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SINGULAR BOUNDARY PROPERTIES OF HARMONIC FUNCTIONS AND FRACTAL ANALYSIS**

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Abstract

This paper shows an important relation between the fractal analysis and the boundary properties of harmonic functions. It is proved that the multifractal analysis of a finite measure μ on $\mathbb{R}^d$ determines the (non-tangential) boundary increasing properties of $P\mu$, the Poisson integral of μ which is harmonic on $\mathbb{R}_+^{d+1}$. Some examples are given.

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§1. Introduction

Consider the half-space $\mathbb{R}_+^{d+1} = \{(x,r); x \in \mathbb{R}^d, r > 0\}$ $(d \geq 1)$. Its boundary is $\partial\mathbb{R}_+^{d+1} = \{(x,0); x \in \mathbb{R}^d\}$ which will be identified with $\mathbb{R}^d$. The Poisson kernel with respect to $\mathbb{R}_+^{d+1}$ is

$$P_y(x,r) = \Gamma \frac{r}{(|x-y|^2 + r^2)^{\frac{d+1}{2}}},$$

where $(x,r) \in \mathbb{R}_+^{d+1}, y \in \mathbb{R}^d = \partial\mathbb{R}_+^{d+1}$, and Γ is a constant depending only on d.

Let μ be a finite positive measure on $\mathbb{R}^d$. It is well known that the Poisson integral of μ

$$P\mu(x,r) = \int P_y(x,r)d\mu(y)$$

is a positive harmonic function on $\mathbb{R}_+^{d+1}$, and that locally (near a finite boundary point) any positive harmonic function differs from a Poisson integral by a harmonic function with local boundary values 0 (see the remark below).

Many general boundary properties of $u = P\mu$ have been studied, especially for $\mu = f(x)dx$ with $f \in L^p(\mathbb{R}^d)$ (see for example [8, 9]). However, when μ is singular with respect to the Lebesgue measure $\mathcal{L}_d$ we know that (apply [1, Corollary 6.7] to the harmonic functions u and 1)

$$\lim_{r\to 0+} u(x,r) = \begin{cases} 0 & \text{for } \mathcal{L}_d - a.e.\ x \in \mathbb{R}^d, \\ +\infty & \text{for } \mu - a.e.\ x \in \mathbb{R}^d. \end{cases}$$

Therefore, we must study the asymptotic behaviour of u as $r \to 0$. We shall first define the degree of u at a boundary point $x \in \mathbb{R}^d$ in a natural way, and then establish the relation between the degree and the local Lipschitz exponent of μ at x. We show that the

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multifractal analysis of μ is exactly the decomposition of the boundary according to the degrees of u. Consequently some singular boundary properties can be deduced from the multifractal analysis.

Remark 1.1. Instead of the Poisson kernel, we can use the Martin kernel $K_y(x,r)$, $(x,r) \in \mathbb{R}_+^{d+1}$, $y \in \mathbb{R}^d \bigcup \{\infty\}$:

$$K_y(x,r) = \begin{cases} (1+|y|)^{\frac{d+1}{2}} \frac{r}{(|x-y|^2+r^2)^{\frac{d+1}{2}}} & \text{if } y \in \mathbb{R}^d, \\ r & \text{if } y = \infty, \end{cases}$$

the Martin boundary of $\mathbb{R}_+^{d=1}$ being $\Delta = \mathbb{R}^d \bigcup \{\infty\}$.

Then by the Martin representation theory or by a Riesz-Herglotz-type representation[3], any positive harmonic function u can be uniquely represented as

$$u = K\mu_u = \int_\Delta K_y d\mu_u(y),$$

where μ_u is a finite measure on Δ.

If we study the local properties of u near a point $(b,0) \in \partial\mathbb{R}_+^{d+1}$, we can suppose that μ_u is supported by $B(b,1)$. Then

$$u = P\mu, \quad \text{with } d\mu(y) = \Gamma^{-1} 1_{B(b,1)}(y)(1+|y|^2)^{\frac{d+1}{2}} d\mu_u(y).$$

Note that $1 \le 1+|y|^2 \le 1+(|b|+1)^2$ $(y \in B(b,1))$. Hence all our results have similar statements when the Martin kernel is applied.

