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TO DETERMINE THAT ALL THE ROOTS OF A TRANSCENDENTAL POLYNOMIAL HAVE NEGATIVE REAL PARTS**

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Abstract

A transcendental polynomial which has all its roots lying in the left half-plane is called stable. In this paper, the stability of transcendental polynomials which are nearly as general as those discussed by L. S. Pontryagin are investigated through direct analytic method. For application of the method presented in this paper, the asymptotic stability of some differential-difference equations is investigated.

Keywords Transcendental polynomial, Direct analytic method, Asymptotic stability
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§1. Introduction

Let

$$h(z,t)=\sum_{m,n}a_{mn}z^mt^n, \tag{1.1}$$

where $a_{mn},z,t\in C$, and $m,n\in N^+\equiv\{0,1,2,\cdots\}$. From [16], we have the following:

Definition 1.1. *We call the term $a_{rs}z^rt^s$ the principal term of $h(z,\tau)$ if $a_{rs}\neq 0$ and the exponents r and s attain their maximum, that is, for each other term $a_{mn}z^mt^n$ in* (1.1), *for $a_{mn}\neq 0$, we have either $r>m, s>n$ or $r=m, s>n$, or $r>m, s=n$.*

We mention here that in this paper if $a_{rs}z^rt^s$ is the principal term of $h(z,t)$, so called is $a_{rs}z^re^{sz}$ of $h(z,e^\tau)$. Clearly, not every polynomial has a principal term, e.g., $h(z,t)=z+t$.

Definition 1.2. *For an entire function $H(z), z_0$ is said to be a p-root* (*n-root*) *of $H(z)$ if $H(z_0)=0$ and* $\operatorname{Re} z_0>0$ ($\operatorname{Re} z_0<0$).

For clearness, we indicate that in this paper the pure imaginary root means by nonzero one.

Let

$$N(H(z))\equiv\{\lambda\in C|H(\lambda)=0\},\quad C^-\equiv\{\lambda\in C|\operatorname{Re}\lambda<0\}.$$

Definition 1.3. *An entire function $H(z)$ is said to be stable if $N(H(z))\subset C^-$.*

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From [8] or [13], we know, generally, the problem of asymptotic stability of differential - difference equations is equvalent to the problem of stability of transcendental polynomials of the kind $h(z, e^z)$.

For $H(z) = h(z, e^z)$, in the early 1940's, L.S. Pontryagin[16] gave several theorems for the judgement of its stability. But, due to the difficulties of calculation, these results are not easy to be applied. Since then, many scholars, such as Chebotarev, Neimark, Michailov, Kababov, Tzypkin and Yuanxun Qin, etc., studied this problem from different directions and got some useful judging rules (see [2, 5, 18] and the literature therein). But the computation is still not quite direct and convenient. Since 1980's, direct analytic methods (see, e.g., [1, 4, 9, 14, 19–21]) have been used and some easy-checking results have been obtained. Enlightened by the method in [21], we generalize it and apply to the above problem in a more systematic and stricter way. In this paper, we lead in a time-delay parameter τ to construct a new transcendental polynomial and analyze the robustness of transecndental polynomial concerning τ by using implicit function theorem (see, e.g., [6]) and some kind of continuity about its roots with respect to its coefficients. The method we use here is analogous to D-decomposition method in [5].

For $h(\cdot,\cdot)$ given by (1.1), we define

$$H(z, \tau) \equiv h(z, e^\tau), \tag{1.2}$$

where $\tau \geq 0$ is a time-delay parameter. The following lemmas (see [16]) tell us that in order to study the stability of $H(z) = H(z, 1)$, it is enough to take some kinds of transcendental polynomials into consideration.

Lemma 1.1. *If polynomial $h(z, t)$ has no principal term, then $H(z, 1) = h(z, e^z)$ possesses an infinity of roots with arbitrarily large positive real parts.*

Lemma 1.2. *Suppose polynomial $h(z, t)$ has principal term $a_{rs}z^r t^s$. By $\chi_*^{(s)}(t)$ we denote the coefficient of z^r in the polynomial $h(z, \tau)$. If the function $\chi_*^{(s)}(e^z)$ has one p-root, then $H(z, 1) = h(z, e^z)$ has an unbounded set of p-roots. If all the roots of the function $\chi_*^{(s)}(e^z)$ are n-roots, then $H(z, 1) = h(z, e^z)$ can have only a bounded number of p-roots.*

It is worth mentioning that in this paper the roots are enumerated in multiplicity.

For simplicity, we assume

(A1) $h(z, t)$ is a polynomial with real coefficients which has principal term $a_{rs}z^r t^s$ ($a_{rs} > 0$).

(A2) $N\Big(\frac{\partial^r h(z,t)}{\partial z^r}\Big|_{t=e^z} \Big) \subset C^-$.

Actually in (A2), $\frac{\partial^r h(z,t)}{\partial z^r}\Big|_{t=e^z} = r!\chi_*^{(s)}(e^z)$.

Proposition 1.1. *If $H(z, 0) = h(z, 1)$has root zero or odd number of positive real roots, then $H(z, \tau)$ is not stable for all $\tau \geq 0$.*

Proof. If $h(0, 1) = 0$, it is easy to know $H(0, \tau) = 0$ ($\forall \tau \geq 0$). So $h(z, \tau)$ ($\tau \geq 0$) is not stable when $h(z, 1)$ has root zero. If $h(z, 1)$ has odd number of positive real roots, then it is necessary that $H(0, 0) < 0$. Since from (1.2), for any given $\tau \geq 0$, $H(z, \tau) > 0$ for $z > 0$ large enough, from continuity we know that $H(z, \tau)$ has a positive real root. Hence, $H(z, \tau)$ is not stable.

In fact, if $h(0, 1) \leq 0, H(z, \tau)$ must not be stable. So we also assume

(A3) $h(0,1) > 0$.

Sometimes, we write $\chi_*^{(s)}(t)$ (without loss of generality, we assume $a_{rs} > 0$) as

$$\chi_*^{(s)}(t) \equiv a_{rs}(t-t_1)(t-t_2)\cdots(t-t_s), \tag{1.3}$$

where

$$t_j = e^{\alpha_j + i\beta_j} \quad (\alpha_j < 0,\ 0 \le \beta_j < 2\pi,\ j = 1, 2, \cdots, s). \tag{1.4}$$

For definiteness, we make some definitions here: If $a > 0, \tan^{-1}\frac{a}{0} \equiv \frac{\pi}{2}, \tan^{-1}\frac{-a}{0} \equiv -\frac{\pi}{2}$. For $a \ge b$,

$$a \bigvee b \equiv \max\{a, b\} = a; \quad (a, b) = \emptyset.$$

$[\alpha]$ denotes the greatest integer not larger than α. Also for $x \in \boldsymbol{R}^n$, we denote $g(x)|f(x)(g(x) \nmid f(x))$ to mean that polynomial $f(x)$ is divisible (indivisible) by polynomial $g(x)$.

