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ON FACTORIZATION THEOREMS OF PLURIHARMONIC MAPS INTO THE UNITARY GROUP***

Cheng Qiyuan* Dong Yuxin**

Abstract

The authors give some constructive factorization theorems for pluriharmonic maps from a Kaehler manifold into the unitary group $U(N)$ and obtain some optimal upper bounds of minimal uniton numbers.

Keywords Pluriharmonic map, Kaehler manifold, Unitary group, Factorization, Uniton number

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§0. Introduction

Let M be a Kaehler manifold and N be a Riemannian manifold. A smooth map $\varphi : M \to N$ is called pluriharmonic if the $(0,1)$-exterior derivative $D"\partial\varphi$ of $\partial\varphi$ vanishes identically. The notion of pluriharmonic maps is a natural extension of harmonic maps from Riemann surfaces. There are many beautiful results on harmonic maps from surfaces (see [1, 5]). It is interesting and important to generalize them to results for pluriharmonic maps from Kaehler manifolds. In [9], Ohnita and Valli extended the famous work of Uhlenbeck[6] to the case of pluriharmonic maps. They investigated the factorization for pluriharmonic maps from compact complex manifolds to the unitary group. By the methods of [2], Ohnita and Udagawa studied also the factorization for pluriharmonic maps into some Grassmann manifolds (see [8]). However, the problem for explicit construction of any pluriharmonic map into $U(N)$ or a general Grassmannian is still open. The first step towards this problem is to give a constructive factorization.

The purpose of this paper is to give some construction factorization theorems for pluriharmonic maps from Kaehler manifold into $U(N)$ or $G_k(C^N)$ (see Theorem 3.1 and Theorem 3.2). In fact, if $M = S^2$, Theorem 3.1 was obtained by Wood[10]. As in [6, 9], we associate with every pluriharmonic map a unique integer $m(\varphi)$, the minimal uniton number which reflects the level of complexity of pluriharmonic map. We obtain some optimal upper bounds of minimal uniton numbers. Some of the above results generalize those in [4, 8, 10, 11].

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*Department of Applied Mathematics, Beijing Institute of Technology, Beijing 100081, China.
**Department of Mathematics, Hangzhou University, Hangzhou 310028, China.
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§1. Preliminaries

Let M be a connected Kaehler manifold and N be a connected Riemannian manifold. Let $\varphi : M \to N$ be a smooth map from M to N. The differential $d\varphi : TM \to \varphi^{-1}TN$ extends by complex linearity to $d\varphi : TM^C \to \varphi^{-1}TN^C$. Relative to the complex structure J of M we have a decomposition $TM^C = TM^{(1,0)} \oplus TM^{(0,1)}$. By restricting $d\varphi$ to each factor we define the bundle maps $\partial\varphi : TM^{(1,0)} \to \varphi^{-1}TN^C$ and $\overline{\partial}\varphi : TM^{(0,1)} \to \varphi^{-1}TN^C$. Using the induced connection $\bigtriangledown^{\varphi}$ and the $\overline{\partial}$-operator of $TM^{(1,0)}$, we define the $(0,1)$-exterior derivative of $\partial\varphi$ by $(D''_{\overline{W}}\partial\varphi)(Z) = \bigtriangledown^{\varphi}_{\overline{W}}(\partial\varphi(Z)) - \partial\varphi(\overline{\partial}_{\overline{W}}Z)$ for each $Z, W \in C^{\infty}(TM^{(1,0)})$. Then φ is called pluriharmonic if φ satisfies $D''\partial\varphi = 0$.

Lemma 1.1.[9] *A smooth map φ from a Kaehler manifold M to a Riemannian manifold N is pluriharmonic if and only if, for any holomorphic curve $\tau : C \to M$, the composite $\varphi \circ \tau$ is always harmonic.*

Let $\underline{C}^N = M \times C^N$ denote the trivial complex bundle equipped with the standard Hermitian metric $\langle\ ,\ \rangle$ on each fibre. Let $U(N)$ denote the unitary group and $u(N)$ its Lie algebra. Denote by μ the Maurer-Cartan form of $U(N)$ which is a left-invariant $u(N)$-valued 1-form on $U(N)$. Let $\varphi : M \to U(N)$ be a smooth map. Set $\alpha_\varphi = 1/2\varphi^*\mu$, which is a $u(N)$-valued 1-form on M. Then we decompose α_φ into $(1,0)$ and $(0,1)$ parts with respect to M: $\alpha_\varphi = \alpha'_\varphi + \alpha''_\varphi$, where α'_φ and α''_φ are sections of $T^*M^{(1,0)} \otimes \mathrm{End}(\underline{C}^N)$ and $T^*M^{(0,1)} \otimes \mathrm{End}(\underline{C}^N)$ respectively.