We notice also that similar results still hold when $\mathbb{R}_+^{d+1}$ is replaced by the unit ball of $\mathbb{R}^{d+1}$. In this case the Poisson kernel and the Martin kernel are the same.

We shall suppose without loss of generality in what follows that μ is a probability measure, and $u = P\mu$.

§2. Preliminary

With the above notations, we define the degree of u at x as follows:

$$\deg(u;x) = \inf\{s \ge 0;\ u(x,t) = O(t^{-s}),\ t \downarrow 0\},$$

which characterizes the asymptotic behaviour of u at the point x as r tends to zero.

Remark 2.1. It is easy to check that $u(x,t) = O(t^{-d})$, so that $\deg(u;x) \in [0,d]$. Note also that if $u(x,t) = O(t^{-s})$ and if $s' > s$, then $u(x,t) = O(t^{-s'})$.

The local Lipschitz exponent of μ at x is defined as

$$\mathrm{Lip}(\mu;x) = \sup\{\alpha \ge 0;\ \mu(B(x,r)) = O(r^\alpha),\ r \downarrow 0\},$$

where $B(x,r)$ denotes the open ball of $\mathbb{R}^d$, centred at x and with radius r.

Lemma 2.1. *We have*

$$\deg(u;x) = \begin{cases} \limsup\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r}, & \textit{if the right side is nonnegative;} \\ 0, & \textit{otherwise.} \end{cases}$$

$$\mathrm{Lip}(\mu;x) = \liminf_{r\to 0+} \frac{\log \mu(B(x,r))}{\log r}.$$

Proof. Let $s = \deg(u;x)$. Then by definition for any $\varepsilon > 0$

$$u(x,r) = O(r^{-s-\varepsilon}),$$

which means $\exists\, C > 0$ and $r_0 \in]0,1[$ such that for $0 < r < r_0$

$$u(x,r) \le Cr^{-s-\varepsilon}.$$

It follows that

$$\frac{\log u(x,r)}{-\log r} \le \frac{\log C}{-\log r} + s + \varepsilon$$

and then

$$\limsup_{r\to 0+} \frac{\log u(x,r)}{-\log r} \le s + \varepsilon$$

so that $\limsup\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r} \le s$ since ε is arbitrary.

Conversely, let $s = \limsup\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r}$. Suppose first $s \ge 0$. Then

$$\frac{\log u(x,r)}{-\log r} < s + \varepsilon \ \text{ if } \ r \ \text{ is small enough,}$$

which implies $u(x,r) \le r^{-s-\varepsilon}$, then $\deg(u,x) \le s$.

Now if $s < 0$, we may take $\varepsilon < |s|$. We get in the same way

$$u(x,r) \le r^{-s-\varepsilon} \le 1,$$

which gives $\deg(u;x) = 0$. Thus the fist assertion of the lemma is proved.

The second assertion can be proved in the same way.

Remark 2.2. By the Harnack Principle (see Lemma 2.4 below) the above limit for u can be replaced by the non-tangential limit.

Remark 2.3. Since μ is supported by $\boldsymbol{R}^d$, the dimension of μ (see [5] and [6]) $\dim \mu \le d$. Thus $\mathrm{Lip}(\mu;x) \le d$ for $\mu - a.e.\ x \in \boldsymbol{R}^d$ (see [6]).

Lemma 2.2. $\forall x \in \boldsymbol{R}^d,\ r > 0$

$$u(x,r) \ge Cr^{-d}\mu(B(x,r)),$$

where C is a positive constant depending only on d. In particular, we have

$$u(x,r) = O(r^{-s}) \Longrightarrow \mu(B(x,r)) = O(r^{d-s}).$$

Proof. Note that for $y \in B(x,r)$, $P_y(x,r) \ge Cr^{-d}$, hence

$$u(x,r) = \int_{R^d} P_y(x,r)d\mu(y) \ge \int_{B(x,r)} P_y(x,r)d\mu(y) \ge Cr^{-d}\mu(B(x,r)).$$

It follows from Lemma 2.2 that

$$\frac{\log u(x,r)}{-\log r} \ge \frac{C}{-\log r} + d - \frac{\log \mu(B(x,r))}{\log r}, \quad 0 < r < 1,$$

which yields the following corollary.

