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CONSTRUCTION OF SOLUTIONS TO M-D RIEMANN PROBLEMS FOR A 2×2 QUASILINEAR HYPERBOLIC SYSTEM**

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Abstract

The author studies M-D Riemann problems for a quasilinear nonstrictly hyperbolic system. The initial data are taken as three different constants in three sections divided by three rays starting from the origin. From each direction of these rays two waves coming from infinity are allowed. All possible local singularity structures are carefully studied and classified. Then based on such analysis, existence and global singularity structure of the solution are obtained under some assumptions.

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§1. Introduction

Recently, the study of nonlinear hyperbolic system of conservation laws in multidimensional case has been extensively developed (for instance see [1-4]). Correspondingly, the study on multidimensional Riemann problems is then more interesting and attractive than before (see [5] and its references). The M-D Riemann problem was first put forward by R.Courant and K.O.Friedrichs in [6]. In the scalar equation case this problem has been discussed by D. Wagner[7], W. B. Lindquist[8], T. Zhang and L. Xiao[9].

In the study of M-D Riemann problems for hyperbolic systems the following system

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} ug_1(u,v) \\ vg_1(u,v \end{pmatrix}_x + \begin{pmatrix} ug_2(u,v \\ vg_2(u,v) \end{pmatrix}_y = 0 \tag{1.1}$$

attracts people's special attention. Such a system arises in oil recovery, elastic theory and magneto-hydrodynamics (see [10]). Generally, it is a nonstrictly hyperbolic system. The M-D Riemann problem of the system is much more complicated than the scalar case. Meanwhile, it is simpler than the problem for general hyperbolic system, because in the study of self-similar solutions to the system the appearance of domain of mixed type can be avoided. That is why we are interested in the M-D Riemann problem for (1.1) and regard such a study as a necessary step to more complicated and practical cases.

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When $g_1 = u, g_2 = v$, the system becomes

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} u^2 \\ uv \end{pmatrix}_x + \begin{pmatrix} uv \\ v^2 \end{pmatrix}_y = 0. \tag{1.2}$$

Some first attack to its M-D Riemann problems was given in [11, 12]. In this paper we will give a detailed discussion on the singularity structure of its solutions and a way to construct the solution of M-D Riemann problems. The initial data of our problem are taken as

$$U|_{t=0} = (u_0(x, y), v_0(x, y)) \tag{1.3}$$

which are composed of different constants (u_i, v_i) in three angular domains

$$\Omega_i : \quad \theta_{i-1} < \theta < \theta_i, \tag{1.4}$$

where $\theta_i = -\frac{\pi}{2} + \frac{2}{3} i\pi$. We notice that the lines carrying discontinuity of the initial data form a capital letter "Y", hence the problem is also called Y-shape Riemann problem. With such initial data the problem will have a complicated flowery singularity structure. Since (1.2) and (1.3) are invariant under dilation $t \to \alpha t, x \to \alpha x, y \to \alpha y$ with $\alpha > 0$, we may only consider the self-similar solutions of (1.2). The singularity structure of the solution on the plane $\{t = 1\}$ represents the structure on the whole space R^3, and then it is enough to give the picture of such a structure on $\{t = 1\}$.

We will prove the existence of the solution to Riemann problem by constructive method. By introducing the variables $\xi = x/t, \eta = y/t$, and regarding (u, v) as functions of $\xi = x/t, \eta = y/t$, we can deduce our problem (1.2),(1.3) to

$$\begin{pmatrix} 2u - \xi & 0 \\ v & u - \xi \end{pmatrix} U_\xi + \begin{pmatrix} v - \eta & u \\ 0 & 2v - \eta \end{pmatrix} U_\eta = 0, \tag{1.5}$$

$$\lim_{r \to \infty} U(r, \theta) = U_i \text{ if } \theta_{i-1} < \theta < \theta_i. \tag{1.6}$$

Then the main task is to give a global wave graph on the whole (ξ, η) plane. By the property of finite propagation speed for hyperbolic system the wave for sufficiently large $|(\xi, \eta)|$ can be determined as a 1-D problem. Then according to the analysis in §2, generally there are two waves come from infinity in each direction $\theta = \theta_i$. Therefore the problem becomes to match all these waves coming from infinity. In order to do that we start from the classification of nodes, which is formed by intersection of elementary waves. For Euler system J.Glimm and others have shown the close relation between the classification of nodes and solution to M-D Riemann problems in [13], which is helpful to our study. For nontrivial nodes for the solutions to (1.5), which is defined in §3, our conclusion on its classification is

Theorem 1.1. *Any non-trivial node can be classified as the following four types:* (a) SSS, (b) $JSJS$, (c) $JJSJ$ *or* JJS, (d) JJR.

The theorem will be proved in section 3, while the picture of all types of nodes have been shown in Fig. 1. Based on this classification we get a clear understanding on interaction of different elementary waves. There are five basic cases for interaction of elementary waves, i.e. $R \otimes S$, $R \otimes J$, $S \otimes S, S \otimes J$ and $J \otimes J$. In these cases all local singularity structures of solutions are described. We will discuss them in §4, and will also indicate two kinds of non-existence. Combining these local structures we can construct solutions to (1.5),(1.6) on the whole (ξ, η) plane. The solutions are composed of elementary waves and constant states. Hence we have

Theorem 1.2. *Under the assumptions* (A_1), (A_2), *the solution to* (1.5), (1.6) *does exist, and it is composed of elementary waves and constant states. Moreover, the solution can be actually constructed.*

The assumptions (A_1), (A_2) will be formulated in §5. Briefly speaking, these conditions are set to avoid the non-existence described in §4. Moreover, the method of constructing the solution will also be given in §5. In the end of this paper we give an example by Figure 2, which shows the complete singularity structure of solutions of (1.5), (1.6) (or (1.2), (1.3)) with three contacts, two simple waves and one shock coming from infinity. The singularity structure in other cases can also be obtained similarly.