§2. Preliminaries

For $H(z,\tau)$ given by (1.2), we introduce a polynomial $(1-\theta z)^{2s} h(z, \frac{(1+\theta z)^2}{(1-\theta z)^2})$, namely,

$$G(z,\theta) \equiv \sum_{m,n} a_{mn} z^m (1-\theta z)^{2s-2n} (1+\theta z)^{2n} \quad (\theta \ge 0), \tag{2.1}$$

where θ is a parameter. It is worth mentioning that $G(z,\theta)$ is an algebraic polynomial, not a transcendental one.

The following lemmas are used to determine the pure imaginary roots of $H(z,\tau)$ as well as the corresponding time delays at which the number of p-roots may alter.

Lemma 2.1. *For $(y,\tau,\theta) \in (\boldsymbol{R}^+)^3, H(iy,\tau)$ is equivalent to $G(iy,\theta) = 0$, where*

$$\tau y = 4\tan^{-1}(y\theta) \pmod{2\pi}. \tag{2.2}$$

Its proof is a modification of the one in [21].

Lemma 2.2. *For $f(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n (a_0, a_1, \cdots, a_n \in \boldsymbol{R}, a_0 \ne 0), f(z)$ has pure imaginary roots iff*

$$a_n \ne 0, \quad \Delta_{n-1} \equiv \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & \cdots & \vdots \\ a_5 & a_4 & a_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{2n-3} & a_{2n-4} & a_{2n-5} & \cdots & a_{n-1} \end{vmatrix} = 0, \tag{2.3}$$

where Δ_{n-1} is the $(n-1)$-order leading principal minor of Hurwitz matrix corresponding to polynomial $f(z)$. Here, we define $\Delta_{n-1} \equiv a_n$ if $n \le 1$.

Its proof can be found in [7].

For a given $\theta \ge 0, G(z,\theta)$ is a polynomial of z, so its Hurwitz determinant $\Delta_*(\theta)$ can be defined. Let

$$\Delta_*(\theta) \equiv \begin{cases} \Delta_{r-1}, & \text{when} \quad \theta = 0; \\ \Delta_{r+2s-1}(\theta), & \text{when} \quad \theta > 0. \end{cases} \tag{2.4}$$

Generally speaking, $\Delta_*(\theta)$ is not right continuous at $\theta = 0$, e.g., $H(z,\tau) = (z+1)^2(e^{\tau z}+1)$. From (2.1) and (2.4) respectively, we have $\Delta_*(0) = 4$ and $\Delta_*(\theta) = 0$ $(\theta > 0)$.

In order to study the stability of $H(z) = H(z,1)$, we consider the problem in the following order. First, we lead in a time-delay parameter τ and get $H(z,\tau)$; then we enumerate the changing number of p-roots of $H(z,\tau)$ as τ varies from 0 to 1. Since $G(z,\theta)$ is obtained by substituting $e^{\tau z} = \frac{(1+\theta z)^2}{1-\theta z)^2}$ into $(1-\theta z)^{2s}H(z,\tau)$, we know $\Delta_*(\theta)$ is determined definitely by the coefficients of $H(z,1)$. In our method, $\Delta_*(\theta)$ is utilized to get the set of pure imaginary roots of $H(z,\tau)$ as well as the corresponding set of time delays. If $\Delta_*(\theta) \equiv 0$ $(\theta > 0)$, e.g., $H(z,1) = z^2 + a^2$ or $e^z + 1$, our method is not applicable. So, it is necessary to assume

(A4)' $\Delta_*(\theta) \not\equiv 0$ $(\theta > 0)$.

Meanwhile, since implicit function theorem is used in most cases of this paper, we also assume

(A5) $H(z,\tau)$ has no multiple pure imaginary roots for each $\tau \geq 0$.

The following propositions are related to (A4)'. They demonstrate that under assumption (A1)-(A2), (A4)' is equivalent to the assumption below:

(A4) For every $a \geq 0, (z^2 + a^2) \nmid h(z,t)$.

Relatively speaking, it is easier to test (A4) than (A4)'. Let

$$Y_\tau \equiv \{y > 0 | H(iy,\tau) = 0\} \tag{2.5}$$

and

$$Y \equiv \bigcup_{\tau \geq 0} Y_\tau. \tag{2.6}$$

Now, obviously, the set of pure imaginary roots of $H(z,\tau)$ is $\{\pm yi | y \in Y\}$. Since the imaginary roots of $H(z,\tau)$ appear in conjugate pairs, it is enough to consider $\{yi | y \in Y\}$. First, we have

Lemma 2.3. *If Y is finite, then* (A4) *is equivalent to* (A4)'.

Proof. If for certain $a > 0, (z^2 + a^2) | h(z,t)$, then $H(z,\tau) = h(z, e^{\tau z})$ has root ai for each $\tau \geq 0$. Thus, we get $\Delta_*(\theta) \equiv 0$ (for $\theta \geq 0$) from Lemma 2.2. This proves that if (A4)' holds then (A4) also holds.

Conversely, if Y is finite and $\Delta_*(\theta) \equiv 0$ (for $\theta > 0$), we let $\theta = \theta_n > 0, n = 1,2,3,\cdots$, such that $G(iy_n, \theta_n) = 0$ and $\theta_n \to 0$ (as $n \to \infty$). From Lemma 2.1, we get

$$H\Big(iy_n, \frac{4}{y_n}\tan^{-1}(\theta_n y_n)\Big) = 0.$$

Since Y is finite, $\{y_n\}$ must have a subsequence of which all elements take the same value. Without loss of generality, we suppose $y_n \equiv y_0$ for every $n \in N$. Therefore when $\theta_n \to 0, \frac{4}{y_0}\tan^{-1}(\theta_n y_0) \to 0$. Considering that $H(iy_0,\tau)$ is an analytic function of τ, which has only isolated zeros if it is not a constant, we conclude $H(iy_0,\tau) \equiv 0$ (for $\tau \geq 0$). Now, it is easy to derive that $(z^2 + y_0^2) | h(z,t)$. Thus, we prove that if (A4) holds then (A4)' also holds.

The equivalence of (A4) and (A4)' is now proved.

On the other hand, if Y is infinite, the above proposition may not hold, e.g., $h(z,t) = t+1$. Now, we may ask: Under what condition is Y finite?

Let

$$H(z,\tau) = h(z, e^{\tau z}) \equiv \sum_{k=0}^{s} p_k^{(s)}(z) e^{k\tau z}, \tag{2.7}$$

where $p_k^{(s)}(z)$ $(0 \le k \le s)$ is a real polynomial with its order not greater than r. Successively define

$$p_k^{(j)}(z) \equiv \det \begin{pmatrix} p_{j-k}^{(j+1)}(-z) & p_{k+1}^{(j+1)}(z) \\ p_{j+1}^{(j+1)}(-z) & p_0^{(j+1)}(z) \end{pmatrix}, \quad 0 \le k \le j \le s-1, \tag{2.8}$$

and

$$\widetilde{Y_0} \equiv \begin{cases} (0,\infty), & \text{if} \quad p_0^{(0)}(z) \equiv 0; \\ \{y>0 | p_0^{(0)}(iy)=0\}, & \text{if} \quad p_0^{(0)}(z) \not\equiv 0. \end{cases} \tag{2.9}$$

We have the following

Proposition 2.1. *$Y \subset \widetilde{Y_0}$. If* (A1)–(A2) *hold, then $\widetilde{Y_0}$ is finite; therefore, Y is finite.*

Its proof is given in Section 6.