Lemma 1.2.[9] *A smooth map $\varphi : M \to U(N)$ is pluriharmonic if and only if*

$$\overline{\partial}\alpha'_\varphi + [\alpha'_\varphi \wedge \alpha''_\varphi] = 0, \tag{1.1}$$

or, equivalently, if and only if

$$\partial\alpha''_\varphi + [\alpha'_\varphi \wedge \alpha''_\varphi] = 0. \tag{1.2}$$

Here the Lie bracket is that of $u(N)$.

This can be interpreted as follows. Set $D_\alpha = d + \alpha_\varphi$. Then D_α produces a holomorphic vector bundle structure in $\underline{C}^N$, provided φ is pluriharmonic (see [9, Lemma 2.2]). The condition (1.1) means that α'_φ is a holomorphic section of $T^*M^{(1,0)} \otimes \mathrm{End}(\underline{C}^N, D_\alpha)$. Similarly (1.2) means that α''_φ is an antiholomorphic section of $T^*M^{(0,1)} \otimes \mathrm{End}(\underline{C}^N, D_\alpha)$. Note that the map φ is constant if and only if $\alpha'_\varphi = 0$ (or, equivalently, $\alpha''_\varphi = 0$). It is easy to see that α''_φ is minus the adjoint of α'_φ, that is, $(\alpha''_\varphi)^* = -\alpha'_\varphi$.

Let $\varphi : M \to U(N)$ be a smooth map. Set $\alpha_\varphi = 1/2\varphi^*\mu = \alpha'_\varphi + \alpha''_\varphi$. Set, for each $\lambda \in C^* \setminus \{0\}$,

$$\alpha_\lambda = (1 - \lambda^{-1})\alpha'_\varphi + (1 - \lambda)\alpha''_\varphi. \tag{1.3}$$

We know that the general linear group $GL(N, C)$ is the complexification of the unitary group $U(N)$. Denote by μ_C the Maurer-Cartan form of $GL(N, C)$. We consider the following linear differential equations

$$\Phi^*_\lambda\mu_C = \alpha_\lambda \tag{1.4}$$

of smooth maps $\Phi_\lambda : M \to GL(N, C)$ for each $\lambda \in C^*$. By (1.3), (1.4) can be written as

$$\partial\Phi_\lambda = (1 - \lambda^{-1})\Phi_\lambda\alpha'_\varphi, \quad \overline{\partial}\Phi_\lambda = (1 - \lambda)\Phi_\lambda\alpha''_\varphi, \quad \lambda \in C^*. \tag{1.5}$$

It is easy to verify that the integrability condition of (1.4) or (1.5) is equivalent to the pluriharmonicity of φ. Thus, if φ is pluriharmonic, we can solve (1.4) in any simply connected complex domain $U \subset M$, and a solution $\Phi : C^* \times U \to GL(N, C)$ is uniquely determined by prescribing $\Phi_\lambda(p) = h(\lambda)$ for any base point $p \in U$ and any smooth map $h : C^* \to GL(N, C)$. Note that Φ_1 is always a constant map, we lose nothing by assuming

$$\Phi_1 = I. \tag{1.6}$$

From (1.5) we have $\partial(\Phi_\lambda \Phi^*_{\sigma(\lambda)}) = \overline{\partial}(\Phi_\lambda \Phi^*_{\sigma(\lambda)}) = 0$, where $\sigma(\lambda) = (\overline{\lambda})^{-1}$ and $*$ denote the conjugate transpose matrix. By (1.6), we get

$$\Phi_\lambda \Phi^*_{\sigma(\lambda)} = I. \tag{1.7}$$

From now on we consider only the case that the extended solution satisfying $\Phi(\cdot, p) : C^* \to GL(N, C)$ is holomorphic. Hence $\Phi_\lambda(x) = \Phi(\lambda, x)$ is holomorphic in $\lambda \in C^*$ for each fixed $x \in U$. From (1.5) and (1.6), we see that $\Phi_{-1} = Q\varphi$ for some $Q \in U(N)$ constant. Following [6, 9], we call Φ an extended solution of φ. If M is simply connected, we can choose $U = M$ and thus we have a global extended solution $\Phi : C^* \times M \to GL(N, C)$.

