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PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEMS**

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Abstract

The authors establish the existence of nontrival periodic solutions of the asymptotically linear Hamiltonian systems in the general case that the asymptotic matrix may be degenerate and time-dependent. This is done by using the critical point theory, Galerkin approximation procedure and the Maslov-type index theory introduced and generalized by Conley, Zehnder and Long.

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§1. Introduction and Main Results

We study 1-periodic solutions to the following systems:

$$\dot{z} = JH'(t, z), \tag{HS}$$

where $H'(t, z)$ denotes the gradient of H with respect to the variable z, $\dot{z} = dz/dt$, and $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ with I_N being the identity matrix in $\mathbf{R}^N$ and N being a positive integer. We assume:

(H1) $H \in C^2([0,1] \times \mathbf{R}^{2N}, \mathbf{R})$ is a 1-periodic function in t and satisfies

$$|H''(t, z)| \le a_1|z|^s + a_2, \quad \forall (t, z) \in \mathbf{R} \times \mathbf{R}^{2N}, \text{ where } s \in (1, \infty), \quad a_1, a_2 > 0,$$

(H2) $H'(t, z) = B_0(t)z + o(|z|)$ as $|z| \to 0$ uniformly in t,

(H3) $H'(t, z) = B_\infty(t)z + o(|z|)$ as $|z| \to \infty$ uniformly in t,

where $B_0(t)$ and $B_\infty(t)$ are $2N \times 2N$ symmetric, continuous 1-periodic matrix functions.

In case $B_\infty(t)$ is "nondegenerate", i.e., 1 is not a Floquet multiplier of the linear system $\dot{y} = JB_\infty(t)y$, the problem (HS) has been studied by many authors. We refer to the papers by Amann-Zehnder[1], Chang[2], Conley-Zehnder[3], Li Liu[4], Long-Zehnder[5], Long[6] and the bibliography therein. However, only few papers have treated the case that $B_\infty(t)$ is "degenerate". For example, in [7] and [8], K. C. Chang and A.Szulkin considered the case that $B_\infty(t)$ is constant; and in [9], the first author considered the case that $B_\infty(t)$ is "finitely degenerate" (see Remark 2.1).

The purpose of this paper is to study the existence of nontrival periodic solutions of (HS) in the general case that $B_\infty(t)$ is degenerate and continuous 1-periodic in t. In Sectioin 2,

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for a given continuous 1-periodic and symmetric matrix function $B(t)$, we assign a pair of integers $(i,n) \in \mathbf{Z} \times \{0,\cdots,2N\}$ to it, and call the pair (i,n) the Maslov-type index of $B(t)$, just as in [6]. Using the results in [5,6] and the Galerkin approximation method[4], we establish the relation theorem (Theorem 2.1) between the Maslov-type index and Morse index. The main idea comes from [5, 6, 8]. In Section 3, based on the minimax principle[7,10], we prove the following main results.

Set $G(t,z) = H(t,z) - \frac{1}{2}(B_\infty(t)z,z)$, and we denote by (i_0,n_0) and (i_∞,n_∞) the Maslov-type indices of $B_0(t)$ and $B_\infty(t)$ respectively. One of the main results reads as:

Theorem 1.1. *Suppose that H satisfies* (H1)–(H3) *and $G'(t,z)$ is bounded. Then* (HS) *has a nontrival solution in each of the following two cases:*

(i) $i_\infty \notin [i_0, i_0+n_0]$, *and either $n_\infty = 0$ or $G(t,z) \to -\infty$ as $|z| \to \infty$ uniformly in t.*

(ii) $i_\infty + n_\infty \notin [i_0, i_0+n_0]$, *and either $n_\infty = 0$ or $G(t,z) \to +\infty$ as $|z| \to \infty$ uniformly in t.*

Theorem 1.1 generalizes the corresponding results in [1, 3, 4, 5, 6, 8, 11], where $B_\infty(t)$ is restricted to either being constant matrix or being nondegenerate.

In Section 4, we consider the periodic solutions of strong resonant Hamiltonian systems. This is largely motivated by Chang[7,12]. Our result reads as:

Theorem 1.2. *Suppose H satisfies* (H1)–(H3) *and*

$$G(t,z) \to 0, \quad |G'(t,z)| \to 0 \quad \text{as } |z| \to \infty \text{ uniformly in } t. \tag{1.1}$$

Then (HS) *has a nontrival solution if one of the following three cases occurs:*

(1) $\int_0^1 H(t,0)\,dt = 0$.

(2) $\int_0^1 H(t,0)\,dt > 0$ *and* $i_\infty \notin [i_0, i_0+n_0]$.

(3) $\int_0^1 H(t,0)\,dt < 0$ *and* $i_\infty + n_\infty \notin [i_0, i_0+n_0]$.

Theorem 3.1 of [7] may be regarded as the special case of Theorem 1.2, where $B_\infty(t)$ is restricted to constant matrix and $|H''(t,z)|$ is bounded.

In Section 5, as an appendix, we give some results which were proved in [13, 14] and used to prove Theorem 2.3 in Section 2. We sketch the proof briefly.

§2 Maslov-Type Index and Morse Index

Maslov-type index was introduced and generalized by Conley-Zehnder[3], Long-Zehnder[5] and Long[6]. Here we repeat it briefly, for more details we refer to [6].

Let $W = Sp(N,\mathbf{R}) = \{M \in \mathcal{L}(\mathbf{R}^{2N}) : M^TJM = J\}$. We define

$$\mathcal{P} = \{\gamma \in C^1([0,1],W) : \gamma(0) = I, \dot\gamma(1) = \dot\gamma(0)\gamma(1), \text{ and } J\gamma(t)\gamma^{-1}(t) \text{ is symmetric for each } t\}.$$

For every $\gamma \in \mathcal{P}$, we define its Maslov-type index $(i(\gamma), n(\gamma)) \in \mathbf{Z} \times \{0,\cdots,2N\}$ as follows:

$$n(\gamma) = \dim\ker(\gamma(1) - I).$$

If $n(\gamma) = 0$, $i(\gamma)$ is defined just the same as the one in [3,5]. If $n(\gamma) \neq 0$, according to the following Lemma 2.1 proved by Long[6], we define $i(\gamma) = i(\gamma_{-v})$ for $v \in (0,1]$.

Lemma 2.1. *For every $\gamma \in \mathcal{P}$, $n(\gamma) \neq 0$, there exists $h \in C^1([-1,1]\times[0,1],W)$, which we denote by $h(v,t) = \gamma_v(t)$ for $(v,t) \in [-1,1]\times[0,1]$, such that*

(i) $\gamma_v \in \mathcal{P}$, $\gamma_0 = \gamma$ *and* $\gamma_v \to \gamma$ *in* $C^1([0,1], W)$ *as* $v \to 0$.