Corollary 2.1.

$$\liminf_{r\to 0+} \frac{\log u(x,r)}{-\log r} \ge d - \limsup_{r\to 0+} \frac{\log \mu(B(x,r))}{\log r},$$

$$\limsup_{r\to 0+} \frac{\log u(x,r)}{-\log r} \ge d - \liminf_{r\to 0+} \frac{\log \mu(B(x,r))}{\log r}.$$

Now we show that the converse of Lemma 2.2 also holds.

Lemma 2.3. *Let $0 \le s \le d,\ x \in \boldsymbol{R}^d$. Then*

$$\mu(B(x,r)) = O(r^s) \Longrightarrow u(x,r) = O(r^{s-d}).$$

Proof. By hypothesis, there exist $C > 0$ and $r_0 > 0$ such that

$$\mu(B(x,r)) \leq Cr^s,\ 0 < r \leq 2r_0. \tag{2.1}$$

Note that

$$u(x,r) = \int_{R^d \setminus B(x,r_0)} P_y(x,r)d\mu(y) + \int_{B(x,r_0)} P_y(x,r)d\mu(y)$$

and that the first integral tends to 0 as $r \to 0$. It is sufficient to show

$$\int_{B(x,r_0)} P_y(x,r)d\mu(y) = O(r^{s-d}). \tag{2.2}$$

Let $0 < r < r_0$, and let n be the integer such that

$$2^{n-1}r < r_0 \leq 2^n r.$$

Denote $I_0 = B(x,r)$ and

$$I_j = B(x, 2^j r) \setminus B(x, 2^{j-1}r) \ \text{ for } 1 \leq j \leq n.$$

Then we have

$$\int_{B(x,r_0)} P_y(x,r)d\mu(y) \leq \int_{B(x,2^n r)} P_y(x,r)d\mu(y) = \sum_{j=0}^{n} \int_{I_j} P_y(x,r)d\mu(y). \tag{2.3}$$

If $y \in I_j (j \geq 1)$, then $|x-y| \geq 2^{j-1}r$ and $P_y(x,r) \leq C_j r^{-d}$, where $C_j = \frac{\Gamma}{(2^{d+1})^{j-1}}$.

If $y \in I_0$, then $P_y(x,r) \leq C_0 r^{-d}$, with $C_0 = \Gamma$. Hence by (2.1) and (2.3) we have successively

$$\begin{aligned}\int_{B(x,r_0)} P_y(x,r)d\mu(y) &\leq \sum_{j=0}^{n} C_j r^{-d}\mu(I_j) \leq \sum_{j=0}^{n} C_j r^{-d}\mu(B(x,2^j r)) \\ &\leq \sum_{j=0}^{n} CC_j r^{-d} 2^{js} r^s = r^{s-d}\sum_{j=0}^{n} CC_j 2^{js} \leq \widehat{C} r^{s-d},\end{aligned}$$

where $\widehat{C} = C\Gamma + C\Gamma 2^{d+1} \sum_{j=0}^{\infty} \frac{1}{(2^{d+1-s})^j}$ is a finite positive constant depending only on C, s and d. This establishes (2.2) which proves the lemma.

Remark 2.4. Similar result as Lemma 2.3 holds for more general case[10] .

The following Harnack Principle (or Harnack Inequality) will be useful (see [10, p.16]).

Lemma 2.4. *Let $x_0 \in \mathbb{R}^{d+1}$ and $r > 0$. Suppose that u is a positive harmonic function defined on $B(x_0, r)$. Then $\forall x,\ y \in B(x_0, r/2)$, $u(x)/u(y) \leq 3^{d+1}$.*

§3. Main Results

We can now prove the following

Theorem 3.1. *Let $u = P\mu$ be a positive harmonic function on $\mathbb{R}^{d+1}_+$, and $x \in \mathbb{R}^d$. Then we have*

$$\deg(u;x) = \begin{cases} d - \mathrm{Lip}(\mu;x), & \textit{if}\ \mathrm{Lip}(\mu;x) \leq d; \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. We have first by Lemma 2.1 and Corollary 2.1 that

$$\deg(u;x) \geq d - \mathrm{Lip}(\mu;x).$$

Let us prove the opposite inequality:

$$\deg(u;x) \le d - \mathrm{Lip}(\mu;x).$$

Let $s = \mathrm{Lip}(\mu;x)$. If $s = 0$, this inequality is trivial by Remark 2.1. If $s > d$, then it follows that

$$\mu(B(x,r)) = O(r^d)\ (r \downarrow 0),$$

which implies by Lemma 2.3 $u(x,r) = O(1)$, hence $\deg(u;x) = 0$.