Note that a pluriharmonic map from a Riemann surface is just a harmonic map. By Lemma 1.1 and (1.4), we immediately see the following

Lemma 1.3. *Let $\varphi : M \to U(N)$ be a pluriharmonic map and Φ_λ be an extended solution of φ. Let $\tau : C \to M$ be a holomorphic curve. Then $\Phi_\lambda \circ \tau$ is an extended solution of $\varphi \circ \tau$ and every extended solution of $\varphi \circ \tau$ is obtained in this way.*

We recall the bijective correspondence between complex subbundles η of the vector bundle $\underline{C}^N = M \times C^N$ with rank k and smooth maps $\Pi_\eta - \Pi^\perp_\eta : M \to G_k(C^N) \subset U(N)$, where Π_η (respectively $\Pi^\perp_\eta$) denotes Hermitian projection onto η (respectively its orthogonal complement $\eta^\perp$ in $\underline{C}^N$). We assume that $\Phi : C^* \times M \to GL(N, C)$ is an extended solution of pluriharmonic map φ. For a smooth map $\Pi - \Pi^\perp : M \to G_k(C^N)$, that is, $\Pi^2 = \Pi^* = \Pi$, set $\Psi_\lambda = \Phi_\lambda(\Pi + \lambda\Pi^\perp) : M \to GL(N, C)$ for each $\lambda \in C^*$. Note that $\Psi_{-1} : M \to U(N)$.

Lemma 1.4.[9] *The map Ψ is an extended solution if and only if the subbundle η of $\underline{C}^N$ satisfies the following:*

(1) *η is invariant by α'_φ,*

(2) *η is a holomorphic subbundle of the holomorphic vector bundle $(\underline{C}^N, D_\alpha)$.*

Such a subbundle η or the corresponding map $\Pi - \Pi^\perp$ is called a uniton for φ. The procedure of making a new pluriharmonic map $\psi = \Psi_{-1}$ (respectively a new extended solution Ψ_λ) from a given pluriharmonic map φ (respectively a given extended solution Φ_λ) is called the addition of a uniton. It is important to introduce the notion of unitons with the singularity set, since we work over higher dimensional complex manifold.

Definition.[9] *Let $\varphi : M \to U(N)$ be a pluriharmonic map from an m-dimentional Kaehler manifold. We call η a meromorphic uniton for φ if η is a smooth uniton for φ defined over $M \setminus S_\eta$, where S_η is an analytic subset of M with $\dim_C S_\eta \leq m - 2$.*

Lemma 1.5. Let $\varphi : M \to U(N)$ be a pluriharmonic map. Then

(1) $\mathrm{Im}\alpha'_\varphi$ is a meromorphic uniton for φ if α'_φ is considered as a bundle homomorphism $\alpha'_\varphi : T^*M^{(1,0)} \otimes \underline{C}^N \to \underline{C}^N$.

(2) $\mathrm{Ker}\alpha'_\varphi$ is a meromorphic uniton for φ if α'_φ is considered as a bundle homomorphism $\alpha'_\varphi : \underline{C}^N \to T^*M^{(1,0)} \otimes \underline{C}^N$.

Proof. We consider α'_φ as a bundle homomorphism $\alpha'_\varphi : T^*M^{(1,0)} \otimes \underline{C}^N \to \underline{C}^N$. Set $\mathrm{Im}\alpha'_\varphi = \bigcup_{x\in M} \mathrm{Im}(\alpha_\varphi)_x$. Let $t = \max\{\dim \mathrm{Im}(\alpha'_\varphi)_x; x \in M\}$. It induces a bundle homomorphism $\wedge^t\alpha'_\varphi : \wedge^t(T^*M^{(1,0)} \otimes \underline{C}^N) \to \wedge^t\underline{C}^N$. From Lemma 1.2, we see that α'_φ and $\wedge^t\alpha'_\varphi$ are holomorphic. Set $V = \{x \in M; (\wedge^t\alpha'_\varphi)_x = 0\} = \{x \in M; \dim(\mathrm{Im}\alpha'_\varphi)_x < t\}$, which is an analytic subset of M. Then, $\underline{\mathrm{Im}}\alpha'_\varphi$ is a holomorphic subbundle of $\underline{C}^N$ over $M \setminus V$. Let $V = V_1 + V_2$ be a decomposition of V into the union of components of codimension 1 and the union of components of codimension at least 2. If $x \in V_1$, then there exists a neighborhood U of x and a holomorphic function w on U such that $V_1 \cap U$ is defined by $w = 0$. Thus we have, near x, $\wedge^t\alpha'_\varphi = w^k\sigma$, where σ is a local holomorphic section of $\mathrm{Hom}(\wedge^t(T^*M^{(1,0)} \otimes \underline{C}^N), \wedge^t\underline{C}^N)$. The image of σ defines a holomorphic subbundle of $\underline{C}^N$ of rank t over V_1 around x. In this way, $\underline{\mathrm{Im}}\alpha'_\varphi$ extends to a holomorphic subbundle of $(\underline{C}^N, D_\alpha)$ over $M \setminus V_2$. By Lemma 1.4, we have (1). In a similar way, we can prove (2).