(ii) $n(\gamma_v) = 0$ *for all* $v \neq 0$ *(in this case* $i(\gamma_v)$ *is well-defined). Moreover,*

$$i(\gamma_v) = i(\gamma_{v'}), \quad i(\gamma_{-v}) = i(\gamma_{-v'}) \quad \text{for all} \quad v, v' \in (0,1].$$

(iii) $i(\gamma_v) - i(\gamma_{-v}) = n(\gamma)$, *if* $v \in (0,1]$.

For a given continuous 1-periodic and symmetric matrix function $B(t)$, let $\gamma(t)$ be the fundamental solution matrix of the linear Hamiltonian systems:

$$\dot{y} = JB(t)y \tag{2.1}$$

with $\gamma(0) = I$. Then $\gamma(t) \in \mathcal{P}$ and the Maslov-type index $(i(\gamma), n(\gamma))$ is defined. We also call $(i(\gamma), n(\gamma))$ the Maslov-type index of $B(t)$.

Let $S^1 = \mathbf{R}/(2\pi\mathbf{Z})$, $E = W^{1/2,2}(S^1, \mathbf{R}^{2N})$. Recall that E is a Hilbert space with norm $\|\cdot\|$ and inner product $\langle\ ,\ \rangle$, and E consists of those $z(t)$ in $L^2(S^1, \mathbf{R}^{2N})$ whose Fourier series

$$z(t) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(2\pi nt) + b_n \sin(2\pi nt))$$

satisfies

$$\|z\|^2 = |a_0|^2 + \frac{1}{2}\sum_{n=1}^{\infty} n(|a_n|^2 + |b_n|^2) < \infty,$$

where $a_j, b_j \in \mathbf{R}^{2N}$. We define two selfadjoint operators $A, B \in \mathcal{L}(E)$ by extending the bilinear forms

$$\langle Ax, y\rangle = \int_0^1 (-J\dot{x}, y)\, dt, \quad \langle Bx, y\rangle = \int_0^1 (B(t)x, y)\, dt \tag{2.2}$$

on E. Then B is compact (cf. [5]). Using the Floquet theory, we have

$$n(\gamma) = \dim\ker(A - B). \tag{2.3}$$

Let $B_\infty(t)$ be the matrix functioin in (H3) with the Maslov-type index (i_∞, n_∞), and B_∞ be the operator, defined by (2.2), corresponding to $B_\infty(t)$. Then by (2.3) we have

$$n_\infty = \dim\ker(A - B_\infty).$$

Let $\cdots \le \lambda'_2 \le \lambda'_1 < 0 < \lambda_1 \le \lambda_2 \le \cdots$ be the eigenvalues of $A - B_\infty$, and let $\{e'_j\}$ and $\{e_j\}$ be the eigenvectors of $A - B_\infty$ corresponding to $\{\lambda'_j\}$ and $\{\lambda_j\}$ respectively.

For $m \ge 0$, set $E_0 = \ker(A - B_\infty)$, $E_m = E_0 \oplus \operatorname{span}\{e_1, \cdots, e_m\} \oplus \operatorname{span}\{e'_1, \cdots, e'_m\}$ and P_m to be the orthogonal projection from E to E_m. Then $\{P_m\}$ is an approximation scheme with respect to the operator $A - B_\infty$, i.e., $(A - B_\infty)P_m = P_m(A - B_\infty)$ and $P_m x \to x$ as $m \to \infty$ for any $x \in E$. In the following we denote $T^{\#} = (T_{ImT})^{-1}$, and we also denote by $M^+(\cdot)$, $M^-(\cdot)$ and $M^0(\cdot)$ the positive definite, negative definite and null subspaces of the selfadjoint linear operator defining it, respectively.

Lemma 2.2. *For any continuous* 1*-periodic and symmetric matrix function* $B(t)$*, there exists an* $m^* > 0$ *such that for* $m \ge m^*$,

$$\dim\ker(P_m(A - B)P_m) \le \dim\ker(A - B).$$

Proof. There is an $m_1 > 0$ such that for $m \ge m_1$,

$$\dim\, P_m \ker(A - B) = \dim\ker(A - B). \tag{2.4}$$

For otherwise, there exist $x_j \in \ker(A-B)\cap(I-P_{m_j})E$ such that $\|x_j\|=1$. Notice that $(A-B_\infty)x_j=(I-P_{m_j})(B-B_\infty)x_j$. Then we have

$$\|(A-B_\infty)x_j\| \geq \|(A-B_\infty)^{\#}\|^{-1} > 0,$$

and

$$\|(I-P_{m_j})(B-B_\infty)x_j\| \leq \|(I-P_{m_j})(B-B_\infty)\| \to 0$$

as $j\to\infty$, a contradiction. Thus (2.4) holds.

Take $m \geq m_1$, let $X_m = P_m \ker(A-B)$ and $E_m = X_m \oplus Y_m$. Then we have

$$Y_m \subset \operatorname{Im}(A-B).$$

Let $d=\frac{1}{4}\|(A-B)^{\#}\|^{-1}$. Since B and B_∞ are compact, we have

$$\|(I-P_m)(B-B_\infty)\| \to 0 \quad \text{as} \quad m\to+\infty.$$

Hence there is an $m_2 \geq m_1$ such that for $m \geq m_2$,

$$\|(I-P_m)(B-B_\infty)\| \leq 2d. \tag{2.5}$$

For $m \geq m_2$, $\forall y \in Y_m$, we have

$$y=(A-B)^{\#}(A-B)y=(A-B)^{\#}(P_m(A-B)P_m y+(P_m-I)(B-B_\infty)y).$$

This implies that

$$\|y\| \leq \frac{1}{2d}\|P_m(A-B)P_m y\|. \tag{2.6}$$

Hence by (2.4) and (2.6) we have

$$\dim\ker\, P_m(A-B)P_m \leq \dim\, X_m = \dim\ker\,(A-B).$$

Theorem 2.1. *For any continuous* 1*-periodic and symmetric matrix function* $B(t)$ *with the Maslov-type index* (i_0,n_0)*, there exists an* $m^*>0$ *such that for* $m\geq m^*$ *we have*

$$\begin{aligned} &\dim\, M_d^+(P_m(A-B)P_m) = m+i_\infty-i_0+n_\infty-n_0,\\ &\dim\, M_d^-(P_m(A-B)P_m) = m-i_\infty+i_0,\\ &\dim\, M_d^0(P_m(A-B)P_m) = n_0, \end{aligned} \tag{2.7}$$

where $d=\frac{1}{4}\|(A-B)^{\#}\|^{-1}$, $M_d^+(\cdot)$, $M_d^-(\cdot)$ *and* $M_d^0(\cdot)$ *denote the eigenspaces corresponding to the eigenvalue* λ *belonging to* $[d,+\infty),(-\infty,-d]$ *and* $(-d,d)$ *respectively.*

Proof. Case 1, $n_0=0$. By (2.3) we have $\dim\ker\,(A-B)=0$.