Generally, for $H(z,\tau)$ given by (1.2), from Lemma 2.1, we have

$$Y = \bigcup_{\theta\in\Theta} \{y>0 | G(iy,\theta)=0\}, \tag{2.10}$$

where

$$\Theta \equiv \{\theta > 0 | \Delta_*(\theta) = 0\}. \tag{2.11}$$

Also, we need the following

Lemma 2.4. *If there exist $0 \le \alpha < \beta$ and a real analytic function $y(\tau)$, such that $H(iy(\tau),\tau)=0$ for $\tau\in[\alpha,\beta]$, then $\Delta_*(\theta)\equiv 0$ for $\theta>0$.*

Its proof is given in Section 6.

This lemma denies the possibility that implicit funciton $z=z(\tau)$ (such that $H(z(\tau),\tau)=0$) will move through the imaginary axis along the real axis as τ varies. In addition, since nonpositive p-roots always appear in double, we conclude that the number increment (or decrement) of p-roots is even.

§3. Main Results

In this section, we will present the main results of this paper. Denote by Ω_τ $(\tau \ge 0)$ the number of p-roots of $H(z,\tau)$. We have the following

Theorem 3.1. *If* (A1)–(A5) *hold, then, for certain τ_0, we have the following four steps to determine Ω_{τ_0}.*

Step 1. *From* (2.10)–(2.11), *calculate $Y \equiv \{y_j | 1 \le j \le m^*\}$.*

Step 2. *For each $y_j \in Y (1 \le j \le m^*)$, calculate*

$$\{\tau_j^\nu | H(iy_j, \tau_j^\nu)=0, 0 \le \tau_j^\nu y_j < 2\pi\} \equiv \{\tau_j^\nu | 1 \le \nu \le \nu_j\},$$

and therefore obtain

$$\widetilde{M} \equiv \Big\{\tau_{j,k}^\nu \equiv \tau_j^\nu + \frac{2k\pi}{y_j} \Big| j=1,2,\cdots,m^*; \nu=1,2,\cdots,\nu_j; k\in N^+\Big\}.$$

Step 3. *For each $\tau_{j,k}^\nu \in \widetilde{M}\cap[0,\tau_0)$, there exists $z(\tau)=\operatorname{Re} z(\tau)+i\operatorname{Im} z(\tau)$ in the neighborhood of $\tau_{j,k}^\nu$, such that $H(z(\tau),\tau)=0$ and $z(\tau_{j,k}^\nu)=iy_j$, where $\operatorname{Re} z(\tau)$ and $\operatorname{Im} z(\tau)$ are real analytic functions of τ. Differentiate both sides of $H(z,\tau)=0$ concerning τ and calculate $\frac{d\operatorname{Re} z(\tau)}{d\tau}\Big|_{\tau=\tau_{j,k}^\nu}$, $\frac{d^2\operatorname{Re} z(\tau)}{d\tau^2}\Big|_{\tau=\tau_{j,k}^\nu}$ and so on, till we get the least number $\mu=\mu(j,k,\nu)$ such that $d_\mu = \frac{d^\mu \operatorname{Re} z(\tau)}{d\tau^\mu}\Big|_{\tau=\tau_{j,k}^\nu} \ne 0$.*

Step 4. *Define*

$$\sigma(y_j, \tau_{j,k}^{\nu}) = \sigma_+(y_j, \tau_{j,k}^{\nu}) - \sigma_-(y_j, \tau_{j,k}^{\nu}),$$

where

$$(\sigma_+(y_j, \tau_{j,k}^{\nu}), \sigma_-(y_j, \tau_{j,k}^{\nu})) \equiv \begin{cases} (1,1), & \text{if } \mu \text{ is even and } d_\mu > 0; \\ (0,0), & \text{if } \mu \text{ is even and } d_\mu < 0; \\ (1,0), & \text{if } \mu \text{ is odd and } d_\mu > 0; \\ (0,1), & \text{if } \mu \text{ is odd and } d_\mu < 0. \end{cases}$$

Stipulate that $\sigma_\pm(y, \tau) = 0$, *if* $y \notin Y_\tau$ *and* $\sigma_-(y, 0) = 0$ *for* $y \in Y_0$. *Thus, we get*

$$\Omega_{\tau_0} = \Omega_0 + 2\sum_{j=1}^{m^*}\sum_{\nu=1}^{\nu_j}\sum_{0\le \nu_{j,k}^{\nu}<\tau_0} \sigma(y_j, \tau_{j,k}^{\nu}) - 2\sum_{y\in Y_{\tau_0}} \sigma_-(y, \tau_0).$$

Moreover, $H(z, \tau_0)$ *is stable iff* $\tau_0 \notin \widetilde{M}$ *and* $\Omega_{\tau_0} = 0$.

Its proof will be given in Section 4.

Remark 3.1. Step 1 is used to obtain all the possible pure imaginary roots. We will see from Section 4, as τ varies, the change for the number of p-roots can only occur when $H(z, \tau)$ has pure imaginary roots. If $Y = \emptyset$, then $\Omega_\tau = \Omega_0$ for $\tau \geq 0$. Also from (A3), we notice that Ω_0 must be an even number. Since p-roots always appear in pairs, Ω_τ is also an even number. Step 2 is used to calculate all the possible time delays at which the number of p-roots may alter. From Lemma 2.2, it is easy to know $\nu_j \geq 1$ $(j = 1, 2, \cdots, m^*)$. In Step 3, assumption (A5) and implicit function theorem guarantee the existence of analytic function $z(\tau)$, while assumption (A4) and Lemma 2.5 make it sure that we need only finite steps to get μ. In Step 4, for the case μ is even and $d_\mu > 0$, it means that in the neighborhood of $\tau_{j,k}^{\nu}$, as τ increases from $\tau < \tau_{j,k}^{\nu}$ to $\tau > \tau_{j,k}^{\nu}$, p-root $z(\tau)$ turns into iy_j at $\tau_{j,k}^{\nu}$ and later becomes a p-root again. Similar discussions can be implemented on other cases.