§2. Pluriharmonic Maps of Finite Uniton Numbers

Let $\varphi : M \to U(N)$ be a pluriharmonic map and Φ_λ be an extended solution of φ. Since Φ_λ is holomorphic in $\lambda \in C^*$, we may expand Φ_λ in a Laurent series: $\Phi_\lambda = \sum_{s=-\infty}^{+\infty} T_s\lambda^s$, where $T_s : U \to gl(N, C)$. We say that a pluriharmonic map $\varphi : M \to U(N)$ has at most uniton n if there exists a global extended solution $\Phi : C^* \times M \to GL(N, C)$ such that

(i) Φ has the Laurent expansion in $\lambda \in C^*$ of the form $\Phi_\lambda = \sum_{s=0}^{n} T_s\lambda^s, \quad T_n \not\equiv 0$,

(ii) $\Phi_1 = I$, (iii) $\Phi_\lambda\Phi^*_{\sigma(\lambda)} = I$, (iv) $\Phi_{-1} = Q\varphi$ for some $Q \in U(N)$.

Here $T_s : M \to gl(N, C)$.

Note that the uniton numbers n can be enlarged in a fake way by multiplying an extended solution Φ_λ by a holomorphic map $\widetilde{Q} : C^* \to GL(N, C)$ (constant in $p \in M$) with $\widetilde{Q}_1 = I$. Set $m(\varphi) = \min\{n; \varphi \text{ has an } n\text{-extended solution}\}$. We call $m(\varphi)$ the minimal uniton number of φ.

Assume that M is a compact Kaehler manifold. From Theorem 4.2 in [9], we know that a pluriharmonic map $\varphi : M \to U(N)$ has finite uniton number if φ has a global extended solution Φ. In particular, any pluriharmonic map $\varphi : M \to U(N)$ from a simply connected compact Kaehler manifold M always has finite uniton number.

Let $f : M \to G_k(C^N)$ be a holomorphic map. Denote by $O^s(f)$ the s-th osculating spaces along f. Then, there exists a unique positive integer n such that $O^{n-1}(f) \subset O^n(f)$ and $O^n(f) = O^{n+1}(f)$. Set $W = \cup R_s$, where R_s is the singular set of $O^s(f)$ with $\mathrm{codim}_C R_s \geq 2$. Then we obtain $n+1$ holomorphic subbundles $\eta_s = O^s(f)$ of $\underline{C}^N$ over $M\setminus W$, $s = 0, 1, \cdots, n$. If f is full, we have

$$\eta_s \subset \eta_{s+1}, \quad \partial C^\infty(\eta_s) \subset C^\infty(\eta_{s+1}), \quad \overline{\partial} C^\infty(\eta_s) \subset C^\infty(\eta_s), \quad \eta_n = (M \setminus W) \times C^N. \quad (2.1)$$

Proposition 2.1. *Let Π_s be the Hermitian projection on η_s, $s = 0, 1, \cdots, n$, where η_s satisfies* (2.1). *Then*

$$\Phi_\lambda = \Pi_0 + \sum_{s=1}^{n} \lambda^s(\Pi_s - \Pi_{s-1})$$

is an extended solution of a pluriharmonic map $\varphi = \Phi_{-1}$.