Since B and B_∞ are compact, there exists an $m^*>0$ such that for $m\geq m^*$,

$$\|(I-P_m)(B_\infty-B)\|+\|(B_\infty-B)(I-P_m)\| \leq \frac{1}{2}\|(A-B)^{-1}\|^{-1}.$$

Since $P_m(A-B)P_m=(A-B)P_m+(P_m-I)(B_\infty-B)P_m$, for $m\geq m^*$ we have

$$\|P_m(A-B)P_m x\| \geq \frac{1}{2}\|(A-B)^{-1}\|^{-1}\|x\| \quad \text{for any} \quad x\in E_m.$$

Hence we have

$$M_d^\star(P_m(A-B)P_m)=M^\star(P_m(A-B)P_m), \quad \text{where } \star=+,-,0.$$

Notice that

$$\begin{aligned} A-B &= P_m(A-B)P_m + (I-P_m)(A-B_\infty) \\ &\quad + (I-P_m)(B_\infty - B) + P_m(B_\infty - B)(I-P_m) \\ &= A - (B_\infty + P_m(B-B_\infty)P_m) + (I-P_m)(B_\infty - B) + P_m(B_\infty - B)(I-P_m). \end{aligned}$$

By Theorem 5.1, Theorem 5.2 and Definition 5.1, we have

$$\begin{aligned} I(B, B_\infty) &= I(B_\infty + P_m(B-B_\infty)P_m, B_\infty) \\ &= \dim M^+(P_m(A-B)P_m) - \dim M^+(P_m(A-B_\infty)P_m) - n_\infty. \end{aligned}$$

Hence $\dim M^+(P_m(A-B)P_m) = I(B,B_\infty) + m + n_\infty = i_\infty - i_0 + m + n_\infty$.

Similarly, $\dim M^-(P_m(A-B)P_m) = m - i_\infty + i_0$.

Case 2, $n_0 > 0$. Let γ be the fundamental solution matrix of (2.1) and γ_v be the things described in Lemma 2.1. For $-1 \le v \le 1$, we define

$$B_v(t) = -J\dot\gamma_v(t)\gamma_v^{-1}(t), \quad 0 \le t \le 1.$$

By Lemma 2.1 we have $B_0(t) = B(t)$, $n(\gamma_v) = 0$ for $v \ne 0$, and $\|B_v - B\| \to 0$ as $v \to 0$, where B_v is the operator, defined by (2.2), corresponding to $B_v(t)$.

Choose $0 < v_0 \le 1$ such that for $v = \pm v_0$, $\|B - B_v\| \le \frac{1}{2}d$. By Case 1, there exists an $m_1 \ge 0$ such that for $m \ge m_1$,

$$\begin{aligned} &M^+(P_m(A-B_v)P_m) = m + i_\infty - i(\gamma_v) + n_\infty, \\ &M^-(P_m(A-B_v)P_m) = m - i_\infty + i(\gamma_v), \qquad (2.8) \\ &M^0(P_m(A-B_v)P_m) = 0. \end{aligned}$$

By Lemma 2.2 there exists an $m^* \ge m_1$ such that for $m \ge m^*$,

$$\dim M_d^0(P_m(A-B)P_m) \le n_0. \qquad (2.9)$$

For otherwise, there exists $y \in M_d^0(P_m(A-B)P_m) \cap Y_m$, $\|y\| = 1$, where

$$E_m = P_m \ker(A-B) \oplus Y_m, \quad \dim P_m \ker(A-B) = n_0.$$

Then $\|P_m(A-B)P_m y\| \le d\|y\|$, a contradiction to (2.6).

Since $P_m(A-B_v)P_m = P_m(A-B)P_m + P_m(B-B_v)P_m$, by Lemma 2.1 and (2.7), for $m \ge m^*$ we have

$$\begin{aligned} M_d^+(P_m(A-B)P_m) &\le M^+(P_m(A-B_{v_0})P_m) = m + i_\infty - i_0 - n_0 + n_\infty, \\ M_d^+(P_m(A-B)P_m) &\ge M^+(P_m(A-B_{-v_0})P_m) - M_d^0(P_m(A-B)P_m) \\ &= m + i_\infty - i_0 + n_\infty - M_d^0(P_m(A-B)P_m). \end{aligned}$$

By (2.9), we have $M_d^0(P_m(A-B)P_m) = n_0$ and

$$M_d^+(P_m(A-B)P_m) = m - i_\infty - i_0 - n_0.$$

Similarly, we have $M_d^-(P_m(A-B)P_m) = m - i_\infty + i_0$.

Remark 2.1. (i) We say that $B(t)$ is admissible for $B_\infty(t)$ if $\|(P_m(A-B)P_m)^\#\|^{-1} \ge d$ for m large enough and some $d > 0$ independent on m. It is easy to show that $B(t)$ is admissible for $B_\infty(t)$ iff $\dim\ker(P_m(A-B)P_m) = \dim\ker(A-B)$ for m large enough.

(ii) If $B_\infty(t) = 0$, then $\{P_m\}$ is the usual approximation scheme with respect to the operator A. In this case, we can prove Theorem 2.1 similarly. If $B(t)$ is admissible for 0, Theorem 2.1 is the same as Theorem 6 in [6]. It is easy to show that if $B(t)$ is constant or $B(t)$ is nondegenerate or $B(t)$ is "finitely degenerate" (i.e. $\ker(A-B) \subset E_{m_0}$ for some $m_0 \geq 0$), then $B(t)$ is admissible for 0.

Lemma 2.3. *Suppose that $f \in C^2(\mathbf{R}^n, \mathbf{R})$ and there are symmetric matrix L on $\mathbf{R}^n$ and constants $r > 0$, $d > 0$ such that*

$$|f'(x) - Lx|/|x| \to 0 \quad as \quad |x| \to 0,$$

$$|f''(x) - L| < \frac{1}{2}d, \qquad \forall x \in V_{2r} = \{x \in \mathbf{R}^n : |x| \leq 2r\}.$$

If $f'(x) \neq 0$ for any $x \in C_r = \{x \in \mathbf{R}^n : r \leq |x| \leq 2r\}$, then for any $\epsilon > 0$ there exists a $g \in C^2(\mathbf{R}^n, \mathbf{R})$ such that

(1) $g(x) = f(x)$ for $|x| \geq 2r$, $g'(x) \neq 0$ for $x \in C_r$, and $|f(x) - g(x)| < \epsilon$ for $x \in \mathbf{R}^n$.

(2) $g(x)$ has only finite number of nondegenerate critical points, say $\{x_1, \cdots, x_{m_0}\}$, in V_r satisfying

$$\dim M_d^-(L) \leq \dim M^-(g''(x_j)) \leq \dim M_d^-(L) + \dim M_d^0(L), \quad for \quad j = 1, 2, \cdots, m_0.$$

Proof. Just the same as the proof of [4, Theorem 1.3], we repeat it briefly.