Let

$$E_{\tau_0}^- = \max_{1\le j\le m^*}\max_{1\le \nu\le \nu_j}\Big\{\tau_j^{\nu} + \frac{2\pi}{y_j}\Big[\frac{(\tau_0 - \tau_j^{\nu})y_j}{2\pi}\Big]\Big\},$$

$$E_{\tau_0}^+ = \min_{1\le j\le m^*}\min_{1\le \nu\le \nu_j}\Big\{\tau_j^{\nu} + \frac{2\pi}{y_j}\Big[\frac{(\tau_0 - \tau_j^{\nu})y_j}{2\pi}\Big] + \frac{2\pi}{y_j}\Big\}.$$

From the theorem above, we have the following

Corollary 3.1. *For* $\tau_0 \geq 0$, *if* $H(z, \tau_0)$ *has no pure imaginary root, then there exists a maximum time-delay interval* O_{τ_0}, *which contains* τ_0, *and for each* $\tau \in O_{\tau_0}, H(z, \tau)$ *and* $H(z, \tau_0)$ *have the same number of p-roots and no pure imaginary root as well. Here,* O_{τ_0} *can be expressed as*

$$O_{\tau_0} \equiv \begin{cases} \mathbb{R}^+, & \text{if } Y = \emptyset; \\ (E_{\tau_0}^-, E_{\tau_0}^+), & \text{if } Y \neq \emptyset \text{ and } E_{\tau_0}^- \geq 0; \\ \big[0, E_{\tau_0}^+\big), & \text{if } Y \neq \emptyset \text{ and } E_{\tau_0}^- < 0. \end{cases} \tag{3.1}$$

In Section 4, we will prove Theorem 3.1 under the assumption (A5). But in fact, we find that sometimes (A5) may be reduced. For example, we only need the condition "$H(z, \tau)$ has no multiple pure imaginary root in $0 \leq \tau < \tau_0$" to study the stability of $H(z, \tau_0)$. More generally, if $H(z, \tau)$ has multiple pure imaginary roots, our method is still feasible. But under the circumstances, we should introduce perturbation method, the efficiency of which can be proved through a limit process. Let

$$\widetilde{H}(z,x) \equiv \sum_{m,n} a_{mn}(x) z^m e^{nz}, \tag{3.2}$$

where $a_{mn}(x)$ is a real continuous function, $x \in \boldsymbol{R}^{n_1} (n_1 \in N)$. If at $x = x_0, \widetilde{H}(z,x_0)$ has principal term $a_{rs}(x_0) z^r e^{sz}$, we denote by $\chi_*^{(s)}(e^z, x_0)$ the coefficient term of z^r. We have the following

Lemma 3.1. *If $\widetilde{H}(z,x_0)$ has principal term $a_{rs}(x_0) z^r e^{sz} (a_{rs}(x_0) > 0)$, but no pure imaginary root, and $N(\chi_*^{(s)}(e^z,x_0)) \subset C^-$, then $\exists \epsilon > 0, M = M(x_0,\epsilon) > 0$ and $c = c(x_0,\epsilon) > 0$, such that when $|x - x_0| < \epsilon, \widetilde{H}(z,x)$ has principal term and no pure imaginary root, $N(\chi_*^{(s)}(e^z,x_0)) \subset C^-$, and all the p-roots and pure imaginary roots of $\widetilde{H}(z,x)$ lie in the open disk $|z| < M$. In addition, $\widetilde{H}(z,x)$ and $H(z,x_0)$ have the same number of p-roots in $|z| \le M$ and all the roots of $\widetilde{H}(z,x)$ in $|z| > M$ satisfy* $\operatorname{Re} z < -c$.

Its proof is similar to the first part of the proof of Lemma 4.1. It can be done by taking into account the continuity of the coefficients (similarly see e.g. [15]) of $\widetilde{H}(z,x)$. Due to the space limit, we omit it here.

Corollary 3.2. *If $\widetilde{H}(z,x_0)$ has principal term and no pure imaginary root, $N(\chi_*^{(s)}(e^z,x_0)) \subset C^-$, and meanwhile, there exists a real sequence $\{x_n\}$ such that $x_n \to x_0 (n \to \infty)$ and $\widetilde{H}(z,x_n)$ has h p-roots but no pure imaginary root, then $\widetilde{H}(z,x_n)$ has h p-roots.*

Its proof is direct from Lemma 3.4.

When $x \in \boldsymbol{R}^2$, let $x = (\tau,\epsilon), \widetilde{H}(z,x) = \widetilde{H}(z,\tau,\epsilon)$ and $\chi_*^{(s)}(e^z,x) \equiv \chi_*^{(s)}(e^z,\tau,\epsilon)$. We have the following corollary, which constructs the theory of our perturbation method.

Corollary 3.3. *Suppose $\alpha(\epsilon), \beta(\epsilon)$ are two real continuous functions of ϵ in the neighborhood of $\epsilon = 0$, such that when $\tau \in (\alpha(\epsilon),\beta(\epsilon))$, $\widetilde{H}(z,\tau,\epsilon)$ has positive principal term as well as h p-roots and no pure imaginary root, $N(\chi_*^{(s)}(e^z,\tau,\epsilon)) \subset C^-$. Meanwhile, if $\alpha(\epsilon) \to \alpha$, $\beta(\epsilon) \to \beta (\alpha < \beta)$ as $\epsilon \to 0$ and for each $\tau \in (\alpha,\beta)$, $\widetilde{H}(z,\tau,0)$ has principal term and $N(\chi_*^{(s)}(e^z,\tau,0)) \subset C^-$, then $\widetilde{H}(z,\tau,0)$ has h p-roots for each $\tau \in (\alpha,\beta)$.*

Proof. For $\tau \in (\alpha,\beta)$, we know $\exists \epsilon_1 > 0$ such that $\tau \in (\alpha(\epsilon),\beta(\epsilon))$ if $|\epsilon| < \epsilon_1$. Since $\widetilde{H}(z,\tau,\epsilon)$ has principal term as well as h p-roots and no more imaginary root, and $N(\chi_*^{(s)}(e^z,\tau,\epsilon)) \subset C^-$, and in addition, $\widetilde{H}(z,\tau,0)$ has principal term and $N(\chi_*^{(s)}(e^z,\tau,0)) \subset C^-$, we obtain, from Corollary 3.5, that $\widetilde{H}(z,\tau,0)$ has h p-roots.

For $H(z,\tau) = \sum\limits_{m,n} a_{mn} z^m e^{n\tau z}$, when $\tau > 0$, let $\tilde{z} = \tau z$. Through some simplifications, we have

$$\widetilde{H}(\tilde{z},\tau) = \sum_{m,n} a_{mn} \tau^{r-m} \tilde{z}^m e^{n\tilde{z}}. \tag{3.3}$$

Obviously, (3.3) is a special case of (3.2) and the stability of $H(z,\tau)$ $(\tau > 0)$ is equivalent to that of $\widetilde{H}(\tilde{z},\tau)$.

§4. Proof of Theorem 3.1.

In this section, we will prove Theorem 3.1. In order to do so, we first present some lemmas.

Lemma 4.1. *Suppose* (A1)-(A5) *hold. Then* $\exists\epsilon > 0$ *such that for* $0 < \tau \le \epsilon, H(z,\tau)$ *has no pure imaginary root but* $\Omega_0^+ \equiv \Omega_0 + 2\sum\limits_{y\in Y_0} \sigma_+(y,0)$ *p-roots.*

Proof. First, we prove the following several results.