§2. Basic Facts

For reader's convenience let us briefly recall some basic facts on the system (1.2) first. Denoting $U = {}^t(u, v)$, we can write the system (1.2) as

$$U_t + \begin{pmatrix} 2u & 0 \\ v & u \end{pmatrix} U_x + \begin{pmatrix} v & u \\ 0 & 2v \end{pmatrix} U_y = 0. \tag{2.1}$$

For given μ, ν the eigenvalues of its characteristic matrix are $\lambda_1 = \mu u + \nu v, \lambda_2 = 2(\mu u + \nu v)$. The system (2.1) is strictly hyperbolic, if and only if $\mu u + \nu v \neq 0$. In this paper we always require $(u, v) \neq (0, 0)$ in order to avoid strong degeneracy.

The Rankine-Hugoniot condition for (1.2) is

$$\phi_t[U] + \phi_x[uU] + \phi_y[vU] = 0 \quad \text{on } \phi(t, x, y) = 0, \tag{2.2}$$

where $\phi(t, x, y)$ is the equation of the surface bearing the discontinuity of the solution, and $[\ ]$ represents the jump of corresponding functions. (2.2) implies

$$\phi_t + u^{\pm}\phi_x + v^{\pm}\phi_y = 0, \text{ and } [u]/[v] = -\phi_y/\phi_x \tag{2.3}$$

which represents a contact J, or

$$\phi_t + (u^+ + u^-)\phi_x + (v^+ + v^-)\phi_y = 0 \text{ and } u^+/v^+ = u^-/v^- \tag{2.4}$$

which represents a shock S. Besides, on shock S the entropy condition should be assigned. Denote the normal direction of S as $(-\sigma, \mu, \nu)$ with $\mu^2 + \nu^2 = 1$, and let it point from (u^-, v^-) to (u^+, v^+). Then the entropy condition is

$$\lambda_1(u^-, v^-; \mu, \nu) < \sigma, \lambda_2(u^+, v^+; \mu, \nu) < \sigma < \lambda_2(u^-, v^-; \mu, \nu) \quad \text{if} \quad \lambda_1 < \lambda_2,$$

$$\sigma < \lambda_1(u^+, v^+, \mu, \nu), \lambda_2(u^-, v^-; \mu, \nu) < \sigma < \lambda_2(u^+, v^+; \mu, \nu) \quad \text{if} \quad \lambda_1 > \lambda_2.$$

By the property of finite propagation speed for hyperbolic system the solution to (1.2), (1.3) for sufficiently large $(x/t, y/t)$ can be determined as a one dimensional problem. Its explicit expression is given as follows.

Noticing that the system (1.2) is invariant under the rotation around t-axis, we may assume that the initial data have the form

$$(u, v)|_{t=0} = \begin{cases} (u^+, v^+), & x > 0, \\ (u^-, v^-), & x < 0. \end{cases} \tag{2.5}$$

For $u^+ > u^- > 0$, the solution is

$$(u,v)=\begin{cases}(u^-,v^-), & x<u^-t,\\ (u^-,v^+u^-/u^+), & u^-t<x<2u^-t,\\ (x/2t,xv^+/(2tu^+), & 2u^-t<x<2u^+t,\\ (u^+,v^+), & x>2u^+t.\end{cases} \tag{2.6}$$

Generally, a simple wave and a contact appear in this solution, while the simple wave disappears if $u^+=u^-$, and the contact disappears if $u^+/v^+=u^-/v^-$.

For $u^- > u^+ > 0$, the solution is

$$(u,v)=\begin{cases}(u^-,v^-), & x<u^-t,\\ (u^-,v^+u^+/u^-), & u^-t<x<(u^++u^-)t,\\ (u^+,v^+), & x>(u^++u^-)t.\end{cases} \tag{2.7}$$

Generally, a shock and a contact appear in this solution, while the shock disappreas if $u^+=u^-$, and the contact disappears if $u^+/v^+=u^-/v^-$.

The cases $0>u^->u^+$ and $0>u^+>u^-$ are similar.

Remark 2.1. Here we emphasize that in the case $u^+u^-<0$ the solution to the Cauchy problem of system (1.2), (2.5) does not exist. For instance, take

$$(u,v)|_{t=0}=\begin{cases}(-1,1), & x>0,\\ (1,1), & x<0.\end{cases} \tag{2.8}$$

The first equation of (1.2) can be solved independently, its solution is simply

$$u=1 \text{ if } x<0, \quad u=-1 \text{ if } x>0.$$

Substituting it into the second equation of (1.2) we obtain $v=1$ in both regions $x<0$ and $x>0$. But this contradicts the R-H condition $\sigma=[uv]$. The contradiction implies the non-existence in the class of piecewise functions.

Now let us turn to the system (1.5). By direct computation we know that (1.5) is also hyperbolic on the plane (ξ,η), and it is strictly hyperbolic if and only if $v\xi-u\eta\neq 0$. Two eigenvalues for (1.5) are $\tilde\lambda_1=(v-\eta)/(u-\xi), \tilde\lambda_2=(2v-\eta)/(2u-\xi)$. Correspondingly, the first family of characteristics is linearly degenerate, and the second one is genuine nonlinear if $v\eta-u\xi\neq 0$.