For any $x \in C_r$, since $f'(x) \neq 0$, we have $|f'(x)| \geq \rho > 0$. Let $g(x) = f(x) + (a, x)h(|x|^2)$, where $a \in \mathbf{R}^n$, $|a| < \min\{\epsilon/2r, \rho/(2 + 64r)\}$, and $h : [0, +\infty) \to [0, 1]$ is a smooth truncated function

$$h(s) = \begin{cases} 0, & s \geq 2r, \\ \text{smooth}, & \frac{3}{2}r \leq s \leq 2r, \qquad \text{satisfying} \quad |h'(s)| \leq 4/r, \\ 1, & s \leq \frac{3}{2}r. \end{cases}$$

Then g satisfies (i), and for any $x \in V_r$, $g''(x) = f''(x)$.

On the other hand, for any $x \in V_r$, $u \in M_d^-(L) \backslash \{0\}$,

$$(f''(x)u, u) \leq (Lu, u) + |f''(x) - L| \|u\|^2 \leq -\frac{1}{2}d|u|^2 < 0.$$

Then $\dim M^-(f''(x) \geq \dim M_d^-(L)$, $\quad \forall x \in V_r$.

Similarly, $\dim M^+(f''(x)) \geq \dim M_d^+(L)$ for $x \in V_r$. Now by Sard's Lemma we can choose the vector $a \in \mathbf{R}^n$ such that g satisfies (ii). The proof is complete.

Lemma 2.4. *Suppose $x_n \in \ker(P_n(A - B)P_n)$, $\|x_n\| \to +\infty$ as $n \to +\infty$, $h \in C([0, 1] \times \mathbf{R}^{2N}, \mathbf{R})$ and $K \subset L^q([0, 1], \mathbf{R}^{2N})$ is compact for $q \geq 1$. Then*

(i) $(h(t, x) \to 0$ as $|z| \to \infty$ uniformly in $t \in [0, 1])$

$\Longrightarrow (\lim\limits_{n\to\infty} \int_0^1 |h(t, x_n + y)|\, dt = 0$ *uniformly in* $y \in K)$.

(ii) $(h(t, z) \to \pm\infty$ as $|z| \to \infty$ uniformly in $t \in [0, 1])$

$\Longrightarrow (\lim\limits_{n\to\infty} \int_0^1 |h(t, x_n + y)|\, dt = \pm\infty$ *uniformly in* $y \in K)$.

Here the limit "$\lim\limits_{n\to\infty}$" is in the sense of subsequence.

Proof. Let $u_n = x_n/\|x_n\|$. It is easy to show that

$$u_n \to z_0 \in \ker(A - B) \text{ (in the sense of subsequence).}$$

We claim that $\forall \epsilon > 0$, there exist $\delta(\epsilon) > 0$, $n^* > 0$ such that for $n \geq n^*$,

$$\text{meas}\,\{t \in [0, 1] : |u_n(t)| < \delta(\epsilon)\} < \epsilon. \tag{2.10}$$

In fact, since $z_0 \in \ker(A - B)$ and $\|z_0\| = 1$, we have

$$\text{meas}\,\{t \in [0, 1] : |z_0(t)| = 0\} = 0.$$

Hence it is easy to show that

(1) $\forall\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that

$$\operatorname{meas}\Omega_0(\epsilon) \equiv \operatorname{meas}\{t \in [0,1] : |z_0(t)| < \delta(\epsilon)\} < \epsilon.$$

(2) $\forall\epsilon > 0$ and $\forall\delta > 0$, there exists $n_1 = n_1(\epsilon,\delta) > 0$ such that for $n \geq n_1$,

$$\operatorname{meas}\Omega_1(\epsilon,\delta) \equiv \operatorname{meas}\{t \in [0,1] : |z_0(t) - u_n(t)| > \delta\} < \epsilon.$$

If the claim is false, then there exists an $\epsilon_0 > 0$, and for any integer $k \geq 1$ there exists an $n_k > k$ such that

$$\operatorname{meas}\Omega_k \equiv \operatorname{meas}\{t \in [0,1] : |u_{n_k}(t)| < \tfrac{1}{k}\} \geq \epsilon_0.$$

Let $\epsilon_1 = \frac{1}{4}\epsilon_0$ in (1). Then there exists $\delta(\epsilon_1)$ such that

$$\operatorname{meas}\Omega_0(\epsilon_1) < \epsilon_1. \tag{2.11}$$

Let $\epsilon_2 = \frac{1}{2}\epsilon_0$, $\delta_2 = \frac{1}{2}\delta(\epsilon_1)$ in (2). Then there exists an $n_2 = n(\epsilon_2,\delta_2) > 0$ such that for $n \geq n_2$,

$$\operatorname{meas}\Omega_1(\epsilon_2,\delta_2) < \epsilon_2.$$

Now we take k large enough such that $\frac{1}{k} < \frac{1}{2}\delta(\epsilon_1)$, $n_k \geq n_2$. It is easy to show that

$$\Omega_k \cap ([0,1]\backslash\Omega_1(\epsilon_2,\delta_2)) \subset \Omega_0(\epsilon_1).$$

By (2.11), we have

$$\begin{aligned}\frac{1}{4}\epsilon_0 = \epsilon_1 &> \operatorname{meas}\Omega_0(\epsilon_1) \geq \operatorname{meas}(\Omega_k \cap ([0,1]\backslash\Omega_1))\\ &\geq \operatorname{meas}\Omega_k - \operatorname{meas}\Omega_1(\epsilon_2,\delta_2) \geq \epsilon_0 - \epsilon_2 = \frac{1}{2}\epsilon_0.\end{aligned}$$

This is a contradiction. Hence (2.9) holds.

Since K is compact, using the same arguments as in the proof of [11, Lemma 3.2], we have that $\forall\epsilon > 0$ there exists $M(\epsilon) > 0$ such that

$$\operatorname{meas}\{t \in [0,1] : |v(t)| > M(\epsilon)\} < \epsilon \quad \text{for any} \quad v \in K. \tag{2.12}$$

By (2.10) and (2.12), using the same arguments as in the proof of [11, Lemma 3.2], we get (i), (ii).

§3. Periodic Solution of (HS)

In this section we establish the periodic solutions of (HS) and prove Theorem 1.1. Just as in Section 2, let $E = W^{1/2,2}(S^1, \mathbf{R}^{2N})$ and $\{P_m\}$ be the approximation scheme with respect to $A - B_\infty$, $E_m = P_m E$. We define

$$f(z) = \frac{1}{2}\langle Az, z\rangle - \int_0^1 H(t,z)\,dt$$

on E. It is well known that $f \in C^2(E, \mathbf{R})$ whenever H satisfies (H1). Looking for the solution (HS) is equivalent to looking for the critical points of f (see [4, 8]).

Let f_m be the restriction of f to the space E_m.