Now let $0 < s \le d$. Then for any $\varepsilon > 0, \varepsilon < s$,

$$\mu(B(x,r)) = O(r^{s-\varepsilon}).$$

Then Lemma 2.3 gives

$$u(x,r) = O(r^{-(d-s+\varepsilon)})$$

so that

$$\deg(u;x) \le d - s + \varepsilon$$

and then by the arbitrary of ε, $\deg(u;x) \le d - s$. The theorem is thus proved.

Corollary 3.1. *If the limit* $\lim\limits_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r}$ *exists and* $\le d$, *then the limit* $\lim\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r}$ *exists and*

$$\lim_{r\to 0+} \frac{\log u(x,r)}{-\log r} = d - \lim_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r}.$$

Proof. By Corollary 2.1 and Theorem 3.1 we have

$$d - \lim_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r} \le \liminf_{r\to 0+} \frac{\log u(x,r)}{-\log r} \le \limsup_{r\to 0+} \frac{\log u(x,r)}{-\log r}$$
$$= d - \lim_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r}.$$

Let μ be a probability measure on $\mathbb{R}^d$. Consider its multifractal decomposition (see [4] and [2]):

$$E^\mu_\alpha = \{x \in \mathbb{R}^d;\ \mathrm{Lip}(\mu;x) = \alpha\}\quad (0 \le \alpha \le d),\quad \text{with}\ \ \dim E^\mu_\alpha = f(\alpha).$$

If we write $D^u_s = \{x \in \mathbb{R}^d;\ \deg(u;x) = s\}$ $(0 \le s \le d)$, then we have immediately

Theorem 3.2. *With the above notations, we have*

$$D^\mu_s = E^\mu_{d-s},\ \textit{and then}\ \dim D^u_s = f(d-s).$$

In particular if μ is unidimensional (*see* [5] *or* [6]) *with* $\dim\mu = \alpha,\ 0 \le \alpha \le d$, *then the corresponding harmonic function u satisfies*

$$\deg(u,x) = d - \alpha\ \ for\ \mu - a.e.\ x \in \mathbb{R}^d.$$

Proof. In fact, by [6] μ is unidimensional if and only if $\mathrm{Lip}(\mu;\cdot) = \alpha\ \ \mu - a.e.$.

Remark 3.1. For the multifractal analysis, see [7] and [4]. In many "self-similar" cases the function $f(\alpha) = \dim E^\mu_\alpha$ can be calculated by means of the Legendre transform (see for example [2]). Then $\dim\{\deg(u;\cdot) = s\}$ is determined by Theorem 3.2. As an application of Theorem 3.2, we give the following example.

Example 3.1. By Remark 2.1 we know that $0 \le \deg(u;x) \le d$. It is not difficult to show that for any $s \in [0,d]$ there exists harmonic function u such that $\deg(u;x) = s$ for some $x \in \mathbb{R}^d$. Futhermore one can pose the following question: Is there a harmonic function u such that the set $\{s;\ \deg(u;x) = s$ for some $x \in \mathbb{R}^d\}$ is the whole interval $[0,d]$? Using the multifractal analysis and Theorem 3.2, we can give a positive answer.

From the Moran fractals[2] we can construct without difficulty a series of measures $\mu_n \in M^1([2n,2n+1]^d)$ $(n>0)$ (we omit the details of the construction) such that

$$\dim E_\alpha^{\mu_n} > 0,\ \forall \alpha \in \Big[\frac{1}{n}, d-\frac{1}{n}\Big].$$

Let $u = P_0 + \sum\limits_{n=1}^{\infty} 2^{-n} P\mu_n$. It is obvious that u is a harmonic function. We claim that the set $\{s;\ \deg(u;x) = s$ for some $x \in \mathbb{R}^d\}$ is the whole interval $[0,d]$. Moreover $\dim\{\deg(u,\cdot) = s\} > 0$ for any $s \in [0,d[$. In fact, we observe that $\deg(u;0) = d$ and that for $x \in [2n,2n+1]^d$, $\deg(u;x) = \deg(P\mu_n;x)$. It suffices then to apply Theorem 3.1.