Consider the solution of (1.5) on the plane (ξ,η). If $\eta=\eta(\xi)$ is a line carrying jump of the solution U of (1.5), then the corresponding surface in the space (t,x,y) is $y-t\eta(x/t)=0$. In that case the slope of $\eta=\eta(\xi)$ satisfies

$$\frac{d\eta}{d\xi}=\frac{\eta-v^+}{\xi-u^+}\Big(=\frac{\eta-v^-}{\xi-u^-}\Big) \qquad \text{(contact)} \tag{2.9}$$

or

$$\frac{d\eta}{d\xi}=\frac{\eta-(v^++v^-)}{\xi-(u^++u^-)} \text{ and } \frac{v^+}{u^+}=\frac{v^-}{u^-} \qquad \text{(shock)}\ . \tag{2.10}$$

(2.9) indicates that a contact J connecting the constant states (u_1,v_1) and (u_2,v_2) is the straight line passing $P_1(u_1,v_1)$ and $P_2(u_2,v_2)$. And for a curved contact, its tangent line is determined by the states on both sides at every point according to (2.9).

(2.10) indicates that a shock S connecting the states (u_1,v_1) and (u_2,v_2) must be a line passing the point (u_1+u_2,v_1+v_2), and for a curved shock S, its tangent line is determined by the states on both sides at every point according to (2.10). On the other hand, the points

$P_1(u_1, v_1)$, $P_2(u_2, v_2)$ and $P_{1+2}(u_1 + u_2, v_1 + v_2)$ must lie on a line h passing through the origin O.

The simple waves R in the plane (ξ, η) are composed of characteristic lines of second class. Since the system (1.5) implies

$$\frac{\partial}{\partial \xi}(\frac{v}{u}) + \frac{v-\eta}{u-\xi}\frac{\partial}{\partial \eta}\Big(\frac{v}{u}\Big) = 0, \tag{2.11}$$

we know that v/u is constant on the first characteristic line, and it is then a constant on the whole region occupied by the simple wave. This means that if the state (u_1, v_1) is connected with the state (u_2, v_2) by a simple wave, then $v_1/u_1 = v_2/u_2 = k$, and $v/u \equiv k$ on the region occupied by this simple wave. For any simple wave we call the characteristics adjacent to constant region the front of simple wave. For any point (ξ, η) in the region of the simple wave, the second characteristic starting from this point stops at $(2u, 2v)$ on the line $h : \eta = k\xi$.

Remark 2.2. We also require that all second characteristics in a simple wave do not intersect before they meet the line h. Therefore, in the whole region occupied by this simple wave all points (u, v) must be located on h; moreover, these points are located in between the origin and the centre wave, because $(u, v) = (0, 0)$ is not allowed.

All characteristics are orientated by the direction from any point on them to the point (u, v) (resp. $(2u, 2v)$) for the first class (resp. for the second class). On any shock the entropy condition means that among four characteristics on both sides of the shock, three characteristics point into the shock, and only one points away from the shock. Such a condition is simply called "three incoming and one outgoing". For any shock two second characteristics on its both sides point into the shock, and the direction of the shock is taken as from the point on it to the point $(u^- + u^+, v^- + v^+)$, where $(u^\pm, v^\pm)$ are the states on the both sides of the shock. For any contact, if at a point on it the directions of characteristics on its both sides coincide, then the direction is also taken as the direction of the contact at this point.

Remark 2.3. The entropy condition also requires that the points (u_1, v_1),(u_2, v_2) on the both sides of a shock lie on the same ray starting from the origin O. Meanwhile, these two points are located in between the origin and the shock.

Remark 2.4. The solutions (2.6), (2.7) of 1-D problem have corresponding expressions on (ξ, η) plane. Here we indicate that on (ξ, η) plane the shock, the contact and all second characteristics in the simple wave are parallel to the line carrying the discontinuity of the initial data. Comparing the place of simple wave R or shock S, the contact J is always located nearer to the origin. Besides, the non-existence showed in Remark 2.1 also has its corresponding version for system (1.5). It means that as a 1-D problem on (ξ, η) plane, if two states U^+, U^- are located on a line m passing through the origin, but they are on different sides to the origin, then the solution does not exist.

§3. Local Singularity Structure

Instead of solving the problem (1.2),(1.3), we solve (1.5) with the condition (1.6). The solution for sufficiently large $|(\xi, \eta)|$ can be determined as a one-dimensional problem, and it has been given in section 2 apart from a rotation of the coordinate system. Therefore the problem becomes to match all waves coming from infinity. To do this we first give a classification of local singularity structure at a node, which is the intersection of wave fronts. Then we consider the interaction of waves from local viewpoint. Finally, we give a way to establish a global wave graph on (ξ, η) plane.

We classify the local singularity structure by its flattened version. A point Q on the (ξ, η) plane is called a non-trivial node of solution U (or simply called a node), if U is discontinuous at Q, and the following hypotheses are satisfied.

(H_1) U is piecewise smooth, it has the form $U = U_j$ for $\varphi_{j-1} < \theta < \varphi_j,\ j = 1, \cdots, n$, where φ_j is the angle between a ray issuing from Q and the horizontal line, $\varphi_n = \varphi_0 + 2\pi$, and each U_j is constant or a centre simple wave with centre Q.

The region, where U is constant, is called constant region. In the simple wave case, the lines $\theta = \varphi(j-1)$ and $\theta = \varphi_j$ are the fronts of the simple wave U.

(H_2) U satisfies R-H conditions on the line, where U has jump. Moreover, U satisfies entropy condition on any shock.

(H_3) The node is non-degenerate. It means $v\xi - u\eta \neq 0$ in a neighbourhood of the node Q; particularly, Q is different from O.