We say that f satisfies the $(\mathrm{PS})_c^*$ condition for $c \in \mathbf{R}$, if any sequence $\{x_m\}$ such that $x_m \in E_m$, $f_m'(x_m) \to 0$ and $f_m(x_m) \to c$ possesses a subsequence convergent in E (cf. [4]).

Now let (i_∞, n_∞) be the Maslov-type index of $B_\infty(t)$ and

$$G(t,z) = H(t,z) - \frac{1}{2}\langle B_\infty(t)z, z\rangle.$$

Lemma 3.1. *If $G'(t,z)$ is bounded, then for any $c \in \mathbf{R}$, f satisfies* $(\mathrm{PS})_c^*$ *and f_m satisfies* $(\mathrm{PS})_c$ *in each of the following three cases:*

(i) $n_\infty = 0$,

(ii) $G(t,z) \to -\infty$ *uniformly in t as* $|z| \to \infty$,

(iii) $G(t,z) \to +\infty$ *uniformly in t as* $|z| \to \infty$.

Proof. Let $\psi(z) = \int_0^1 G(t,z)\,dt$ for $z \in E$. Then

$$f(z) = \frac{1}{2}\langle (A - B_\infty)z, z\rangle - \psi(z).$$

It is easy to show that $\psi'(z)$ is compact and bounded. In view of Lemma 2.4, the proof is just the same as the proof of Lemma 2.1 in [4] and Lemma 7.1 in [8].

Proof of Theorem 1.1. Step 1. Let (i_0, n_0) be the Maslov-type index of $B_0(t)$ and B_0 be the operator, defined by (2.2), corresponding to $B_0(t)$. Let $d = \frac{1}{4}\|(A - B_0)^{\#}\|^{-1}$. Then there exists an $m_1 > 0$ such that Theorem 2.1 holds for $m \ge m_1$.

For $m \ge m_1$, we consider

$$f_m(z) = \frac{1}{2}\langle Az, z\rangle - \int_0^1 H(t,z)dt = \frac{1}{2}\langle (A - B_0)z, z\rangle - \psi_0(z)$$

on E_m, where $\psi_0(z) = \int_0^1 (H(t,z) - \frac{1}{2}(B_0(t)z,z))\,dt$. Since H satisfies (H2) and (H3), using the same arguments as [4, Lemma 3.1], we have

$$\|f'(z) - (A - B_0)z\|/\|z\| \to 0, \quad \|f''(z) - (A - B_0)\| \to 0 \quad \text{as} \quad \|z\| \to 0. \tag{3.1}$$

Noticing that $f_m'(z) = P_m(A - B_0)z - P_m\psi_0'(z)$, we have

$$\|f_m'(z) - P_m(A - B_0)P_m z\|/\|z\| \le \|f'(z) - (A - B_0)z\|/\|z\| \to 0,$$

as $\|z\| \to 0$ and $z \in E_m$. By (3.1) there exists $r > 0$ such that

$$\|f''(z) - (A - B_0)\| < \frac{1}{2}d \ \text{ for } \ z \in V_{2r} = \{z \in E : \|z\| \le 2r\}.$$

Hence we have

$$\|f_m''(z) - P_m(A - B_0)P_m\| \le \|f''(z) - (A - B_0)\| < \frac{1}{2}d \ \text{ for } \ z \in V_{2r} \cap E_m.$$

Now we claim that there exists an $m_2 \ge m_1$ such that for $m \ge m_2$, $f_m'(x) \ne 0$ for $x \in E_m$ and $r \le \|x\| \le 2r$.

For otherwise, there exist $x_j \in E_{m_j}$ such that $r \le \|x_j\| \le 2r$ and $f_{m_j}'(x_j) = 0$. Then by Lemma 3.1, it is easy to show that there is a critical point $x^* \in E$ of f such that $r \le \|x^*\| \le 2r$ and the proof is complete.

Take $m \ge m_2$. By Lemma 2.3, for any $0 < \epsilon \le \frac{1}{2}$, there exists $g_m \in C^2(E_m, \mathbf{R})$ satisfying Lemma 2.3 (1), (2).

For any $z \in V_{2r}$, we have

$$|f(z) - f(0)| \le \frac{1}{2}d \cdot (2r)^2 + \|A - B_0\|(2r)^2.$$

Let $a_0 = |f(0)| + 4r^2(\frac{1}{2}d + \|A - B_0\|) + 1$. Then $|f_m(z)| < a_0$ for any $z \in V_{2r} \cap E_m$ and $f_{ma} = g_{ma}$ for $|a| \ge a_0$, where $f_{ma} = \{x \in E_m : f_m(x) \le a\}$.

Step 2. Let $\psi(z) = \int_0^1 G(t,z)dt$. Then $|\psi'(z)| \le c_1$ and

$$f_m(z) = \frac{1}{2}\langle (A - B_\infty)z, z\rangle - \psi(z), \quad \forall z \in E_m.$$

Set $E^+ = M^+(A - B_\infty)$, $E^- = M^-(A - B_\infty)$, $E_0 = M^0(A - B_\infty)$, $E_m^+ = P_m E^+$ and $E_m^- = P_m E^-$. Then dim $E_m^+ =$ dim $E_m^- = m$.

Let $r_1 = (c_1 + 1)\|(A - B_\infty)^\#\|$ and $D_m = (E_m^+ \cap V_{r_1}) \times (E_m^- \oplus E_0)$. Noticing that $P_m(A - B_\infty) = (A - B_\infty)P_m$, just as in the proof of [10, Lemma II 5.1], we know that f has no critical points outside D_m, and that $-df(x)$ points inward to D_m on ∂D_m.

Now we prove Theorem 1.1 in the Case (ii). By Lemma 2.4, either $\psi(P_0x) = \psi(0)$ whenever $n_\infty = 0$ or $\psi(P_0x) \to +\infty$ as $\|P_0x\| \to \infty$ whenever $G(t, z) \to +\infty$ as $|z| \to \infty$ uniformly in t. Just as in the proof of [10, Lemma II 5.1], there exist $a_1 < a_2 < -a_0$, $r_2 > r_3 > 0$ such that

$$\begin{aligned}(E_m^+ \cap V_{r_1}) \times ((E_m^- \oplus E_0)\backslash V_{r_2}) &\subset f_{ma_1} \cap D_m \\ &\subset (E_m^+ \cap V_{r_1}) \times ((E_m^- \oplus E_0)\backslash V_{r_3}) \subset f_{ma_2} \cap D_m,\end{aligned}$$

and a_1, a_2, r_2, r_3 are independent of m. For any $x \in D_m$, we have

$$\begin{aligned}f_m(x) \le \frac{1}{2}\|A - B_\infty\| r_1^2 - \frac{1}{2}\|(A - B_\infty)^\#\|^{-1}\|x_-\|^2 \\ + c_1(\|x_-\| + r_1) - \psi(P_0x).\end{aligned}$$

It is easy to show that there exists $b \ge a_0$, which is independent of m, such that $f_m(x) < b$ for $x \in D_m$.

