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DUAL ASPECTS OF THE QUASITRIANGULAR BIALGEBRAS AND THE BRAIDED BIALGEBRAS**

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Abstract

It is shown that the dual bialgebra of any quasitriangular bialgebra is braided, and the dual bialgebra of some braided bialgebra is quasitriangular. Also it is proved that every non-degenerate finite dimensional braided (dually, quasitriangular) bialgebra has an antipode.

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§0. Introduction

The quasitriangular Hopf algebras (or "quantum groups" in the strict sense) seem to be developed as a mathematical algebraic tool to solve certain Yang-Baxter equations. Mathematically, the quasitriangular Hopf algebras (generally, bialgebras) H themselves enjoy remarkable properties relating to their bialgebra structures and the distinguished elements. One of those is that the category of all H-modules (i.e., representations of H) forms a braided monoidal category ${}_H\mathrm{Mod}$. A natural question was: what structure on A would induce a braided monoidal structure on ${}^A\mathrm{Comod}$? The answer was provided by Larson-Towber in [3], where they called it the braided bialgebra. Independently, the braided bialgebra (Hopf algebra) was introduced by Majid, he called it the dual quasitriangular bialgebra (Hopf algebra) (see, [6, 7]).

There are numerous dual properties between the quasitriangular bialgebras (Hopf algebras) and the braided bialgebras (Hopf algebras).

In this paper we examine some dual aspects of the quasitriangular bialgebras and the braided bialgebras. In section 1 we recall basic definitions and results used in the sequal. In section 2 we review Sweedler's bialgebra pair (H, H^0). We show that the dual bialgebra of every quasitriangular bialgebra is braided (Theorem 2.1), and the dual bialgebra of braided bialgebra (under a finiteness condition) is quasitriangular (Theorem 2.2). We prove that there exists a Hopf quotient for any braided bialgebra under some conditions. In particular, any finite dimensional non-degenerate braided bialgebra has an antipode (Theorem 2.5), dually, for any finite dimensional non-degenerate quasitriangular bialgebra.

We work over a fixed field K. All maps are K-linear. All modules and comodules are left ones. For $x \in H^*$, $h \in H$ we will write $\langle h, x\rangle$ for $x(h)$.

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§1. Preliminaries

This section contains a review of some background material that will be used freely in the sequel.

We adopt the usual notations for Hopf algebra $(H, m, \eta, \Delta, \epsilon, \mathcal{S})$, or simply H. Here (H, m, η) is a unital algebras, $\Delta : H \to H \otimes H$ the comultiplication, $\epsilon : H \to K$ the counit. These define a bialgebra. By Hopf algebra we mean a bialgebra equipped also with an antipod $\mathcal{S} : H \to H$. For any Hopf algebra H, there exists a dual Hopf algebra H^0, which is the "maximal" Hopf algebra contained in H^*. As a subspace, H^0 consists of all $x \in H^*$ which vanish on a cofinite ideal of H, and the Hopf algebra structure maps m^0, η^0, Δ^0, ϵ^0, and $\mathcal{S}^0$ are induced by Δ, ϵ, m, η and $\mathcal{S}$, respectively (see [10]). The definition works also for H a bialgebra; in this case, H^0 is called the dual bialgebra of H (see [10] for details). A Hopf algebra H is quasitriangular if it possesses an invertible element $R \in H \otimes H$ such that

$$(\Delta \otimes id)(R) = R_{13}R_{23}, \quad (id \otimes \Delta)(R) = R_{13}R_{12}, \tag{1.1}$$

$$\Delta^{op}(h) = R\Delta(h)R^{-1} \quad \text{for all } \ h \in H, \tag{1.2}$$

where

$$R = \sum R^{(1)} \otimes R^{(2)}, \quad R_{12} = \sum R^{(1)} \otimes R^{(2)} \otimes id,$$
$$R_{13} = \sum R^{(1)} \otimes id \otimes R^{(2)}, \quad R_{23} = \sum id \otimes R^{(1)} \otimes R^{(2)},$$

and Δ^{op} is the "twisted" comultiplication of H. We denote a quasitriangular Hopf algebra with the distinguished element R by (H, R) (see [2]). The definition works also for H a bialgebra; in this case, we call (H, R) a quasitriangular bialgebra.