(H_4) U is generic. It means that

1) any centre waves must emanate from a node;

2) at most two waves point to the node; and if there appear two incoming waves, then they must be adjacent;

3) if two contact waves bound a constant region, then at least one of them points to the node.

These hypotheses are set to avoid non-uniqueness. Indeed, there are examples to show the non-uniqueness even for local singularity structure. The classification of such nontrivial nodes are given in Theorem 1.1. Its proof can be deduced through a series of lemmas as follows.

Lemma 3.1. *Any singular line starting from a node cannot be both shock and contact; meanwhile, the front of any simple wave cannot be a shock or a contact.*

The fact is almost obvious, because under the hypothesis H_3 the shock, the contact and the front of simple wave must have different directions at any point.

By this lemma we know that shocks, contacts and simple waves at a node are separated by constant regions.

Lemma 3.2. *At any node the two boundaries of any constant region connot be "R and R" or "S and R", and if the boundaries are "S and S", these S cannot be both outgoing.*

Proof. Using the value of U in this region we draw a second characteristic τ starting from Q (more precisely, from a point sufficiently near to Q in this region). Since τ cannot coincide with the directions of the fronts of two different simple waves, then "R and R" is impossible. For the case "R and S", the direction of front of the simple wave R would be outward with respect to the shock S, but according to the entropy condition the second characteristics must point to the shock. Finally, for the case "S and S", if both two S were outgoing, then τ would be outward for one of them, this violates the entropy condition again.

For our convenience of analyzing the singularity structure of solutions we introduce some notations. Denote by ℓ a ray starting from Q, denote by P the point with coordinates (u, v) representing the state on ℓ. Sometimes we also denote by (P) the constant region, where U is equal to the coordinate of P. When ℓ rotates around the point Q, the point P moves on the plane (ξ, η) correspondingly. For a given sector D with vertex at Q, its vertical sector is denoted by $\tilde{D}$, and its angle is denoted by $\mathrm{Ang}(D)$. For a given ray J starting from Q, the extension in its opposite direction is denoted by $\tilde{J}$. If ℓ_1 intersects ℓ_2, we simply denote as $\ell_1 \nmid \ell_2$.

Continuing our discussion of the properties of nodes, we have

Lemma 3.3. *If ℓ does not run over any contact wave J, then P moves along a ray h starting from the origin. Moreover, the point P moves monotonously.*

Proof. In this case v/u keeps a constant value, hence P is located on a line m passing through the origin O. The point O divides m into m^+ and m^-. Now let ℓ rotate around Q. When ℓ runs over a shock, P jumps on the line h, but it must stay in one side of m according to Remark 2.3. On the other hand, when ℓ runs over a region of simple wave, the point P moves continuously. The point P cannot reach the origin; otherwise, the state on ℓ will be $(0, 0)$, violating the hypothesis (H_3). Meanwhile, P cannot go to infinity either, so it must stay on the same side of m. This half of m will be denoted by h in the sequel. Besides, since R and S cannot be adjacent waves, the monotonicity can be verified for simple wave or for shock separately. The verification is a simple collorary of the equation of simple wave or the entropy condition of shock.

Lemma 3.4. *The case that only one contact issuing from the node Q is impossible.*

Proof Notice that in any sector, containing only S or R, the ratio $k = v/u$ keeps a constant value. Now let the ray ℓ rotate starting from a given place $\theta = \varphi_0$. If it runs over J, then the value u/v changes. Hence we are led to a contradiction when the ray rotates back to its initial place $\varphi_0 + 2\pi$.

Lemma 3.5. *Assume that the sector D formed by the contacts J_1 and J_2 has its angle less than π. If the solution in the sector D is not constant, and there are no other contacts in D, then the origin must be located in $\tilde{D}$.*

Proof. Lemma 3.3 shows when the ray ℓ runs over the region D from J_1 to J_2, the corresponding point P moves on a ray h starting from the origin. Particularly, the points representing the states near J_1 and J_2 in the sector D are on the ray h. Denote the four sectors divided by J_1,J_2 and their extensions $\tilde{J}_1$,$\tilde{J}_2$ as D, D_1, D_2 and $\tilde{D}$. The origin O is not on $J_1, J_2, \tilde{J}_1, \tilde{J}_2$ because of (H_3), so we only need to show that O cannot be in D, D_1 and D_2.

First we assume that $O \in D$. We know that $h \nmid J_1, h \nmid J_2$ is impossible, because of Lemma 3.3. Now suppose $h \nmid J_1, h \nmid \tilde{J}_2$, the points P_1, P_2 are their intersections, and $OP_1 < OP_2$. Rotate ℓ from J_1 to J_2, ℓ could not meet simple wave first; otherwise the simple wave must be incoming and then (H_4) is violated. However, if ℓ meets a shock first, then the entropy condition is not satisfied. Similarly, $h \nmid \tilde{J}_1, h \nmid J_2$ does not happen. The contradiction indicates $O \notin D$.

Now we assume that $O \in D_1$. Then we have $h \nmid J_1, h \nmid J_2$ at P_1, P_2 with $OP_1 < OP_2$, or $h \nmid \tilde{J}_2, h \nmid \tilde{J}_1$ at P_2 , P_1 with $OP_2 < OP_1$. Let us only consider the first case. By the method of constructing shock and simple wave we know that if the region (P') is adjacent to the region (P_2) through a shock or a simple wave, then OP' must stay outside the sector P_1OP_2. But this is impossible, so the contradiction implies $O \notin D_1$. Similarly, we can exclude the case $O \in D_2$. Then the only possible case is $O \in \tilde{D}$.