We claim that there exists an $m_3 \ge m_2$ such that for $m \ge m_3$, f_m has not any critical points in $\{x \in E_m : a_1 \le f_m(x) \le a_2\}$.

For otherwise, there exist $\{x_j\}$ such that $x_j \in E_{m_j}$, $f'_{m_j}(x_j) = 0$ and $a_1 \le f_{m_j}(x_j) \le a_2$. By Lemma 3.1, it is easy to show that there exists $x^* \in E$ such that $f'(x^*) = 0$, $a_1 \le f(x^*) \le a_2 < -a_0 \le f(0)$ and the proof is complete.

Using the same arguments as in [10, Lemma II 5.1] we have

$$\begin{aligned}H_q(g_{mb}, g_{ma_2}) &= H_q(f_{mb}, f_{ma_2}) \\ &\cong H_q(D_m, D_m \cap f_{ma_2}) \cong \delta_{q(m+n_\infty)}.\end{aligned}$$

Since f_m satisfies $(\mathrm{PS})_c$ condition, it is easy to show that g_m also satisfies $(\mathrm{PS})_c$ condition. By Principle II in [6], there exists a critical value $c_m \in (a_2, b)$ of g_m, which is determined by

$$c_m = \inf_{\tau \in [\tau]} \sup_{x \in |\tau|} g_m(x), \quad \text{where} \quad 0 \ne [\tau] \in H_{m+n_\infty}(g_{mb}, g_{ma_2}),$$

where τ is a singular chain in $[\tau]$, and $|\tau|$ is the support of τ.

If the number of critical points of g_m with the critical value c_m, $\#K_{c_m}(g_m) < +\infty$, by Principle II in [6], there exists $x_m \in K_{c_m}(g_m)$ such that the critical group

$$C_{m+n_\infty}(g_m, x_m) \ne 0.$$

By Lemma 2.3 and Theorem 2.1, if $\|x_m\| < r$, we have

$$m - i_\infty + i_0 \le m + n_\infty \le m - i_\infty + i_0 + n_0,$$

a contradiction to the condition that $i_\infty + n_\infty \notin [i_0, i_0 + n_0]$. Hence $\|x_m\| \ge 2r$.

If $\#K_{c_m}(g_m) = +\infty$, by Lemma 2.3, there is a critical point $x_m \in E_m$ of g_m such that $g_m(x_m) = c_m$ and $\|x_m\| \ge 2r$.

But $f_m(z) = g_m(z)$ if $\|z\| \ge 2r$, therefore we have

$$f'_m(x_m) = 0 \quad \text{and} \quad f_m(x_m) = c_m.$$

By Lemma 3.1, it is easy to show that there exists a crtical point $x^* \in E$ of f such that $\|x^*\| \geq 2r$. We have proved our conclusion in Case (ii) .

Similarly we can prove our conclusion in Case (i) and the proof is complete.

Based on the local link idea[4,15] and Remark 2.1, similarly, we can prove the following local link theorem, we omit the details.

Theorem 3.1. *Suppose that H satisfies* (H1)–(H2) *and $G'(t,z)$ is bounded. If $B_0(t)$ is admissible for $B_\infty(t)$* (*cf. Remark* 2.4), *then* (HS) *has a nontrivial solution in each of the following two cases:*

(i) *$i_\infty \neq i_0 + n_0$, $G_0(t,z) = H(t,z) - \frac{1}{2}(B_0(t)z,z) > 0$ for $|z| > 0$ small, and either $n_\infty = 0$ or $G(t,z) \to -\infty$ as $|z| \to \infty$ uniformly in t;*

(ii) *$i_\infty + n_\infty \neq i_0$, $G_0(t,z) < 0$ for $|z|$ small, and either $n_\infty = 0$ or $G(t,z) \to +\infty$ as $|z| \to \infty$ uniformly in t.*

As a direct consequence, we have

Corollary 3.1. *Suppose that $G'(t,z)$ is bounded and $G'(t,z) = o(|z|)$ uniformly in t as $|z| \to 0$. If $n_\infty \neq 0$, then* (HS) *has a nontrivial solution in each of the following two cases:*

(i) *$G(t,z) > 0$ for $|z| > 0$ small, and $G(t,z) \to -\infty$ as $|z| \to \infty$ uniformly in t;*

(ii) *$G(t,z) < 0$ for $|z| > 0$ small, and $G(t,z) \to +\infty$ as $|z| \to \infty$ uniformly in t.*

§4. Strong Resonant Hamiltonian Systems

In this section, we consider the strong resonant Hamiltonian systems (HS) with $G(t,z)$ satisfying (1.1). This is motivated by Chang[7,12].

Lemma 4.1. *Under the assumptions of Theorem* 1.2, *the function f satisfies* $(\mathrm{PS})_c^*$ *for $c \neq 0$. Moreover any* $(\mathrm{PS})_c^*$ *sequence $\{x_m\}$, i.e., $x_m \in E_m$, $f(x_m) \to c$ and $f'_m(x_m) \to 0$, possesses a subsequence* (*still denoted by $\{x_m\}$*) *with the property that either $\{x_m\}$ strongly converges to a critical point of f in E or $c = 0$ and $(I - P_0)x_m \to 0$, $\|P_0 x_m\| \to \infty$.*

Proof. For $z \in E$, let $\psi(z) = \int_0^1 G(t,z)dt$. Then

$$f(z) = \frac{1}{2}\langle (A - B_\infty)z, z\rangle - \psi(z).$$

In view of the fact that $(A - B_\infty)P_m = P_m(A - B_\infty)$, the proof is just the same as [12, Lemma 3.1].

Proof of Theorem 1.2. By Lemma 4.1 and Lemma 2.3, using the same arguments as Step 1 of the proof of Theorem 1.1, there exist $m_2 > 0$, $r > 0$ and $a_0 > 0$ such that for $m \geq m_2$ and for any $0 < \epsilon \leq \frac{1}{2}$ there exists a $g_m \in C^2(E_m, \mathbf{R})$ satisfying Lemma 2.3 (1), (2), and $f_{ma} = g_{ma}$ for $|a| \geq a_0$.

Now we shall apply the abstract theorem on strong resonance problem in [8].

Let $S^{n_\infty} = E_0 \cup \{\infty\}$. We extend the function f_m to the enlarged space:

$$\tilde{f}_m(u,v) = \begin{cases} f_m(u,v) = \frac{1}{2}\langle (A - B_\infty)u, u\rangle - \psi(u,v), & (u,v) \in P_m E_0^\perp \times E_0, \\ \frac{1}{2}\langle (A - B_\infty)u, u\rangle \quad , & (u,\infty) \in P_m E_0^\perp \times \{\infty\}. \end{cases} \tag{4.1}$$

Let $r_1 = (c_1 + 1)\|(A - B_\infty)^\#\|$ and $b \geq a_0$ be the constants described in Step 2 of the proof of Theorem 1.1. Then $f_m(x) < b$ for $x \in (E_m^+ \cap V_{r_1}) \times E_m^- \times S^{n_\infty}$.