Proof. For $\varphi = \Phi_{-1}$, we define α_λ by (1.3). Let $\tau : C \to M$ be any holomorphic curve. From Theorem 10.1 of [6], we see that $\Phi_\lambda \circ \tau = \left\{\Pi_0 + \sum_{s=1}^{n} (\Pi_s - \Pi_{s-1})\right\} \circ \tau$ is an extended solution to the harmonic map $\varphi \circ \tau$, i.e.,

$$(\Phi_\lambda \circ \tau)^* \mu_C = (1 - \lambda^{-1})\alpha'_{\varphi\circ\tau} + (1 - \lambda)\alpha''_{\varphi\circ\tau} = \tau^*\alpha_\lambda.$$

Since $\tau : C \to M$ is arbitary, by Lemma 1.1, we see that φ is pluriharmonic and $\Phi_\lambda^* \mu_C = \alpha_\lambda$, i.e., Φ_λ is an extended solution of φ.

Proposition 2.2. *Assume that M is a compact Kaehler manifold with $c_1(M) > 0$. Then any pluriharmonic map $\varphi : M \to U(N)$ has finite uniton number.*

Proof. Since M is a compact Kaehler manifold with $c_1(M) > 0$, the solution of Yau to Calabi conjecture[13] ensures the existence of a Kaehler metric on M with positive Ricci corvature. From [12] we know that a compact Kaehler manifold with positive Ricci curvature is simply connected. Hence φ has finite uniton number.

Proposition 2.3. *Let $\varphi : M \to U(N)$ be a pluriharmonic map from a Kaehler manifold. If φ has finite uniton number, then for any holomorphic curve $\tau : C \to M$, $\varphi \circ \tau$ has finite uniton number and $m(\varphi\circ\tau) \le m(\varphi)$, where $m(\cdot)$ denotes the minimal uniton number of the map. Furthermore, set $\Gamma = \{\tau : C \to M \text{ is a holomorphic curve }\}$. Then $\max_{\tau\in\Gamma}\{m(\varphi \circ \tau\} = m(\varphi)$.*

Proof. This follows directly from Lemma 1.3.

In the case that M is an arbitary compact Kaehler manifold, not all pluriharmonic maps have finite uniton numbers. If $\dim_C M = 1$, there are many harmonic maps with infinite uniton numbers (see [3, 10]). The above proposition may be used to construct pluriharmonic maps of infinite uniton number from a compact Kaehler mnifold M with $\dim_C M > 1$.

Example. Set $M = S^2 \times T^2$. Let $\psi : T^2 \to CP^2$ be the Clifford minimal torus (see [10]). Then we know that $c \circ \psi : T^2 \to U(3)$ is a harmonic map with infinite uniton number (see [4]), where $c : CP^2 \to U(3)$ is the Cartan embedding. Let $f : S^2 \to U(N-3)$ be any harmonic map. Then the map $\varphi = (f, c\circ\psi) : M \to U(N)$ is a pluriharmonic map with infinite uniton number.

§3. Factorization Theorems

Let $\varphi : M \to U(N)$ be a pluriharmonic map. We say that φ is factorable (as the product of k uniton factors) if we can write

$$\varphi = \varphi_0(\beta_1 - \beta_1^\perp)\cdots(\beta_k - \beta_k^\perp), \tag{3.1}$$

where $\varphi_0 : M \to U(N)$ is a constant map, $k \in \{0, 1, 2, \cdots\}$, and, for each $i = 1, \cdots, k$, β_i is a uniton for $\varphi_{i-1} = \varphi_0(\beta_1 - \beta_1^\perp)\cdots(\beta_{i-1} - \beta_{i-1}^\perp)$.

Given an extended solution $\Phi_\lambda = \sum_{s=0}^{n} T_s\lambda^s$ of some pluriharmonic map with finite uniton number. Set $V_0 = V_0(\Phi_\lambda) =$ linear closure of $\{v \in C^N : v = T_0(q)w, q \in M, w \in C^N\}$.

Theorem 3.1. *Let M^m be a connected Kaehler manifold and $\varphi : M \to U(N)$ be a pluriharmonic map of finite uniton number. Then φ has a unique factorization*

$$\varphi = \varphi_0(\beta_1 - \beta_1^\perp)\cdots(\beta_k - \beta_k^\perp), \tag{3.2}$$

over $M \setminus S$ into $\varphi_0 \in U(N)$ and $\beta_i - \beta_i^\perp : M \to Gr(C^N)$ $(i = 1, 2, \cdots, k)$, where

(a) *S is an analytic subset of M with* $\dim_C S \le m-2$,

(b) *each* $\varphi_i = \varphi_0(\beta_1 - \beta_1^\perp)\cdots(\beta_i - \beta_i^\perp) : M \setminus S \to U(N) (i = 1, 2, \cdots, k)$ *is a pluriharmonic map,*

(c) *β_i is a meromorphic uniton for φ_{i-1} and* $\beta_i^\perp = \underline{\mathrm{Im}}(\alpha'_\varphi)(i = 1, \cdots, k)$,

(d) $\beta_1 : M \setminus S \to Gr(C^N)$ *is a holomorphic map,*

(e) $m(\varphi) \le k < N$.