Lemma 3.6. *Assume that the contacts J_1, J_2 issuing from Q are on the same line. On a half-plane there is not any other contact for the solution U, then U must be constant on this half-plane.*

Still let ℓ rotate over the half plane. Then the corresponding point P moves on a ray h starting from O. Since h intersects J_1 and J_2 (or their extensions) at the same point, the monotonicity of P shows that P just stays at this point. It means that U is constant.

Lemma 3.7. *If there are only two contacts J_1, J_2 meeting at Q, they form a sector D_1 with angle less that π and a sector D_2 with angle greater than π, then we have*

1) *there is at most one shock in D_1,*

2) *there is at most one shock or one centre wave in D_2.*

Proof. If U is not constant in D_1, then the origin O is in its vertical region $\tilde{D}_1$ according to Lemma 3.5. The ray h defined in Lemma 3.3 intersects $J_1, \tilde{J}_2$ (or $J_2, \tilde{J}_1$). Therefore, if there is a simple wave in D_1, it must be incoming, and this is not allowed. Furthermore, we indicate that there could not be more than one shock in D_1; otherwise, when ℓ rotates from J_1 to J_2, the point P representing the state on ℓ does not move monotonously.

If U is not constant in D_2, an outgoing centre wave is allowed. According to Lemma 3.2 we only need to exclude the possibility that two or more shocks appear in D_2. Next we will show that appearance of two shocks in D_2 is impossible.

First we consider the case that both shocks point to the node. Since $h \nmid \tilde{J}_1$ or $h \nmid \tilde{J}_2$ implies that the number of incoming waves is greater than 3, the point 2 of (H_4) is violated. So we can only have $h \nmid J_1 = h \nmid J_2$. However, in this case the point 3 of (H_4) is violated, because $O \notin \tilde{D}_1$ implies U is constant according to Lemma 3.3.

The case that both shocks point away from the node is also impossible according to Lemma 3.2, so we consider the case $h \nmid S_1, h \nmid S_2$, i.e. one shock is inward, and the other is outward. In this case we could not have $h \nmid \tilde{J}_1, h \nmid \tilde{J}_2$ because it leads to the number of incoming waves equal to 3. We could not have $h \nmid J_1, h \nmid J_2$ either, because $O \notin D_1$ implies U is constant, and then the point 3 of (H_4) is violated. However, $h \nmid J_1, h \nmid \tilde{J}_2$ (or vice versa) is also impossible, because according to the entropy condition J_1 is in between O and S_1, $\tilde{J}_2$ is in between O and $\tilde{S}_2$, and then S_1 and S_2 cannot be in D_2 together.

Lemma 3.8. *If three contacts emanate from Q, then*

1) *U is constant in one sector with angle less than π,*

2) *there are two contacts lying on the same line, and the solution U is constant on one of the half plane bounded by this line.*

Proof. Denote by $D_i (i = 1, 2, 3)$ the sector formed by rotating a ray from J_i to J_{i+1} $(J_4 = J_1)$. Among three D_i at least two sectors, say D_1 and D_2, have angle less than π. If U is not constant in both D_1 and D_2, then by Lemma 3.5 the origin O is in $\tilde{D}_1 \cap \tilde{D}_2$, but this is an empty set. So we may assume U is constant in D_1, and U is actually equal to the coordinates of Q. We also confirm that U is not constant in D_2; otherwise, the contact between D_1 and D_2 would disappear. This does not coincide with the assumption of this lemma.

Now consider the remaining sector D_3. By the reason mentioned above, the angle of D_3 could not be less than π. Thus we need to show that the angle of D_3 could not be greater than π either.

Now suppose $\mathrm{Ang}(D_3) > \pi$. Then as mentioned above U would not be constant in D_3. We know $O \in \tilde{D}_2$ by Lemma 3.5, which also implies $O \notin \tilde{D}_1$. Observe the place of the ray h. It cannot intersect J_1 and J_2 because of the hypothesis $(\mathrm{H}_4)_3$. However, when it intersects $\tilde{J}_1$ and $\tilde{J}_2$, both contacts J_1 and J_2 point into the node Q; meanwhile, there must be one shock in the region D_2 or D_3 pointing into Q. This leads to a contradiction to the hypothesis $(\mathrm{H}_4)_2$.

Therefore, the only remaining possibility is $\mathrm{Ang}(D_3) = \pi$, that is, J_1 and J_3 are on the same line. Then Lemma 3.6 gives the second conclusion of this lemma.

Proof of Theorem 1.1. By using all above lemmas we only need to show that there are at most three contacts meet at a node. In fact, if we have four contacts meeting at a node, they form four sectors $D_i (i = 1, \cdots, 4)$ bounded by contacts. Choose any three sectors among these four, then according to the argument in Lemma 3.8 there are two contacts lying on the same line and having opposite directions. Therefore, among all D_i we have two pairs of contacts, which are on the same line. It is obviously impossible. Combining with Lemma 3.4 we know that the number N of contacts starting from a node should be $0, 2$ or 3.

If $N = 0$, we can only have shocks at the node Q according to Lemma 3.2. Moreover, the number of incoming shocks is not more than 2 by (H_4), and the number of outgoing shocks is at most one. This is the case (a) in Theorem 1.1.

For $N = 2$ we can use the results in Lemma 3.7. The case "one shock appears in each D_i" corresponds to the type (b), "only one shock appears in one of two sectors" corresponds to the type $(c)_2$, "only one centre wave in D_2" corresponds to the type (d) of Theorem 1.1 respectively.

The case $N = 3$ has been discussed in Lemma 3.8. It corresponds to the type $(c)_1$ of Theorem 1.1. Thus the proof of Theorem 1.1 is complete.