Example 3.2. Recall that the classical Cantor's set $K \subset \mathbb{R}^1$ is defined as follows: $K = \bigcap\limits_{j=0}^{\infty} E_j$, where $E_0 = [0,1], \cdots, E_{j+1} = \frac{1}{3}E_j \bigcup \Big(\frac{1}{3}E_j + \frac{2}{3}\Big)$ $(j \ge 0)$. It is clear that $E_{j+1} \subset E_j$ and that E_j consists of 2^j intervals $I_j(k) = [a_j(k), b_j(k)]$ $(k = 0, \cdots, 2^j-1)$ of length 3^{-j}, with

$$a_j(0) = 0, \cdots, a_j(k+1) > 3^{-j} + a_j(k). \tag{3.1}$$

For each $j \ge 0$, define the probability measure μ_j on $[0,1]$ as the uniform distribution on E_j. Then in $M^1[0,1]$ (the spaces of probability measures on $[0,1]$ endowed with the weak topology), $\{\mu_j\}$ has an adherent measure $\mu \in M^1[0,1]$. It follows easily from the construction that μ is supported by K and $\mu(I_j(k)) = 2^{-j},\ j \ge 0,\ k = 0, \cdots, 2^j-1$.

Consider the corresponding harmonic function $u = P\mu$. By Theorem 3.1 the boundary properties of u can be deduced from the fractal properties of μ. We can also get information about μ by the boundary behaviour of u.

Let $s = \frac{\log 2}{\log 3}$. We claim that $u(x,r) \sim r^{s-1},\ x \in K,\ r < \frac{1}{3}$. That is, there exist two absolute constants $C_1,\ C_2 > 0$ such that

$$C_1 r^{s-1} \le u(x,r) \le C_2 r^{s-1},\ x \in K,\ r < \frac{1}{3}. \tag{3.2}$$

The first of the above inequalities is easy to prove. Let us prove the second.

Let $x \in K$ and let N be the integer such that

$$3^{-N-1} < r \le 3^{-N}. \tag{3.3}$$

Then $x \in I_N(K)$ for an integer $K \in \{0, \cdots, 2^N-1\}$. Since $|x - a_N(K)| \le 3r$, we deduce from the Harnack's inequality (Lemma 2.4) that

$$u(x,r) \sim u(a_N(K), r).$$

For convenience, we suppose $K = 0$, that is, $a_N(K) = 0$, the demonstration for $K \in \{1, \cdots, 2^N-1\}$ is similar. Therefore, we have only to establish (3.2) for $x = 0$.

In fact, using the fact that $a_N(k) \ge k3^{-N} \ge kr$ (which follows from (3.1) and (3.3)) and that

$$\sup_{y \in I_N(k)} P_y(0,r) \le \Gamma \frac{r}{r^2 + (a_N(k))^2},$$

we have successively

$$u(0,r) = \sum_{k=0}^{2^N-1} \int\limits_{I_N(k)} P_y(0,r)d\mu(y) \le \Gamma 2^{-N} r \sum_{k=0}^{2^N-1} \frac{1}{r^2(1+k^2)} \le \frac{2^{-N}}{r}\widehat{C}, \tag{3.4}$$

where $\widehat{C} = \sum\limits_{k=0}^{\infty} \frac{\Gamma}{1+k^2}$.

Finally noting that $3^s = 2$, we have then by (3.3) $2^{-N} \le 3^s r^s$. This combined with (3.4) gives $u(0,r) \le C' r^{s-1}$, $C' = 3^s \widehat{C}$, which proves our assertion.

An immediate consequence is the following

$$\deg(u;x) = 1-s, \text{ then } \operatorname{Lip}(\mu;x) = s \ (\forall x \in K).$$

Remark 3.2. Using Corollary 4.1 below, we can also get $\lim\limits_{r\to o+} \frac{\log\mu(B(x,r))}{\log r} = s,\ x \in K$.