Proof. From [6, 9], we know that there exists a unique extended solution $\Phi_\lambda = \sum_{s=0}^{n} T_s \lambda^s$ of φ with $V_0(\Phi) = C^N$ and $n = m(\varphi)$. By Lemma 1.5, $\mathrm{Im}\alpha'_\lambda$ is a meromorphic uniton for φ defined over $M \setminus S_0$, where S_0 is an analytic subset of M with $\dim_C S_0 \le m-2$. Thus $\Phi^{(1)} = \lambda^{-1}\Phi_\lambda(\alpha_0 + \lambda(\alpha_0)^\perp) : M \setminus S_0 \to GL(N, C)$ is an extended solution of $\varphi^{(1)} = \varphi(\alpha_0 - (\alpha_0)^\perp)$, where $\alpha_0 = \mathrm{Im}(\alpha'_\varphi)$. From (1.5) we have

$$T_0 \alpha'_\varphi = 0, \tag{3.3}$$

$$\overline{\partial} T_0 = T_0 \alpha''_\varphi, \qquad \partial T_0 = T_0 \alpha'_\varphi - T_1 \alpha'_\varphi. \tag{3.4}$$

By (3.3), we see that $\Phi_\lambda^{(1)}$ has the following form

$$\Phi_\lambda^{(1)} = \sum_{s=0}^{n} T_s^{(1)} \lambda^s, \tag{3.5}$$

where $T_s^{(1)} = T_s(\alpha_0)^\perp + T_{s+1}\alpha_0$, $s = 0, 1, 2, \cdots, n$. Since $T_0^{(1)}(\alpha_0)^\perp = T_0(\alpha_0)^\perp$, $T_0\alpha_0 = 0$, we have $V_0(\Phi_\lambda^{(1)}) = C^N$ and $\mathrm{rank}(T_0^{(1)}) \ge \mathrm{rank}(T_0)$. Set $M' = \{x \in M;\ \mathrm{rank}(T_0)$ and $\mathrm{rank}(T_0^{(1)})$ are maximal$\}$, which is a connected dense open subset of M. We can show that $\mathrm{rank}(T_0) < \mathrm{rank}(T_0^{(1)})$ on M'. If equality holds, we must have

$$\mathrm{Im}(T_1\alpha_0) = \mathrm{Im}(T_1\alpha'_\varphi) \subset \mathrm{Im}(T_0). \tag{3.6}$$

We conclude from (3.4) and (3.5) that $\mathrm{Im}(T_0)\mid_{M'}$ is a holomorphic and antiholomorphic subbundle of $M' \times C^N$. This implies that $\mathrm{Im}(T_0)$ is a constant subspace $V_0 \subset C^N$. By assumption, $V_0 = C^N$. However, from the reality condition (1.7), we have $T_0 T_n^* = 0$, i.e., we get $T_n = 0$, a contradiction. Thus we show that $\mathrm{rank}(T_0) < \mathrm{rank}(T_0^{(1)})$ on M'.

Iterating the above construction we get a sequence of extended solution

$$\Phi_\lambda, \Phi_\lambda^{(1)}, \cdots, \Phi_\lambda^{(i+1)} = \lambda^{-1}\Phi_\lambda^{(i)}(\alpha_i + \lambda(\alpha_i)^\perp), \cdots \tag{3.7}$$

where $\Phi_\lambda^{(i)}$ is an extended solution of the pluriharmonic map $\varphi^{(i)} = \varphi^{(i-1)}(\alpha_{i-1} - (\alpha_{i-1})^\perp)$ and $\alpha_i = \mathrm{Im}\alpha'_{\varphi^{(i)}}$ is a meromorphic uniton for $\varphi^{(i)}$ defined over $M \setminus S_i$ with $\dim_C S_i \le m-2$. We write $\Phi_\lambda^{(i)} = \sum_{s=0}^{n} T_s^{(i)} \lambda^s$. If $\varphi^{(i)}$ is not a constant, then $\mathrm{rank}(T_0^{(i+1)}) > \mathrm{rank}(T_0^{(i)})$ on an open dense subset of M. This implies that there exists an integer k such that $\mathrm{rank}(T_0^{(k+1)}) = N$. From (1.7), we have $\Phi_\lambda^{(k+1)} = T_0^{(k+1)} = I$. By (3.7), we get