§4. Interaction of Elementary Waves

As mentioned in the introduction we observe the construction of the problem (1.5), (1.6) as a problem of matching waves. Now let us work on interaction of two waves by using the analysis of nodes in the last section. Since any simple wave is composed of characteristics of second class, two simple waves will just merged into one simple wave, if they are adjacent. Therefore, we need to consider five kinds of interaction, i.e. $R \otimes S, R \otimes J, S \otimes S, J \otimes S, J \otimes J$.

By taking a rotation we may assume two elementary waves propagate from down to up, and they meet on top of the figure. The left and right waves are called first and second waves respectively. The three regions (right to the first wave, in between two waves and left to the second waves) are denoted by (1),(2) and (3) separately.

1) $R \otimes S$.

The front of the simple wave R is a second characteristic carrying weak singularity of U. Since the solution itself is continuous on the front, and the R-H conditions only involve the value U, the intersection of this front and shock does not form a non-trivial node. In this case the shock generally becomes a curved shock, which can be determined by

$$\frac{d\eta}{d\xi} = \frac{\eta - (v^- + v^+)}{\xi - (u^- + u^+)}, \quad \eta(\xi_Q) = \eta_Q, \tag{4.1}$$

where $(u^+, v^+) = (u(\xi, \eta), v(\xi, \eta))$ is the solution in the domain of simple wave, and (u^-, v^-) represents the state on the other side of the shock. The simple wave stops propagating after its meeting the shock.

2) $R \otimes J$.

As mentioned above the intersection of the front of a simple wave and a contact does not form a non-trivial node either. When interaction happens, the contact generally becomes a curved contact, which can be determined by

$$\frac{d\eta}{d\xi} = \frac{\eta - v^+}{\xi - u^+}, \quad \eta(\xi_Q) = \eta_Q, \tag{4.2}$$

where $(u^+, v^+) = (u(\xi, \eta), v(\xi, \eta))$ is the solution in the domain of simple wave. Different from the above case, the simple wave will still propagate after its cutting the contact J. On the curved J as a boundary of the region of new simple wave we have two relations

$$v = ku, (\eta - v) = \frac{d\eta}{d\xi}(\xi - u). \tag{4.3}$$

By using (4.3) we may determine the value U at the point (ξ, η), and then determine the direction of characteristics of second class passing (ξ, η). The value of U on this characteristic equals its value at (ξ, η), then the simple wave crossing over the contact J is constructed. Here we emphasize that when $k = \frac{d\eta}{d\xi}$ happens at some point, the above technique does not work, and the local solution caused by such an interaction does not exist.

Now we consider the interaction of waves with strong discontinuity. In each case if we flatten the local singularity structure, then the intersection of waves is a non-trivial node.

3) $S \otimes S$.

Assume that two incoming shocks S_{12}, S_{23} meet at Q. Then we get a node of type (a) according to the classification in Theorem 1.1. Denote the states next to two incoming shocks S_{12} and S_{23} by $P_1(u_1, v_1), P_2(u_2, v_2)$ and $P_3(u_3, v_3)$ with P_2 located between two shocks. By (2.4) we have

$$v_1/u_1 = v_2/u_2 = v_3/u_3. \tag{4.4}$$

Hence the points P_1, P_2, P_3 are on the same ray starting from O, and $P_2 \in \overline{P_1P_3}$. Take P_a such that $OP_a = OP_1 + OP_3$. Then the outgoing shock S_{13} is just $\overline{QP_a}$.

4) $S \otimes J$.

When an incoming shock meets a contact at Q, generally a new contact and a new shock are formed after interaction. This situation belongs to the type (b) in Theorem 1.1. Assume that the place of the shock and the contact are given as Figure 2, and they connect the regions (1), (2) and (3). Notice that any state connected with the state (1) by a simple wave or a shock must be on the line OP_1. Hence if OP_1 intersects QP_3 at $P_{1'}$, then the state (3) can be connected with the state (1') by the contact $J_{1'3}$: $QP_{1'}$, and (1') is also connected with the state (1) by the shock. The crucial point is whether OP_1 intersects QP_3 or not. When they intersect, the local singularity structure is thus constructed. Otherwise, the solution with such local structure does not exist. Besides, we also indicate that if O is located in the region (1) or (2), the reasonable local wave graph also does not exist.

5) $J \otimes J$.

If contacts J_{12} connecting the states (1) (2) and J_{23} connecting the states (2) (3) meet together, then the intersection Q coincides with the point P_2 representing the state (2). Now if the direction OP_1 coincides with OP_3, then we are led to the structure JJR or JJS in Theorem 1.1. Otherwise we are led to the structure $JJSJ$ or non-existence. More precisely, suppose that the origin O is in the domain (3) (or (1)), and the line P_2P_3 intersects the line OP_1 at $P_{1'}$. Let $OP_a = OP_1 + OP_2$. By combining P_2 and P_a we obtain a shock $S_{11'}$ starting from P_2, and by extending J_{23} we obtain contact $J_{31'}$. Particularly, if O, P_1 and P_3 are collinear, then the contact J_3 disappears. However, if the line OP_1 does not intersect the line P_2P_3, then the solution does not exist.

On the other hand, if the origin O is in the domain (2), and OP_1, OP_3 coincide, we may construct a centred rarefaction wave R_{13} starting from P_2, which connects the state (1) and (3). Otherwise, if O, P_1 and P_2 are not collinear, then the solution does not exist. Similarly, when O is in the vertical domain of (2), the solution does not exist either.