1. $\exists M_1 > 0$ such that all the roots of $H(z,0)$ as well as the pure imaginary roots and p-roots of $H(z,\tau)(\tau > 0)$ lie in the disk $|z| \le M_1$, and all the roots of $H(z,\tau)$ $(\tau > 0)$ in $|z| > M_1$ satisfy $\operatorname{Re} z \le -\frac{\epsilon_0}{2\tau}$, where $\epsilon_0 = \min\limits_{1\le j\le s} |\alpha_j|$.

It is easy to verify that $\exists M_0 > 0$ such that all the roots of $H(z,0)$ lie in $|z| \le M_0$. Now, we set

$$M_1 = \max\Big\{1, M_0, \max_{m<r,n\le s}\Big\{\frac{r(s+1)|a_{mn}|}{\epsilon}\Big\}\Big\},$$

where

$$\epsilon = a_{rs}\min_{1\le j\le s}\Big(e^{\frac{\alpha_j}{2}} - e^{\alpha_j}\Big)^s.$$

If $|z| > M_1$, then

$$\frac{1}{|z|} < \min\Big\{1, \min_{m<r,n\le s}\Big\{\frac{\epsilon}{r(s+1)|a_{mn}|}\Big\}\Big\}.$$

Therefore

$$\sum_{m<r,n\le s} |a_{mn}||z|^{m-r} < \epsilon < a_{rs}\prod_{i=1}^{s}(1 - e^{\alpha_j}),$$

that is

$$\sum_{m<r,n\le s} |a_{mn}||z|^{m} < \epsilon|z|^r < a_{rs}\prod_{i=1}^{s}(1 - e^{\alpha_j})|z|^r. \tag{4.1}$$

If z is a root of $H(z,\tau)$ such that $|z| > M_1$ and $\operatorname{Re} z \ge 0$, from (1.3), we have

$$a_{rs}(1 - t_1e^{-\tau z})(1 - t_2e^{-\tau z})\cdots(1 - t_se^{-\tau z}) = -\sum_{m<r,n\le s} a_{mn}z^m e^{(n-s)\tau z}. \tag{4.2}$$

Taking modular forms on both sides and making some estimations, we get

$$a_{rs}(1 - e^{\alpha_1})(1 - e^{\alpha_2})\cdots(1 - e^{\alpha_s})|z|^r \le \sum_{m<r,n\le s} |a_{mn}||z|^m,$$

which obviously contradicts (4.1). Thus, all the pure imaginary roots and p-roots of $H(z,\tau)$ $(\tau > 0)$ lie in $|z| \le M_1$.

Suppose z is a root of $H(z,\tau)$ in $|z| > M_1$. From the above conclusion, we know $\tau > 0$ and $\operatorname{Re} z < 0$. Now turn (4.2) into

$$a_{rs}(e^{\tau z} - t_1)(e^{\tau z} - t_2)\cdots(e^{\tau z} - t_s) = -\sum_{m<r,n\le s} a_{mn}z^m e^{n\tau z}. \tag{4.3}$$

Since $|e^{\tau z} - t_j| \ge |e^{\tau \operatorname{Re} z} - e^{\alpha_j}|$ $(1 \le j \le s)$, estimating (4.3) by utilizing inequality (4.1), we get

$$a_{rs}\Big(\min_{1\le j\le s}|e^{\tau\operatorname{Re} z} - e^{\alpha_j}|\Big)^s|z|^r \le \sum_{m<r,n\le s}|a_{mn}||z|^m < \epsilon|z|^r.$$

Thus, we have inequality

$$\min_{1\le j\le s}|e^{\tau\operatorname{Re} z} - e^{\alpha_j}|^s < \min(e^{\frac{\alpha_j}{2}} - e^{\alpha_j})^s,$$

from which, we can easily obtain $\mathrm{Re}\, z \le -\frac{\epsilon_0}{2\tau}$.

2. Let $M = M_1 + 1$. Then $\exists \epsilon_1 > 0$ such that when $0 < \tau \le \epsilon_1, H(z,\tau)$ and $H(z,0)$ have the same number of p-roots in the disk $|z| \le M$. In addition, all the roots of $H(z,\tau)$ keep certain kind of continuity with respect to τ.

It is easy to prove $\exists \epsilon_2 > 0$ such that when $|z| = M, |H(z,0)| \ge \epsilon_2 > 0$. Since when $\tau \to 0$, $H(z,\tau) \to H(z,0)$ uniformly in $|z| \le M$, we know $\exists \epsilon_1 > 0$ ($\epsilon_1 < \epsilon_2$) such that when $0 < \tau \le \epsilon_1$, $|H(z,\tau) - H(z,0)| < \epsilon_2$. Now, we know on the contour $|z| = M$, $|H(z,\tau) - H(z,0)| < |H(z,0)|$. By Rouché's theorem, we conclude that $H(z,\tau)$ and $H(z,0)$ have the same number of p-roots in the disk $|z| \le M$.

Let

$$B_M(\tau) \equiv N(H(z,\tau)) \cap \{z \in C \| z| \le M\}.$$

For $\tau_1 \ge 0, \tau_2 \ge 0$, we define the usual Hausdorff distance

$$\mathrm{dis}(B_M(\tau_1), B_M(\tau_2)) \equiv \max\Big\{\sup_{z \in B_M(\tau_2)} \mathrm{dis}(z, B_M(\tau_1)), \sup_{z \in B_M(\tau_1)} \mathrm{dis}(z, B_M(\tau_2))\Big\}.$$

The continuity we indicate here is actually the one of $B_M(\tau)$ to τ in the meaning of Hausdorff distance.

Now we come to the right continuity of p-roots of $H(z,\tau)$ at $\tau = 0$. For each $z_j \in N(H(z,0)) \equiv \{z_j | j = 1, 2, \cdots, r\}$ and arbitrary $\epsilon^* > 0$, we can find a disk $C_j = \{z \in C \| z - z_j| \le 2\epsilon_j^*\}$, where $\epsilon_j^* < \frac{\epsilon^*}{2}$ $(1 \le j \le r)$ is a positive number such that $C_j \cap N(H(z,0)) = \{z_j\}$. For $D_j \equiv \{z \in C \| z - z_j| \le \epsilon_j^*\}$, similarly, we know, $\exists \delta_j(\epsilon_j^*) > 0$ such that when $0 \le \tau \le \delta_j(\epsilon_j^*)$, $H(z,\tau)$ and $H(z,0)$ have the same number of roots in D_j. Let $\delta(\epsilon^*) = \min\limits_{1 \le j \le r} \delta_j(\epsilon_j^*)$. We obtain $\mathrm{dis}(B_M(\tau), B_M(0)) \le 2 \max\limits_{1 \le j \le r} \epsilon_j^* < \epsilon^*$ when $0 \le \tau < \min\{\epsilon_1, \delta(\epsilon^*)\}$. Thus we prove the right continuity we mean.