According to a theorem due to Chang[10,12], we have

$$H_q(P_mE_0^\perp \times S^{n_\infty}, \tilde{f}_{md}) \cong H_q((E_m^+ \cap V_{r_1}) \times E_m^- \times S^{n_\infty}, \tilde{f}_{md})$$
$$\cong H_{q-m}(S^{n_\infty})$$

for $-d \geq a_0$ large enough and independent of m. There is a pair of subordinate classes $[\sigma_{m1}] < [\sigma_{m2}]$ with

$$[\sigma_{m1}] \in H_m(P^mE_0^\perp \times S^{n_\infty}, \tilde{f}_{md}) \text{ and } [\sigma_{m2}] \in H_{m+n_\infty}(P_mE_0^\perp \times S^{n_\infty}, \tilde{f}_{md}).$$

Let

$$c_{mi} = \inf_{\tau \in [\sigma_{mi}]} \sup_{x \in |\tau|} \tilde{f}_m(x), \qquad i = 1, 2.$$

Then $d \leq c_{m1} \leq c_{m2} \leq b$. In the sense of subsequence, we have

$$c_i = \lim_{m \to \infty} c_{mi}, \qquad d \leq c_1 \leq c_2 \leq b.$$

Now by [12, Proposition 3.2], if $K_0(f)$ is compact (otherwise, our proof is complete), there is a constant $\epsilon_0 > 0$ such that either $c_2 > \epsilon_0$ is a critical value of f or $c_1 \leq \epsilon_0$ is a critical value of f.

In Case (1), $f(0) = 0$. Thus there must be at least one $c_i \neq 0$ for $i = 1, 2$, which is a critical value of f, and f has a nontrival critical point.

In Case (2), $f(0) < 0$. If $c_2 > \epsilon_0$, the proof is complete. If $c_1 \leq -\epsilon_0$, there must be an $m_3 \geq m_2$ such that for $m \geq m_3$, $c_1 - \frac{1}{4}\epsilon_0 \leq c_{m1} \leq c_1 + \frac{1}{4}\epsilon_0$. Take $m \geq m_3$ and $\epsilon = \min\{\frac{1}{2}, \frac{1}{4}\epsilon_0\}$. By Lemma 2.3 there exists a $g_m \in C^2(E_m, \mathbf{R})$ satisfying Lemma 2.3 (1), (2). It is easy to show that g_m satisfies the conclusion of [12, Lemma 1.1], and we can extend g_m to $\tilde{g}_m : P_mE_0^\perp \times S^{n_\infty} \to \mathbf{R}$, just as (4.1), which satisfies $\tilde{f}_{md} = \tilde{g}_{md}$, $|\tilde{f}_m(u,v) - \tilde{g}_m(u,v)| \leq \epsilon \leq \frac{1}{4}\epsilon_0$. Hence $[\sigma_{m1}] \in H_m(P_mE^\perp \times S^{n_\infty}, \tilde{g}_{md})$.

Let $c_{m1}^* = \inf_{\tau \in [\sigma_{m1}]} \sup_{x \in |\tau|} \tilde{g}_m(x)$. Then

$$c_{m1} - \frac{1}{4}\epsilon_0 \leq c_{m1}^* \leq c_{m1} + \frac{1}{4}\epsilon_0 \leq c_1 + \frac{1}{2}\epsilon_0 \leq -\frac{1}{2}\epsilon_0 < 0.$$

Therefore c_{m1}^* is a critical value of g_m.

If $\#K_{c_{m1}^*}(g_m) = +\infty$, then there exists a critical point $x_m \in E_m$ of g_m such that $\|x_m\| \geq 2r$.

If $\#K_{c_{m1}^*}(g_m) < +\infty$, by Principle I of [7], there is a critical point $x_m \in E_m$ of g_m such that $g_m(x_m) = c_{m1}^*$ and the critical group

$$C_m(g_m, x_m) = C_m(\tilde{g}_m, x_m) \neq 0.$$

By Lemma 2.3 (2) and Theorem 2.1, if $||x_m|| < r$, then we have

$$m - i_\infty + i_0 \leq m \leq m - i_\infty + i_0 + n_0,$$

a contradiction to the condition that $i_\infty \notin [i_0, i_0 + n_0]$.

Hence $||x_m|| \geq 2r$. But $f_m(z) = g_m(z)$ if $||z|| \geq 2r$, therefore we have

$$f_m'(x_m) = 0 \quad \text{and} \quad f^m(x_m) = c_{m1}^*.$$

By Lemma 4.1, it is easy to show that there exists a critical points $x^* \in E$ of f such that $||x^*|| \geq 2r$. Similarly, we prove the case (3). The proof is complete.

As a direct consequence, we have

Corollary 4.1. *Suppose that H satisfies* (H1)–(H3) *and* (1.1). *If* $[i_0, i_0+n_0]\cap[i_\infty, i_\infty+n_\infty]=\emptyset$, *then* (HS) *has a nontrivial solution.*

§5. Appendix

Let $\mathcal{L}_s(E)$ denote the space of the bounded selfadjoint linear operators from E to E and let $\mathcal{L}_c(E)$ denote the space of the bounded linear compact operators from E to E.

Let $Q\in\mathcal{L}_s(E)$, $S\in\mathcal{L}_s(E)\cap\mathcal{L}_c(E)$; $\dim\ker Q<+\infty$ and $Q+S$ is invertible. Set $P^+: E\to M^+(Q)$ and $P_s^+: E\to M^+(Q+S)$ are orthogonal projections.

Lemma 5.1. $P_s^+-P^+\in\mathcal{L}_c(E)$.

Proof. By [16, Problem VI 2.36 and Lemma VI 5.6], we have

$$P_s^+=\frac{1}{2}(U_s^2(0)+U_s(0)),\quad P^+=\frac{1}{2}(U^2(0)+U(0)),$$
$$U(0)=s-\lim_{\substack{r\to0\\t\to+\infty}}U_{r,t}(0),\quad U_s(0)=s-\lim_{\substack{r\to0\\t\to+\infty}}U_{s,r,t}(0),$$

where

$$U_{r,t}(0)=\frac{2}{\pi}\int_r^t(Q^2+y^2)^{-1}Q\,dy.$$
$$\begin{aligned}U_{s,r,t}(0)&=\frac{2}{\pi}\int_r^t((Q+S)^2+y^2)^{-1}(Q+S)\,dy\\&=U_{r,t}(0)+\frac{2}{\pi}\int_r^t((Q+S)^2+y^2)^{-1}S\,dy\\&\quad-\frac{2}{\pi}\int_r^t((Q+S)^2+y^2)^{-1}(S^2+SQ+QS)(Q^2+y^2)^{-1}Q\,dy.\end{aligned}$$

Since $Q+S$ is invertible and $S\in\mathcal{L}_s(E)\cap\mathcal{L}_c(E)$, it is easy to show that there are $K_1,K_2\in\mathcal{L}_c(E)$ such that

$$U_s(0)=U(0)+K_1+K_2,\quad U_s^2(0)=U^2(0)+K_3,$$

where

$$K_3=(K_1+K_2)^2+(K_1+K_2)U(0)+U(0)(K_1+K_2)\in\mathcal{L}_c(E).$$

Hence $P_s^+-P^+=\frac{1}{2}(K_1+K_2+K_3)\in\mathcal{L}_c(E)$.