§5. Solution to Essential M-D Problem

Come back to the problem (1.5),(1.6). We will fix some conditions, under which the solution can be constructed by elementary waves and constant regions. For given $P_1(u_1, v_1)$, $P_2(u_2, v_2)$, $P_3(u_3, v_3)$, let

$$E = \{(\xi, \eta); \xi - \xi_0 = \cot\theta_i(\eta - \eta_0), \quad \text{where} \quad i = 1, 2, 3, (\xi_0, \eta_0) \in \varDelta P_1P_2P_3\}. \tag{5.1}$$

If $O \in E$, then we may make a suitable rotation, so that one of $\theta = \theta_i$ becomes $O\eta$ (η - axis), and $O\eta \cap \varDelta P_1P_2P_3$ is not empty, for instance, P_1, P_2 are in different sides of $O\eta$. In this case we will meet non-existence as mentioned in Remark 2.4. To avoid such non-existence we set the assumption

(A_1) $O \notin E$.

Under the assumption (A_1) by reflection and rotation of the coordinate system we may assume that the points P_1, P_2 and P_3 are located in the sector $V = \frac{\pi}{6} < \theta < \frac{\pi}{2}$. We denote OP_i by β_i for $i = 1, 2, 3$, which has slope ξ_i/η_i and plays an important role in constructing the solution. Besides, in such a placement of the coordinate system we also assume

(A_2) $\eta_1/\xi_1 < \eta_3/\xi_3$.

Next we are going to prove the conclusion of Theorem 1.2. That is, the assumptions (A_1), (A_2) ensure the existence of the solution of (1.5), (1.6). We will prove Theorem 1.2 by constructive method.

As mentioned above we assume $P_1, P_2, P_3 \in V$ according to the condition (A_1). Then we have two waves from infinity in each direction $\theta = \frac{2}{3}(i-1)\pi + \frac{\pi}{6}$ on the plane (ξ, η), because of

$0 \notin E$. If we use T to denote S or R, then we have six waves as $J_{12'}, T_{22'}, J_{2"3}, T_{2"2}, J_{1'3}, T_{1'1}$. In each direction the wave near to the origin is contact according to Remark 2.4.

Lemma 5.1. *On the (ξ, η) plane there is not any bounded region whose boundary is formed only by contacts.*

Proof. Integrating the first equation of system (1.5) we have

$$\iint_D \Big(\frac{\partial u^2}{\partial \xi} + \frac{\partial (uv)}{\partial \eta} - \xi \frac{\partial u}{\partial \xi} - \eta \frac{\partial u}{\partial \eta}\Big) d\xi d\eta = 0,$$

$$\int_{\partial D} (u^2 \cos(n, \xi) + uv \cos(n, \eta) - u\xi \cos(n, \xi) - u\eta \cos(n, \eta)) ds + \iint_D 2u d\xi d\eta = 0,$$

where n denotes outer normal direction. Notice that on any contact $(u-\xi, v-\eta)$ is tangential to its direction, and the fact implies $(u-\xi)\cos(n, \xi) + (v-\eta)\cos(n, \eta) = 0$. Therefore we have $\iint_D u d\xi d\eta = 0$. Similarly, by integrating the second equation of (1.5) we obtain $\iint_D v d\xi d\eta = 0$. However, in phase space U at any point in D should locate on a ray h starting from the origin. This implies that u, v have the same sign on h respectively, and at least one of them does not vanish. Hence we are led to a contradiction.

Lemma 5.2. *If the origin belongs to an unbounded region D with its boundary being composed of contacts, and U is constant at infinity in this region, then U is constant in whole D, and the boundary is composed of two straight contacts.*

Proof. For large $|(\xi, \eta)|$ the boundary ∂D is composed of straight contacts J_1, J_2 connecting D and other regions, so it must be on straight lines passing through P, the interaction of J_1 and J_2. Then for large $|(\xi, \eta)|$, U is equal to the coordinates of P. Denote by D_1 the region, where U is such a constant. Now if $D_1 \neq D$, then U in $D \setminus D_1$ also locates on the ray OP. Let $\Gamma = \partial D_1 \setminus \partial D$, $Q_1 = J_1 \cap \Gamma$. Then Γ near Q_1 is a second characteristic or a shock. According to the value of U, this part of Γ should be a straight line pointing to P' located on the extension OP. However this contradicts $\Gamma \in D$. Hence U must be a constant in the whole D, bounded by two straight contacts J_1 and J_2.

According to Lemma 5.1, if a solution of (1.5),(1.6) exists, then the whole plane will be divided by contacts into three parts. The boundary of each region is composed of contacts extending to infinity. Then we may anticipate that all the three contacts coming from infinity will form a distored letter "Y". Correspondingly, the three parts devided by three legs of this distored letter Y are denoted by $\tilde{\Omega}_i, i = 1, 2, 3$, respectively.

According to Lemma 5.2 the region $\tilde{\Omega_3}$ and U in this region can be easily determined. Let $J_{12'}$ intersect $J_{2"3}$ at P_3. The sector formed by these two contacts with angle $\frac{2}{3}\pi$ can be taken as $\tilde{\Omega}_3$, which is just (P_3).

Then we construct the region $\tilde{\Omega}_1$. When $J_{12'}$ meets $T_{1'1}$, its direction is changed to the one pointing to $P_1' = \Delta_1 \cap J_{1'3}$. $P_1' \in J_{1'3}$ because of (A_2). If $\xi_{P_1} > \xi_{P_3}$, then $T_{1'1}$ is $R_{1'1}$. The continuation J_a of $J_{12'}$ obeys the equation (2.9), which becomes $\frac{d\eta}{d\xi} = \frac{\eta - k_1\xi}{\xi - 2\xi}$ or $\frac{d\eta}{d\xi} = -\frac{\eta}{\xi} + k_1$.