3. $\exists \epsilon_3 > 0$ such that $H(z,\tau)$ has no pure imaginary root if $0 < \tau \le \epsilon_3$.

Under assumptions (A1)–(A4), $\Delta_*(\theta) \not\equiv 0 (\theta > 0)$. Therefore, Θ is also finite. From the process of Steps 1–2 (see Theorem 3.1), we know $T_0 = \{\tau_j^\nu | j = 1, 2, \cdots, m^*; \nu = 1, 2, \cdots, \nu_j\}$ is also finite. Let

$$\epsilon_3 < \tau^0 \equiv \begin{cases} \min\limits_{\tau \in T_0} \tau, & \text{if} \quad T_0 \ne \emptyset, \\ \infty, & \text{if} \quad T_0 = \emptyset. \end{cases}$$

We know $H(z,\tau)$ has no pure imaginary root when $0 < \tau \le \epsilon_3$.

With the results of **1**–**3**, we can prove Lemma 3.1 now.

Let $\epsilon_4 = \min\{\epsilon_1, \epsilon_3\}$. Then when $0 < \tau \le \epsilon_4$, we know $H(z,\tau)$ has no pure imaginary root and $H(z,\tau)$ and $H(z,0)$ have the same number of roots in $|z| \le M$. Moreover, the roots of $H(z,\tau)$ keep right-continuity at $\tau = 0$, and the roots of $H(z,\tau)$ in $|z| > M$ satisfy $\mathrm{Re}\, z \le -\frac{\epsilon_0}{2\tau} \le -\frac{\epsilon_0}{2\epsilon_4}$. Since if $H(z,0)$ has pure imaginary roots, they must be simple, for each $y \in Y_0$, by implicit function theory, we know, $\exists z(\tau) = \mathrm{Re}\, z(\tau) + i\mathrm{Im}\, z(\tau)$ such that $H(z(\tau),\tau) = 0$ and $z(0) = iy$, where $\mathrm{Re}\, z(\tau)$ and $\mathrm{Im}\, z(\tau)$ are real analytic functions of τ. From (A4)' and Lemma 2.5, we have $\mathrm{Re}\, z(\tau) \not\equiv 0$. Thus, $\exists 0 < \epsilon < \epsilon_4$ such that when $0 < \tau \le \epsilon$, for each $y \in Y_0$, $z(\tau) = \mathrm{Re}\, z(\tau) + i\mathrm{Im}\, z(\tau)$ satisfies $|\mathrm{Re}\, z(\tau)| > 0$. Now, determining the changing inclination of each $\mathrm{Re}\, z(\tau)$ at $\tau = 0$ by its dirivatives (stated as in Steps 3 - 4), we conclude that when $0 < \tau \le \epsilon, H(z,\tau)$ has no pure imaginary root and the number of p-roots is $\Omega_0^+ = \Omega_0 + 2 \sum\limits_{y \in Y_0} \sigma_+(y, 0)$.

Similarly, we have the following

Lemma 4.2. *Suppose* (A1)–(A5) *hold. Then for* $\tau_0 > 0$, $\exists \epsilon > 0$ *such that when* $0 < |\tau - \tau_0| < \epsilon, H(z,\tau)$ *has no pure imaginary root; when* $\tau_0 - \epsilon < \tau < \tau_0, H(z,\tau)$ *has* $\Omega_{\tau_0} + 2 \sum\limits_{y \in Y_{\tau_0}} \sigma_-(y, \tau_0)$ *p-roots; when* $\tau_0 < \tau < \tau_0 + \epsilon, H(z,\tau)$ *has* $\Omega_{\tau_0} + 2 \sum\limits_{y \in Y_{\tau_0}} \sigma_+(y, \tau_0)$ *p-roots.*

Now, we are in the stage to prove Theorem 3.1.

Proof of Theorem 3.1. From Lemma 4.1, we know that $\exists \epsilon_0 > 0$ such that when $0 < \tau \leq \epsilon_0$, $H(z,\tau)$ has no pure imaginary root, and its number of p-roots is $\Omega_0^+ = \Omega_0 + 2 \sum\limits_{y \in Y_0} \sigma_+(y, 0)$.

Let

$$\widetilde{T} = \{\tau' > \epsilon_0 | \text{when } \tau \in (\epsilon_0, \tau'), H(z,\tau) \text{ has } \Omega_0^+ \ p\text{-roots and no pure imaginary root}\}$$

and $\tau^* = \sup\{\tau | \tau \in \widetilde{T}\}$. From Lemma 4.2, we claim that $\widetilde{T} = (\epsilon_0, \tau^*)$.

If $\tau^* = \infty$, then it is necessary that $Y = \emptyset$. In this case, $H(z,\tau)$ has Ω_0 p-roots for every $\tau \in \mathbb{R}^+$. Additionally, if $\Omega_0 = 0$, then $H(z,\tau)$ is stable for all $\tau \geq 0$, namely, $H(z,\tau)$ is unconditionally stable.

If $\tau^* < \infty$, then $H(z,\tau^*)$ must have pure imaginary roots. Otherwise from Lemma 4.2, $\exists \epsilon > 0$ such that when $|\tau - \tau^*| \leq \epsilon, H(z,\tau)$ has Ω_0^+ p-roots, which obviously contradicts the difinition of τ^*. Now from Lemma 4.2, $\Omega_{\tau^*} = \Omega_0^+ - 2 \sum\limits_{y \in Y_{\tau^*}} \sigma_-(y, \tau^*)$ and $\exists \epsilon_2 > 0$ such that when $\tau^* < \tau < \tau^* + \epsilon_2$, $H(z,\tau)$ has $\Omega_0^+ + 2 \sum\limits_{y \in Y_{\tau^*}} \sigma_+(y, \tau^*)$ p-roots.

As τ varies from 0 to τ_0, repeating the procedures above, enumerating the number of p-roots as stated in Step 4, we get Theorem 3.1.

§5. Asymptotic Stability of Linear Differential-Difference Equations

In this section, we will mainly discuss the asymptotic stability of differential-difference equation

$$y^{(n)}(t) + cy^{(n)}(t-\tau) + \sum_{j=0}^{n-1} a_j y^{(j)}(t) + \sum_{j=0}^{n-1} b_j y^{(j)}(t-\tau) = 0 \quad (|c| < 1). \tag{5.1}$$

Let

$$p(z,\tau) = p(z)e^{\tau z} + q(z), \tag{5.2}$$

and

$$w(z) = \frac{p'(z)}{p(z)} - \frac{q'(z)}{q(z)}, \tag{5.3}$$

where

$$p(z) = z^n + \sum_{j=0}^{n-1} a_j z^j, \tag{5.4}$$

$$q(z) = cz^n + \sum_{j=0}^{n-1} b_j z^j. \tag{5.5}$$

Clearly, the asympototic stability of equation (5.1) is equivalent to the stability of $p(z,\tau)$.