As the dual setting, a braided Hopf algebra is a pair (A, σ), where A is a Hopf algebra and σ is an invertible (with respect to the convolution product) bilinear form on A satisfying

$$\sigma(m \otimes id) = \sigma_{13} * \sigma_{23}, \quad \sigma(id \otimes m) = \sigma_{13} * \sigma_{12}, \tag{1.3}$$

$$\sigma * m = m^{op} * \sigma, \tag{1.4}$$

where

$$\sigma_{12}(x \otimes y \otimes z) = \sigma(x, y)\epsilon(z),$$
$$\sigma_{13}(x \otimes y \otimes z) = \epsilon(y)\sigma(x, z),$$
$$\sigma_{23}(x \otimes y \otimes z) = \epsilon(x)\sigma(y, z),$$

and m^{op} is the "twisted" multiplication of A. Similarly, we have the concept of braided bialgebra (see [1]).

§2. The Quasitriangular and Braided Duality on Dual Bialgebras

Let H be a bialgebra, H^0 the dual bialgebra of H. In this section we discuss the quasi-triangular and braided relations on H and H^0.

We rewrite the "braided" conditions of section 1 in the element relations. Thus (1.3) and (1.4) are equivalent to

$$\sigma(xy, z) = \sum \sigma(x, z_{(1)})\sigma(y, z_{(2)}), \tag{2.1}$$

$$\sigma(x, yz) = \sum \sigma(x_{(1)}, z)\sigma(x_{(2)}, y), \tag{2.2}$$

and

$$\sum \sigma(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} = \sum y_{(1)} x_{(1)} \sigma(x_{(2)}, y_{(2)}), \tag{2.3}$$

respectively, where x, y, z belong to the braided bialgebra A. Note that (2.1) shows that the map $x \mapsto \sigma(x, -)$ is an algebra map between A and A^*, and (2.2) shows that the map $x \mapsto \sigma(-, x)$ is an antialgebra map between A and A^*.

Let (H, R) be a quasitriangular bialgebra. Then R induces a bilinear form σ_R on H^0. For any x, $y \in H^0$, define

$$\sigma_R(x, y) = \sum \langle R^{(1)}, x \rangle \langle R^{(2)}, y \rangle. \tag{2.4}$$

Moreover, the inverse R^{-1} of R induces the inverse (with respect to the convolution product) $\sigma_{R^{-1}}$ of σ_R. Now we have

Theorem 2.1. *Suppose that (H, R) is quasitriangular. Then (H^0, σ_R) is braided.*

Proof. By the definition, we have to show that (H^0, σ_R) satisfies the conditions (2.1)–(2.3).

$$\begin{aligned}
\sigma_R(xy, z) &= \sum \langle R^{(1)}, xy \rangle \langle R^{(2)}, z \rangle \\
&= \sum \langle \Delta R^{(1)} \otimes R^{(2)}, x \otimes y \otimes z \rangle \\
&= \sum \langle R_{13} R_{23}, x \otimes y \otimes z \rangle \\
&= \sum \langle R^{(1)} \otimes R^{(1)'} \otimes R^{(2)} R^{(2)'}, x \otimes y \otimes z \rangle \\
&= \sum \langle R^{(1)}, x \rangle \langle R^{(1)'}, y \rangle \langle R^{(2)} R^{(2)'}, z \rangle \\
&= \sum \langle R^{(1)}, x \rangle \langle R^{(1)'}, y \rangle \langle R^{(2)}, z_{(1)} \rangle \langle R^{(2)'}, z_{(2)} \rangle \\
&= \sum \sigma_R(x, z_{(1)}) \sigma_R(y, z_{(2)}),
\end{aligned}$$

which shows (2.1). Similarly,

$$\sigma_R(x, yz) = \sum \sigma_R(x_{(1)}, z) \sigma_R(x_{(2)}, y)$$

for (2.2) by using the latter of (1.1).

Also, for any $h \in H$,

$$\begin{aligned}
\langle h, \sum \sigma_R(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} \rangle &= \sum \langle R^{(1)}, x_{(1)} \rangle \langle R^{(2)}, y_{(1)} \rangle \langle h_{(1)}, x_{(2)} \rangle \langle h_{(2)}, y_{(2)} \rangle \\
&= \sum \langle R^{(1)} h_{(1)}, x \rangle \langle R^{(2)} h_{(2)}, y \rangle \\
&= \langle \sum R^{(1)} h_{(1)} \otimes R^{(2)} h_{(2)}, x \otimes y \rangle.
\end{aligned}$$

On the other hand,

$$\langle h, \sum y_{(1)} x_{(1)} \sigma_R(x_{(2)}, y_{(2)}) \rangle = \langle \sum h_{(2)} R^{(1)} \otimes h_{(1)} R^{(2)}, x \otimes y \rangle.$$

From $\Delta^{op}(h) R = R \Delta(h)$, we get

$$\sum \sigma_R(x_{(1)}, y_{(1)}) x_{(2)} y_{(2)} = \sum y_{(1)} x_{(1)} \sigma_R(x_{(2)}, y_{(2)}),$$

which is just (2.3).