Definition 5.1. *Let $B_i\in\mathcal{L}_s(E)\cap\mathcal{L}_c(E)$, $i=1,2$. We define the relative Morse index as*

$$\begin{aligned}I(B_1,B_2)&=\dim\left(M^+(A-B_1)\cap M^-(A-B_2)\right)\\&\quad-\dim\left((M^-(A-B_1)\oplus M^0(A-B_1))\cap(M^+(A-B_2)\oplus M^0(A-B_2))\right).\end{aligned}$$

Theorem 5.1. *Suppose that $B_i,S_i\in\mathcal{L}_s(E)\cap\mathcal{L}_c(E)$ satisfy $M^0(A-B_i-S_i)=\{0\}$, $||S_i||<||(A-B_i)^\#||^{-1}$, for $i=1,2$. Then*

$$\begin{aligned}&I(B_1+S_1,B_2+S_2)-\dim M^0(A-B_1)-\dim M^0(A-B_2)\\&\le I(B_1,B_2)\le I(B_1+S_1,B_2+S_2).\end{aligned}$$

Proof. By P_i^+, P_i^-, P_i^0, P_{is}^+ and P_{is}^- we denote the orthogonal projections from E to $M^+(A-B_i)$, $M^-(A-B_i)$, $M^0(A-B_i)$, $M^+(A-B_i-S_i)$ and $M^-(A-B_i-S_i)$ respectively, $i=1,2$.

Let $T = (P_2^+ + P_2^0)P_1^+ : P_1^+E \to (P_2^+E) \oplus P_2^0E$, $T_s = (P_2^+ + P_2^0)P_{1s}^+ : P_{1s}^+E \to P_2^+E \oplus P_2^0E$.

It is easy to show that the Fredholm index

$$\operatorname{ind} T = I(B_1, B_2), \quad \operatorname{ind} T_s = I(B_1 + S_1, B_2).$$

Let $T_s' = (P_2^+ + P_2^0)P_{1s}^+P_1^+ : P_1^+E \to P_2^+E \oplus P_2^0E$. By Lemma 5.1, $P_{1s}^+ - P_1^+ \in \mathcal{L}_c(E)$. Noticing that

$$T_s(P_{1s}^+P_1^+) = T_s' = T + (P_2^+ + P_2^0)(P_{1s}^+ - P_1^+)P_1^+,$$

we have

$$\operatorname{ind} T = \operatorname{ind} T_s' = \operatorname{ind} T_s + \operatorname{ind}(P_{1s}^+P_1^+).$$

Since $M^0(A - B_1 - S_1) = \{0\}$ and $||S_1|| < ||(A - B_1)^{\#}||^{-1}$, it is easy to show that $-\dim M^0(A - B_1) \le \operatorname{ind}(P_{1s}^+P_1^+) \le 0$. Hence

$$I(B_1 + S_1, B_2) - \dim M^0(A - B_1) \le I(B_1, B_2) \le I(B_1 + S_1, B_2). \tag{5.1}$$

Using the same arguments as above, we have

$$I(B_1 + S_1, B_2 + S_2) - \dim M^0(A - B_2) \le I(B_1 + S_1, B_2) \le I(B_1 + S_1, B_2 + S_2). \tag{5.2}$$

Our conclusion follows from (5.1) and (5.2).

Theorem 5.2. *Let $B_j(t)$ be continuous 1-periodic symmetric matrices in $\mathbf{R}^{2N}$ with the Maslov-type indices (i_j, n_j) and B_j be the operators, defined by* (2.2), *corresponding to $B_j(t)$, for $j = 1, 2$. Then $I(B_1, B_2) = i_2 - i_1 - n_1$.*

Proof. Case 1. $n_1 = n_2 = 0$. By [5, Lemma 3.2], for $j = 1, 2$, there is a continuous family of matrices $B_{js}(t)$ deforming $B_j(t)$ into the standard matrix $B^{i_j}(t)$, $0 \le s \le 1$. Let B_{js} and B^{i_j} be the operators corresponding to $B_{js}(t)$ and $B^{i_j}(t)$ respectively. Then $M^0(A - B_{js}) = 0$ and the Maslov-type index of $B^{i_j}(t)$ is $(i_j, 0)$ for $j = 1, 2$. By Theorem 5.1, $I(B_1, B_2) = I(B^{i_1}, B^{i_2})$. Since $B^{i_1}(t)$ and $B^{i_2}(t)$ are standard matrices, using the same argumants as [13, Lemma 2.8] or the similar arguments as [10, Theorem IV 1.2] we can prove that $I(B^{i_1}, B^{i_2}) = i_2 - i_1$. Hence $I(B_1, B_2) = i_2 - i_1$.

Case 2. $n_1 \ne 0$ or $n_2 \ne 0$. For $j = 1, 2$, let γ_j be the fundamental solution of (2.1) corresponding to $B_j(t)$ and γ_{jv} be the things described in Lemma 2.1.

For $-1 \le v \le 1$, we define

$$B_{jv}(t) = -J\dot{\gamma}_{jv}(t)\gamma_{jv}^{-1}(t), \quad 0 \le t \le 1.$$

Let B_{jv} be the compact operator corresponding to $B_{jv}(t)$, defined by (2.2). By Lemma 2.1 we have $M^0(A - B_{jv}) = 0$ for $v \ne 0$, $\|B_{jv} - B_j\| \to 0$ as $v \to 0$. Now by Theorem 5.1 and Case 1, for $0 < v \le 1$ and close enough to 0, we have

$$I(B_1, B_2) \le I(B_{1v}, B_{2,-v}) = i(\gamma_{2,-v}) - i(\gamma_{1v}) = i_2 - i_1 - n_1.$$

$$\begin{aligned} I(B_1, B_2) &\ge I(B_{1,-v}, B_{2v}) - \dim M^0(A - B_1) - \dim M^0(A - B_2) \\ &= i(\gamma_{2v}) - i(\gamma_{1,-v}) - n_1 - n_2 = i_2 - i_1 - n_1. \end{aligned}$$

Hence $I(B_1, B_2) = i_2 - i_1 - n_1$. The proof is complete.

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