In order to use Theorem 3.1, we need the following lemmas. Lemmas 5.1–5.2 can be obtained by direct calculations.

Lemma 5.1. *For each $\tau \in \widetilde{M}, p(z,\tau)$ has multiple root iy iff both $p(z,\tau)$ and $\tau + w(z)$ have the same root iy.*

Lemma 5.2. *For $p(z,\tau) = p(z)e^{\tau z} + q(z)$, we have*

$$Y = \{y > 0 \| p(iy)| = |q(iy)|\}, \tag{5.6}$$

where Y is defined by (2.6).

To calculate the value of τ_j in Step 2 of Theorem 3.1, we need the following two lemmas.

Lemma 5.3. *For $y_j \in Y, \tau_j$ such that $p(iy_j,\tau_j) = 0$ and $0 \le \tau_j y_j < 2\pi$ is given by*

$$\tau_j = \frac{1}{y_j}(\pi + 2\tan^{-1} K(y_j)), \tag{5.7}$$

where

$$K(y_j) \equiv \frac{p(iy_j) - q(iy_j)}{p(iy_j) + q(iy_j)} i \tag{5.8}$$

is a real number.

Proof. For $y_j \in Y$, since $p(iy_j,\tau) = 0$, we know there exists uniquely τ_j such that $p(iy_j,\tau_j) = 0, 0 \le \tau_j y_j < 2\pi$. From Lemma 2.1 (here $H(z,\tau) = p(z,\tau)$), letting

$$\theta_j = \frac{1}{y_j} tg \frac{\tau_j y_j}{4}, \tag{5.9}$$

we obtain $G(iy_j,\theta_j) = 0$, that is,

$$(1 + iy_j\theta_j)^2 p(iy_j) + (1 - iy_j\theta_j)^2 q(iy_j) = 0.$$

Solving the above equation, we get

$$\theta_j = \frac{K(y_j) + \sqrt{1 + K^2(y_j)}}{y_j},$$

where $K(y_j)$ is given by (5.8). Since $|p(iy_j)| = |q(iy_j)|$, the equality $\operatorname{Im} K(y_j) = 0$ holds. So we know $K(y_j)$ is a real number. Using formula

$$\tan^{-1}(k + \sqrt{1 + k^2}) = \frac{\pi}{4} + \frac{1}{2}\tan^{-1} k \quad (k \in \boldsymbol{R})$$

and (5.9), we finally obtain

$$\tau_j = \frac{\pi + 2\tan^{-1} K(y_j)}{y_j}.$$

However, sometimes, we prefer the following formula. For $y_j \in Y$, we define

$$\phi(y_j) = \arg(p(iy_j)), \quad \text{for} \quad p(iy_j) \ne 0, \tag{5.10}$$

$$\psi(y_j) = \arg(q(iy_j)), \quad \text{for} \quad q(iy_j) \ne 0, \tag{5.11}$$

and let $\phi(y_j) = \phi(y_j + 0), \psi(y_j) = \psi(y_j + 0)$ for y_j such that $p(iy_j) = 0$ or $q(iy_j) = 0$. We also have

Lemma 5.4. *For $y_j \in Y, \tau_j$ such that $p(iy_j,\tau_j) = 0$ and $0 \le \tau_j y_j < 2\pi$ can be expressed by*

$$\tau_j = \frac{1}{y_j}\Big(\psi(y_j) + \pi - \phi(y_j) - 2\pi\Big[\frac{\psi(y_j) + \pi - \phi(y_j)}{2\pi}\Big]\Big). \tag{5.12}$$

Proof. Since $p(iy_j, \tau_j) = 0$, that is,

$$p(iy_j) + q(iy_j)e^{-i\tau_j y_j} = 0,$$

we have

$$\arg(p(iy_j)) = \pi + \arg(q(iy_j)) - \tau_j y_j \pmod{2\pi},$$

that is,

$$\pi + \psi(y_j) - \phi(y_j) = \tau_j y_j \pmod{2\pi}.$$

Thus expression (5.12) follows.

Let

$$F(y) \equiv |p(iy)|^2 - |q(iy)|^2, \tag{5.13}$$

$$p(iy) \equiv p_1(y) + ip_2(y), \tag{5.14}$$

$$q(iy) \equiv q_1(y) + iq_2(y), \tag{5.15}$$

where $p_1(y), p_2(y), q_1(y)$ and $q_2(y)$ are real polynomials. Also denote

$$Y = \{y > 0 | F(y) = 0\} \equiv \{y_j | 1 \le j \le m\}. \tag{5.16}$$

Then for each $y_j \in Y$ $(1 \le j \le m)$ and corresponding $\tau_{j,k} = \tau_j + \frac{2k\pi}{y_j}$, where $0 \le \tau_j y_j < 2\pi, k \in N^+$ (Because of the specialty of (5.2), $\tau^{\nu}_{j,k}$ in Step 2 of Theorem 3.1 has the form $\tau^1_{j,k} (\nu = 1)$, we omit the superscript hereafter), consider $z(\tau) = \mathrm{Re}\, z(\tau) + i\mathrm{Im}\, z(\tau)$ such that $p(z(\tau), \tau) = 0$ and $z(\tau_{j,k}) = iy_j$. We have

Lemma 5.5. *For $y_j \in Y$, if $F'(y_j) \neq 0$, or $F'(y_j) = 0$ but $\tau_{j,k} + w(iy_j) \neq 0$, then iy_j is a simple root of $p(z, \tau_{j,k})$; if $F'(y_j) = \tau_{j,k} + w(iy_j) = 0$, then iy_j is a multiple root of $p(z, \tau_{j,k})$. Meanwhile, for $y_j \in Y$, if $F'(y_j) \neq 0$, then $\frac{d\mathrm{Re}\, z(\tau)}{d\tau}\Big|_{\tau=\tau_{j,k}}$ has the same sign as $F'(y_j)$; if $F'(y_j) = 0$, then $\frac{d\mathrm{Re}\, z(\tau)}{d\tau}\Big|_{\tau=\tau_{j,k}} = 0$ and $w(iy_j) \in \mathbb{R}$. Moreover, if $\tau_{j,k} + w(iy_j) \neq 0$ and $F''(y) \neq 0$, then $\frac{d^2\mathrm{Re}\, z(\tau)}{d\tau^2}\Big|_{\tau=\tau_{j,k}}$ has the same sign as $-(\tau_{j,k} + w(iy_j))F''(y_j)$.*

The first part of Lemma 5.5 comes directly from Lemma 5.1, and the remaining part can be derived through direct but complicated calculations by using $e^{-\tau z} = -\frac{p(z)}{q(z)}$ and (5.3). We omit the proof here.