§4. Further Discussions

It follows from the Harnack's inequality that

$$u(x,r) \sim u(x,r'), \quad \forall x \in \mathbb{R}^d,\ r>0,\ r \le r' \le 2r.$$

From this fact we deduce that the increasing properties of $u(x,r)$ as $r \downarrow 0$ is characterized by the sequence $\{u(x,2^{-n})\}_{n\ge 1}$.

In particular we have

Proposition 4.1. *The limit* $\lim\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r}$ *exists if and only if the limit* $\lim\limits_{n\to\infty} \frac{\log u(x,2^{-n})}{-\log 2^{-n}}$ *exists.*

Note that in general there is no such property for $\lim \frac{\log\mu(B(x,r))}{logr}$ unless μ satisfies some "homogeneous" properties. Then we can show the following further result.

Proposition 4.2. *Let $x \in \mathbb{R}^d$. Suppose that for some positive constant $C < 2^{d+1}$ and $r_0 > 0$,*

$$\mu(B(x,2r)) \le C\mu(B(x,r)),\ \forall r \le r_0. \tag{4.1}$$

Then we have

$$u(x,r) \sim r^{-d}\mu(B(x,r))\ (r \downarrow 0).$$

Proof. We have to show $u(x,r) \le \text{Const.} r^{-d}\mu(B(x,r))$ for r near 0. In fact, we can suppose that μ is supported by $B(x,r_0/2)$. Then for r near 0 and $2^n r \le r_0$, $\mu(B(x,2^n r)) \le C^n \mu(B(x,r))$. We can then proceed as in the proof of Lemma 2.3, where we notice that $\sum\limits_{n=0}^{\infty} \left(\frac{C}{2^{d+1}}\right)^n$ is convergent.

An immediate consequence of Proposition 4.2 is the following supplement of Corollary 3.1.

Corollary 4.1. *If μ satisfies* (4.1)*, then*

$$\liminf_{r\to 0+} \frac{\log u(x,r)}{-\log r} = d - \limsup_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r}.$$

In particular, if $\lim\limits_{r\to 0+} \frac{\log u(x,r)}{-\log r} = \alpha$ *exists, then* $\lim\limits_{r\to 0+} \frac{\log\mu(B(x,r))}{\log r}$ *exists and is equal to* $d-\alpha$.

Remark 4.1. If μ has positive finite density $\lim\limits_{r\to 0+} \frac{\mu(B(x,r))}{r^s}$ $(0 \le s \le d)$, then the condition (4.1) is satisfied with $C = 2^{s+\varepsilon}$.

Now let us estimate $u(x,2^{-n})$ in terms of μ. For simplifying notations we consider the case where $d=1$, $x=0$ and suppose that μ is supported by the interval $[0,1]$.

For $j\geq 0$, let $B_j=B(0,2^{-j})$, and $I_j=B_j\setminus B_{j+1}$, $\mu_j=\mu(I_j)$, $v_j=\mu(B_j)$. Obviously

$$v_n=\sum_{j\geq n}\mu_j+\mu(\{0\}).$$

Then we have

Proposition 4.3. With the above notations, we have

$$2^{-n}u(0,2^{-n})\sim v_n+4^{-n}\sum_{j=0}^{n-1}4^j\mu_j$$

as n tends to $+\infty$.

Proof. For $y\in I_j$, $j<n$, we have by a simple calculation that

$$2^{-n}P_y(0,2^{-n})\sim 4^{j-n},$$

which gives

$$\begin{aligned}2^{-n}u(0,2^{-n})&=\int_{B_n}2^{-n}P_y(0,2^{-n})d\mu(y)+\sum_{j=0}^{n-1}\int_{I_j}2^{-n}P_y(0,2^{-n})d\mu(y)\\&\sim v_n+4^{-n}\sum_{j=0}^{n-1}4^j\mu_j.\end{aligned}$$

Similarly, we can obtain the following proposition, where $\mu\in M^1(\mathbb{R}^d)$, $u=P\mu$, and $B_j=B(x,2^jr)$, $J_j=B_{j+1}\setminus B_j$ $(j\geq 0)$.

Proposition 4.4. $\forall x\in\mathbb{R}^d$, $r>0$,

$$r^du(x,r)\sim\mu(B(x,r))+\sum_{j=0}^{\infty}(2^{-d-1})^j\mu(J_j).$$

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