It is easy to verify that $\frac{d^2\eta}{d\xi^2} < 0$ on J_a. The continuation of J_a, which is called $J_{1'4}$, can reach the point $P_{1'}$. The contacts $J_{1'3}, J_{1'4}, J_a$ and $J_{1'2}$ bound the region $\tilde{\Omega}_1$, and the solution in $\tilde{\Omega}_1$ is then obtained.

Now if $\xi_{P_1} < \xi_{P_3}$, then $T_{1'1}$ is $S_{1'1}$. The intersection of $J_{12'}$ and $S_{1'1}$ forms a node with

$JSJS$ type. After intersection $J_{12'}$ is changed to $J_{1'4}$ immediately, which meets $J_{1'3}$. Then the region $\tilde{\Omega}_1$ and the solution U in it are also obtained.

The remaining part of the plane (ξ, η) is denoted by $\tilde{\Omega}_2$, whose boundary is composed of contacts. The value of U at any point in $\tilde{\Omega}_2$ should be on the line β_2: $v/u = k_2$. Substituting it into (1.5) we obtain the equation $2u\frac{\partial u}{\partial \xi} + 2ku\frac{\partial u}{\partial \eta} - \xi\frac{\partial u}{\partial \xi} - \eta\frac{\partial u}{\partial \eta} = 0$, which can also be obtained in looking for self-similar solution to the scalar equation $\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}u^2 + \frac{\partial}{\partial y}ku^2 = 0$. Therefore, the discussion is essentially similar to [7]. In the region $\tilde{\Omega}_2$ the possible interaction of elementary waves is only $R \otimes S$ or $S \otimes S$. According to the discussion in §4 we can construct all possible waves in $\tilde{\Omega}_i$ successively. Since the construction includes tedious treatment, which is various according to the different location of the points P_1, P_2, P_3, we omit the details. A typical example has been shown in Fig. 2, where the coming waves from infinity are J, S in direction θ_1; J, R in direction θ_2 and J, R in direction θ_3.

Remark 5.1. The condition (A_2) can be somehow released. If $\eta_1/\xi_1 > \eta_3/\xi_3$, then the extension of $J_{12'}$ will meet $J_{2''3}$, so it will interact with the centre wave starting from the point P_3 before meeting $J_{2''3}$. But in this way the non-existence showed in the case 2) in §4 may happen. To avoid such a non-existence we need another condition involving (ξ_i, η_i) for $i = 1, 2, 3$. The explicit form of this condition is

$$\frac{\eta^* - \eta_5}{\xi^* - \xi_5} < k_2, \tag{5.2}$$

where $\xi_5 = (\sqrt{3}\eta_3 + \xi_3)/(\sqrt{3}k_1 + 1), \quad \eta_5 = k_1\xi_5,$

$$\xi^* = \xi_3 + \frac{\xi_1(2\xi_5 - \xi_3)(\eta_3 - k_1\xi_3)}{(\eta_1 + \xi_3/\sqrt{3} - k_1\xi_3)(2\xi_5 - \xi_3) - \xi_1(2\eta_5 - \eta_3)}, \quad \eta^* = \eta_3 + (\xi^* - \xi_3)\frac{2\eta_5 - \eta_3}{2\xi_5 - \xi_3}.$$

The process of obtaining (5.2) is omitted.

References

[1] Majda, A., The existence of multi-dimensional shock fronts, *Mem. Amer. Math. Soc.,* **281** (1983).

[2] Alinhac, S., Existence d'ondes de rarefaction pour des system quasilinaires hyperboliques multidimensionnels, *Comm. PDEs,* **14** (1988), 173–230.

[3] Metivier, G., Interaction de deux chocs pour un systeme de deux loius de conservation en dimension deux d'espace, *Trans. Amer. Math. Soc.,* **296** (1986), 983–1011.

[4] Chen Shuxing, On reflection of multidimensional shock front, *J. Diff. Eqs.,* **80** (1989), 199–236.

[5] Glimm, J., Nonlinear waves: overview and problems, *IMA* (1990), 89–106.

[6] Courant, R. & Friedrichs, K. O., Supersonic flow and shock waves, Interscience Publishers Inc., 1948.

[7] Wagner, D., The Riemann problem in two space dimensions for a single conservation laws, *Math. Ann.,* **14** (1983), 534–559.

[8] Lindquist, B., The scalar Riemann problem in two spatial dimensions: Piecewise smoothness of solutions and its breakdawn, *SIAM J. Anal.,* **14** (1986), 1178–1197.

[9] Zhang, T. & Xiao, L., The Riemann problem and interaction of waves in gas dynamics, Pitman Monograph and Survey in Pure and Applied Math., 41, 1989.

[10] Kerfitz, B. L. & Krenzer, H. C., A system of non-strictly hyperbolic conservation laws arising in elasticity theory, *Arch. Rat. Mech. Anal.,* **72** (1980), 219–241.

[11] Tan, D. C. & Zhang, T., Two dimensional Riemann problem for a hyperbolic system of nonlinear conservation laws (I): Four J cases, *Jour. Diff. Eqs.,* **111**:2(1994), 203–254.

[12] Chen Shuxing & Wang Hui, Weak solution with complete singularity structure for multidimensional Riemann problem of quasilinear hyperbolic system, *Proceedings of International Conference on Nonlinear PDEs,* (1993), 9–23.

[13] Glimm, J., Klingenberg, C., McBryan, O., Plohr, B., Sharp, D. & Yaniv, S., Front Tracking and two dimensional Riemann problems, *Advances in Appl. Math.,* **6**:3 (1985), 259–290.

Case a (SSS) Case b ($JSJS$)

Case c ($JJSJ$) Case d (JJR)

Fig.1

Fig.2