$$I = Q\varphi(\alpha_0 - (\alpha_0)^\perp)\cdots(\alpha_k - (\alpha_k)^\perp), \tag{3.8}$$

over $M \setminus S$ where $S = \cup S_i$ and $Q \in U(N)$. It is easy to verify that if α is a uniton for φ, then $\alpha^\perp$ is a uniton for $\psi = \varphi(\alpha - \alpha^\perp)$ and $\varphi = -\psi(\alpha^\perp - \alpha)$. Hence, from (3.8), we see that φ has the desired factorization. The result (d) follows from Theorem 4.1 in [9]. From the above discussion, we have $\mathrm{rank}(T_0) < \mathrm{rank}(T_0^{(1)}) < \cdots < \mathrm{rank}(T_0^{(k+1)}) = N$. This shows (e).

Corollary 3.1. *Let M be a simply connected compact Kaehler manifold and $\varphi : M \to U(N)$ be a pluriharmonic map. Then φ has the factorization* (3.2).

Remark 3.1. If $M = S^2$, Wood proved that any harmonic map $\varphi : S^2 \to U(N)$ has the factorization (3.2) and he parametrized all harmopnic two-spheres in $U(N)$ using only algebra operations and integral transforms (see [10]).

Corollary 3.2. *Let $\varphi : M \to U(N)$ be a pluriharmonic map of finite uniton number. Then there exists a $Q \in U(N)$ such that $Q\varphi, \varphi Q : M \to SU(N)$.*

Proof. From (3.2), we see that $\det(\varphi)$ =const. over $M \setminus S$ and thus $\det(\varphi)$ =const. over M. Hence there exists a $Q \in U(N)$ such that $\det(Q\varphi) = \det(\varphi Q) = 1$.

Lemma 3.1. *Let $\psi : M \to G_k(C^N)$ be a smooth map and $c : G_k(C^N) \to U(N)$ be the Cartan embedding* (see [6]). *Then $\mathrm{Im}(\alpha_{c(\psi)}) = \mathrm{Im}(A^{\psi}_{(1,0)}) \oplus \mathrm{Im}(A^{\psi^\perp}_{(1,0)})$, where $A^{\psi}_{(1,0)}$ and $A^{\psi^\perp}_{(1,0)}$ denote the ∂-second fundamental forms of ψ and $\psi^\perp$ respectively* (*see* [8] *for details*).

Proof. Since $c(\psi) = c(\psi)^{-1} = \psi - \psi^\perp = 2\psi - I$, we have

$$\alpha'_{c(\psi)} = 1/2(\psi - \psi^\perp)\partial(\psi - \psi^\perp) = (\psi - \psi^\perp)\partial\psi = -\psi^\perp\partial\psi - \psi\partial\psi^\perp. \tag{3.9}$$

From [8], we know that the ∂-second fundamental forms of ψ and $\psi^\perp$ are vector bundle morphisms $A^{\psi}_{(1,0)} : T^*M^{(1,0)} \otimes \psi \to \psi^\perp$ and $A^{\psi^\perp}_{(1,0)} : T^*M \otimes \psi^\perp \to \psi$ defined by

$$A^{\psi}_{(1,0)}(\xi) = \psi^\perp(\partial\xi), \quad \xi \in C^\infty(\psi); \qquad A^{\psi^\perp}_{(1,0)}(\eta) = \psi(\partial\eta), \quad \eta \in C^\infty(\psi^\perp). \tag{3.10}$$

This lemma follows directly from (3.9) and (3.10).

Theorem 3.2. *Let $\psi : M^m \to G_k(C^N)$ be a pluriharmonic map of finite uniton number. Then there is a sequence of pluriharmonic maps $\psi^{(i)} : M \setminus S \to G_{k_i}(C^N)$ $(i = 0, 1, \cdots, l)$ such that* (i) $\psi^{(0)} = \psi$, (ii) *$\psi^{(i)}$ is a transform of $\psi^{(i-1)}$ by adding the uniton $\mathrm{Im}(\alpha'_{c(\psi^{(i)})})$,* (iii) $\psi^{(l)} =$ const., *where S is an analytic subset of M with $\dim_C S \le m - 2$.*

