)<<1$, we have still $\Sigma>0$. So if for certain $l+n<0$ and the cooresponding m, we have $\Sigma<0$, then by continuity, there must exist l^*, n^*, m^* such that $l^*+n^*<0$, $\Sigma^*=0$. This means that $(27)^*$ has an integral line L, and two weak foci O and S_1^*. It is well-known that in this case O and S_1^* must be both centers. But this contradicts $W_1^*<0$.

Remark. Theorem 2 shows in another way that (2,2) distribution of LC for a quadratic system is impossible. (We may add $\delta x(1+ax-y)$ instead of δx in the first equation of (27) to obtain a complete quadratic system).

References

[1] Ye Yanqian, Limit cycles and bifurcation curves for the quadratic differential system $(\mathrm{III})_{m=0}$ having three anti-saddles (I), *Chin. Ana. of Math.,* **17B:**2(1996), 167–174.