This concludes our proof.

Dually, we have

Theorem 2.2. *Suppose that (A, σ) is braided with $\sigma \in (A \otimes A)^0$. Then (A^0, σ) is quasitriangular.*

Proof. Similar to the proof of the above theorem.

Remark. The restricted condition $\sigma \in (A \otimes A)^0$ is to guarantee that $\sigma \in A^0 \otimes A^0$. Dropping the restriction for σ, Majid called it essentially quasitriangular, and Larson-Towber discussed this subject in the context of topological modules.

The above results enable us to do some implications from one to another. For example, from the result that cocommutative bialgebra is quasitriangular, we get the result that commutative bialgebra is braided, and from the result that cocommutative braided bialgebra must be commutative, we imply that commutative quasitriangular bialgebra must be cocommutative, etc.

We next discuss the duality of some special elements.

Recall that $G(A)$ is the set of all group-like elements of the bialgebra A.

Proposition 2.1. *Suppose that (A, σ) is braided, $g \in G(A)$. Then both $\sigma(-, g)$ and $\sigma(g, -) \in G(A^0)$. Moreover, if g is invertible, then both $\sigma(-, g)$ and $\sigma(g, -)$ are invertible (with respect to the convolution product), and*

$$\sigma(-, g)^{-1} = \sigma(-, g^{-1}), \quad \sigma(g, -)^{-1} = \sigma(g^{-1}, -).$$

Proof. Note that under the condition of the proposition, both $\sigma(-, g)$ and $\sigma(g, -)$ belong to $Alg(A, K)$, and $Alg(A, K) = G(A^0)$. The remaining can be verified straight.

Let (H, R) be a quasitriangular Hopf algebra. Denote

$$\underline{u} = \sum (\mathcal{S}R^{(2)})R^{(1)}, \quad \text{and} \quad \underline{v} = \sum R^{(1)}(\mathcal{S}R^{(2)}).$$

The quantum Casimir element of H is $\underline{c} = \underline{uv}$. Then the dual elements of $\underline{u}, \underline{v}$ and $\underline{c}$ are $\tau = \sigma(id \otimes \mathcal{S})\Delta^{op}$, $\lambda = \sigma(id \otimes \mathcal{S})\Delta$, and $\tau * \lambda$, respectively, that is, for any $h \in H$

$$\begin{aligned} \tau: \quad & h \mapsto \sum \sigma(h_{(2)}, \mathcal{S}(x_{(1)})), \\ \lambda: \quad & h \mapsto \sum \sigma(h_{(1)}, \mathcal{S}(h_{(2)})), \\ \tau * \lambda: \quad & h \mapsto \sum \sigma(h_{(2)}, \mathcal{S}(h_{(1)}h_{(3)})). \end{aligned}$$

From these, the dual form of $\mathcal{S}^2(h) = \underline{u}h\underline{u}^{-1}$ is

$$\mathcal{S}^2(x) = \sum \tau(x_{(1)})x_{(2)}\tau^{-1}(x_{(3)}).$$

We have derived

Corollary 2.1.[1,Theorem 1.3] *Let (A, σ) be a braided Hopf algebra. Then*

$$\mathcal{S}^2(x) = \sum \tau(x_{(1)})x_{(2)}\tau^{-1}(x_{(3)}).$$

Hence $\mathcal{S}$ is bijective.

Now suppose that (A, σ) is a braided bialgebra. Denote

$$\sigma_l(A) = \{a \in A \mid \sigma(a, b) = 0 \text{ for all } b \in A\}.$$

Then it is easy to check that $\sigma_l(A)$ is a bi-ideal of A. We call $\sigma_l(A)$ the (left) radical of σ. Similarly, we have the right radical $\sigma_r(A)$ of σ. We call a braided bialgebra (A, σ) non-degenerate, if the left radical $\sigma_l(A) = 0$, which is equivalent to the right radical $\sigma_r(A) = 0$.