Proof. As in the proof of Theorem 3.1, we consider following pluriharmonic maps

$$\varphi^{(0)} = c(\psi), \varphi^{(1)}, \cdots, \varphi^{(i)} = \varphi^{(i-1)}(\alpha_{i-1} - (\alpha_{i-1})^\perp), \cdots$$

over $M \setminus S$, where $\alpha_i = \mathrm{Im}(\alpha_{\varphi^{(i)}})$. From Lemma 3.1, we know that $\varphi^{(0)}(\alpha_0 - (\alpha_0)^\perp) = (\alpha_0 - (\alpha_0)^\perp)\varphi^{(0)}$. This implies that $\varphi^{(1)}$ has image in a Grassmannian; that is, $\varphi^{(1)} = c(\psi^{(1)})$ for some $\psi^{(1)} : M \setminus S \to G_{k_1}(C^N)$. By induction on i and Theorem 3.1, we see that $\varphi^{(i)} = c(\psi^{(i)})$ for some pluriharmonic map $\psi^{(i)} : M \setminus S \to G_{k_i}(C^N)$ and there exists an integer l such that $\psi^{(l)}$ =const.

Remark 3.2. (a) If $\psi : M \to G_k(C^N)$ is a pluriharmonic map from a simply connected compact Kaehler manifold or a compact Kaehler manifold with $c_1(M) > 0$, then we have a similar results (see Proposition 2.2). (b) In [8], Ohnita and Udagawa obtained a generalization of Burstall and Wood[2] for pluriharmonic maps from a Kaehler manifold with $c_1(M) > 0$ to a Grassmannian $G_k(C^N)$ for $k = 2, 3$ and $N \le 12$.

Corollary 3.3. *Let $\psi : M \to CP^{N-1} = G_1(C^N)$ be a pluriharmonic map of finite uniton number. Inductively, define a sequence ψ_i of pluriharmonic maps $\psi_i : M \setminus S \to CP^{N-1}$ by $\psi_i = \mathrm{Im}(A^{\psi_{(i-1)}}_{(1,0)})$ $(i = 1, 2, \cdots)$ with $\psi_0 = \psi$. Then, there exists an integer s such that ψ_s is an anti-holomorphic map from $M \setminus S$ into CP^{N-1}, where $s \le N - 1$.*

Proof. We set $\varphi_i = -\varphi^{(i)}$, where $\varphi^{(i)}$ is as in Theorem 3.2. By using Lemma 3.1, a simply calculation shows that

$$\varphi_i = \mathrm{Im}(A^{\psi_{i-1}}_{(1,0)}) - (\mathrm{Im}A^{\psi_{i-1}}_{(1,0)})^{\perp} = c(\mathrm{Im}(A^{\psi_{i-1}}_{(1,0)}) = c(\psi_i).$$

From Theorem 3.2, we see that $A^{\psi_{l-1}}_{(1,0)} \equiv 0$, that is, ψ_{l-1} is anti-holomorphic. By Proposition 3.2 in [7], we know that $\psi_i : M\backslash \to CP^{N-1}$ (see also [8, Theorem 7.30]).

Remark 3.3. If M is a compact Kaehler manifold with $c_1(M) > 0$ and $\psi : M \to CP^{N-1}$ is a pluriharmonic map, Ohnita and Udagawa[8] proved the above result by using ∂-return map.

Now we hope to estimate the minimal uniton number of a pluriharmonic map with finite uniton number. In [4], the second author and Y. B. Shen proved that if $\varphi : M \to U(N)$ is a harmonic map with finite uniton number from a Riemann surface, then $m(\varphi) \leq \mathrm{rank}(\alpha'_{\varphi})$. From Proposition 2.3, we immediately have the following

Theorem 3.3. *Let $\varphi : M \to U(N)$ be a pluriharmonic map from a compact Kaehler manifold. If φ has finite uniton number, then $m(\varphi) \leq \max_{\tau\in\Gamma} \mathrm{rank}(\alpha'_{\varphi\circ\tau})$.*

Remark 3.4. From [4, 6], we know that $\mathrm{rank}(\alpha'_{\varphi\circ\tau}) \leq N-1$. Hence Theorem 3.3 generalizes Theorem 6.4 in [9].

Corollary 3.4. *Let $\psi : M \to G_k(C^N)$ be a pluriharmonic map of finite uniton number, then $m(\psi) \leq \min\{2k, 2(N-k), N-1\}$.*

Proof. This result follows directly from Lemma 3.1 and Theorem 3.3 (see also [4]).

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