Theorem 2.3. *Let (A, σ) be a finite dimensional non-degenerate braided bialgebra. Then A has an antipode.*

Proof. We first prove that

$$A^* = \{\sigma(a, -) \mid a \in A\}. \tag{2.5}$$

In fact, for any $a \in A$, $\sigma(a,-) \in A^*$. Let $a_1, a_2, \cdots, a_n$ be a base of A. Then $\sigma(a_1,-)$, $\sigma(a_2,-), \cdots, \sigma(a_n,-)$ are K-linear indepependent. Let $k_i \in K$ such that

$$\sum_{i=1}^{n} k_i \sigma(a_i,-) = \sigma(\sum_{i=1}^{n} k_i a_i,-) = 0.$$

Then the assumption of non-degeneracy implies that $\sum\limits_{i=1}^{n} k_i a_i = 0$, and hence $k_i = 0, i = 1, 2, \cdots, n$. Noting that $\dim A^* = \dim A$, we obtain (2.5).

Next, let σ^{-1} be the inverse (with respect to the convolution product). For any $a \in A$, $\sigma^{-1}(a,-) \in A^*$. By (2.5) and again the assumption of non-degeneracy, there exists a unique element $b \in A$ such that $\sigma(b,-) = \sigma^{-1}(a,-)$. We hence define $\mathcal{S} : A \to A$ by

$$\sigma(\mathcal{S}(a),-) = \sigma^{-1}(a,-). \tag{2.6}$$

We have proved that $\mathcal{S}$ is well-defined.

For any a, $b \in A$, we have

$$\begin{aligned} \sigma(\sum a_{(1)}\mathcal{S}(a_{(2)}), b) &= \sum \sigma(a_{(1)}, b_{(1)})\sigma(\mathcal{S}(a_{(2)}), b_{(2)}) \\ &= \sum \sigma(a_{(1)}, b_{(1)})\sigma^{-1}(a_{(2)}, b_{(2)}) \\ &= (\sigma * \sigma^{-1})(a, b) \\ &= \epsilon_{A\otimes A}(a \otimes b) \\ &= \epsilon(a)\epsilon(b) \\ &= \sigma(\epsilon(a), b). \end{aligned}$$

By non-degeneracy, we get $\sum a_{(1)}\mathcal{S}(a_{(2)}) = \epsilon(a)$. Similarly,

$$\sum \mathcal{S}(a_{(1)})a_{(2)} = \epsilon(a).$$

This proves that $\mathcal{S}$ is the antipode of A.

The above theorem ensures that any finite dimensional non-degenerate braided bialgebra is Hopf algebra. For an arbitrary braided bialgebra (A, σ), if the left radical coincides with the right one, then we say that σ is inducible. In this case, we call $\sigma(A)(= \sigma_l(A))$ the radical of σ.

Denote $\sigma(A,-) = \{\sigma(a,-) \mid a \in A\}$, $\sigma^{-1}(A,-) = \{\sigma^{-1}(a,-) \mid a \in A\}$.

Corollary 2.2. *Let (A, σ) be a braided bialgebra where σ is inducible and $\sigma^{-1}(A,-) \subset \sigma(A,-)$. Then $A/\sigma(A)$ is a Hopf algebra.*

Proof. By the assumption, σ induces a braided structure $\bar{\sigma}$ on $\bar{A} = A/\sigma(A)$ by defining

$$\bar{\sigma}(\bar{a}, \bar{b}) = \sigma(a, b).$$

Moreover, $\bar{\sigma}$ is non-degenerate on $\bar{A}$, and $\bar{\sigma}$ is invertible (with respect to the convolution product on $\bar{A}^* \otimes \bar{A}^*$) with

$$\bar{\sigma}^{-1} = \overline{\sigma^{-1}} : (\bar{a}, \bar{b}) \mapsto \sigma^{-1}(a, b).$$

On the other hand, $\bar{\sigma}^{-1}(\bar{A},-) \subset \bar{\sigma}(\bar{A},-)$ from $\sigma^{-1}(A,-) \subset \sigma(A,-)$. Then, the result follows from the above theorem.

We say that a quasitriangular bialgebra (H, R) is non-degenerate, if the induced braided bialgebra (H^0, σ_R) is non-degenerate. In the finite dimensional case, that (H, R) is non-degenerate is equivalent to $\dim H = \mathrm{rank}(R)$. Here

$$\mathrm{rank}(R) = \min\left\{ n \;\middle|\; R = \sum_{i=1}^{n} h_{1i} \otimes h_{2i} \in H \otimes H \right\} \text{ (see [9])}.$$

Thus, we obtain a dual result for quasitriangular bialgebra.

Theorem 2.4. *Let (H, R) be a finite dimensional quasitriangular bialgebra with* $\mathrm{rank}(R) = \dim H$. *Then H is a Hopf algebra.*

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