For $y_j \in Y$, when $F'(y_j) = 0$, from (5.3), we can obtain

$$w(iy_j) = \frac{p_1(y_j)p_2'(y_j) - p_2(y_j)p_1'(y_j)}{p_1^2(y_j) + p_2^2(y_j)} - \frac{q_1(y_j)q_2'(y_j) - q_2(y_j)q_1'(y_j)}{q_1^2(y_j) + q_2^2(y_j)}. \tag{5.17}$$

Now from Theorem 3.1, we have

Proposition 5.1. *Suppose for each $\tau \in \widetilde{M}, p(z, \tau)$ and $\tau + w(z)$ have no common pure imaginary root. Then equation* (5.1) *is asymptotically stable iff $\tau \in O \equiv \bigcup_{i=1}^{J} O_i$, where $O_1, O_2, \cdots, O_J$ are intervals stated as in Corollary* 3.1.

Hence, when $a_0, a_1, \cdots, a_{n-1}, b_0, b_1, \cdots, b_{n-1}$ and c $(|c| < 1)$ are assigned, the time delay τ, at which differential-difference equation (5.1) is asymptotically stable , can be determined from Lemmas 5.1–5.5 and Proposision 5.1. For some cases, we may also need Corollary 3.3 as well.

We point out that, though, for number system, the problem discussed above can be solved with the help of computers, for alphabet system, the problem is not fully solved. But for systems of lower orders, such as order 1 or 2, the results are complete.

Proposition 5.2. *If* $a+b>0, |c|<1$ *and*

$$p_1(z,\tau) \equiv z + a + (cz+b)e^{-\tau z},$$

then $p_1(z,\tau)$ *is stable iff*

$$0 \le \tau < \gamma(a,b,c),$$

where

$$\gamma(a,b,c) \equiv \begin{cases} +\infty, & \text{if} \quad |a| \ge |b|, \\ \frac{2\sqrt{1-c^2}}{\sqrt{b^2-a^2}} \tan^{-1} \frac{\sqrt{(1-c^2)(b^2-a^2)}}{(1-c)(b-a)}, & \text{if} \quad |a| < |b|. \end{cases} \tag{5.18}$$

This result first appeared in [19]. It is easier to be proved by the method above. Due to the space limit, we omit it here.

Let

$$p_2(z,\tau) \equiv z^2 + a_1 z + a_0 + (cz^2 + b_1 z + b_0)e^{-\tau z} (|c|<1). \tag{5.19}$$

Then we have

$$p(iy) = a_0 - y^2 + a_1 yi, \tag{5.20}$$

$$q(iy) = b_0 - cy^2 + b_1 yi, \tag{5.21}$$

and

$$F(y) = (1-c^2)y^4 - (2a_0 - a_1^2 - 2b_0 c + b_1^2)y^2 + a_0^2 - b_0^2. \tag{5.22}$$

Define

$$\Delta \equiv (2a_0 - a_1^2 - 2b_0 c + b_1^2)^2 - 4(1-c^2)(a_0^2 - b_0^2), \tag{5.23}$$

$$\Lambda \equiv 2a_0 - a_1^2 - 2b_0 c + b_1^2, \tag{5.24}$$

$$y_\pm \equiv \sqrt{\frac{\Lambda \pm \sqrt{\Delta}}{2(1-c^2)}}. \tag{5.25}$$

Then

$$Y = \{y > 0 | F(y) = 0\} = \{y_+, y_-\} \cap (0, +\infty). \tag{5.26}$$

Also let

$$\tau_\pm \equiv \frac{1}{y_\pm}(\pi + 2\tan^{-1} K(y_\pm)) \tag{5.27}$$

$$= \frac{1}{y_\pm}\Big(\arg(q(iy_\pm)) + \pi - \arg(p(iy_\pm)) - 2\pi\Big[\frac{\arg(q(iy_\pm)) + \pi - \arg(p(iy_\pm))}{2\pi}\Big]\Big), \tag{5.28}$$

and

$$m_1 \equiv \Big[\frac{2\pi y_+ + y_+ y_- (\tau_+ - \tau_-)}{2\pi(y_+ - y_-)}\Big] \bigvee 1, \tag{5.29}$$

$$m_2 \equiv \Big[\frac{y_+ y_- (\tau_+ - \tau_-)}{2\pi(y_+ - y_-)}\Big] \bigvee 0, \tag{5.30}$$

where

$$
\begin{aligned}
&K(y_\pm)\\
&=\frac{p(iy_\pm)-q(iy_\pm)}{p(iy_\pm)+q(iy_\pm)}i &(5.31)\\
&=\frac{(1-c)y_\pm((a_1c-b_1)y_\pm^2+a_0b_1-a_1b_0))}{(a_0c^2+(a_0-a_1^2-a_1b_1-b_0)c+a_1b_1+b_1^2-b_0)y_\pm^2+a_0b_0+b_0^2-(a_0^2+a_0b_0)c}. &(5.32)
\end{aligned}
$$

We have the following

Theorem 5.1. *If* $a_0+b_0>0$, *then* $p_2(z,\tau)$ *is stable iff*

$$\tau\in T, \tag{5.33}$$

where T *is a real number set listed in the following table corresponding to the various coefficients.*

Table

Line No.	a_1+b_1	Δ	$\lvert a_0\rvert-\lvert b_0\rvert$	Λ	b_0-a_0c	a_1	Y	T
1		<0					$\emptyset$	$\boldsymbol{R}^+$
2			<0				$\{y_+\}$	$[0,\tau_+)$
3			>0	≤ 0			$\emptyset$	$\boldsymbol{R}^+$
4	>0	>0		>0			$\{y_+,y_-\}$	Ω
5			$=0$	≤ 0			$\emptyset$	$\boldsymbol{R}^+$
6				>0			$\{y_+\}$	$[0,\tau_+)$
7		$=0$		≤ 0			$\emptyset$	$\boldsymbol{R}^+$
8				>0			$\{y_+\}(=\{y_-\})$	Ω_{01}
9			≤ 0				$\{y_+\}$	$\emptyset$
10					>0		$\{y_+,y_-\}$	Ω_1
11	$=0$		>0		<0		$\{y_+,y_-\}$	Ω_2
12					$=0$	>0	$\{y_+\}(=\{y_-\})$	Ω_{02}
13						≤ 0	$\{y_+\}(=\{y_-\})$	$\emptyset$
14		≤ 0						
15	<0		≤ 0					$\emptyset$
16		>0	>0	≤ 0				
17				>0			$\{y_+,y_-\}$	Ω_3

where

$$\textit{in Line } 4,\quad \Omega=[0,\tau_+)\bigcup\Big(\bigcup_{k=1}^{m_1}\Big(\tau_-+\frac{2(k-1)\pi}{y_-},\tau_++\frac{2k\pi}{y_+}\Big)\Big),$$

in Line 8, $\Omega_{01}=\boldsymbol{R}\backslash\Big\{\tau_++\frac{2k\pi}{y_+}\Big|k\in N^+\Big\}$, *in Line* 10, $\Omega_1=\bigcup_{k=1}^{m_1}\Big(\tau_-+\frac{2(k-1)\pi}{y_-},\frac{2k\pi}{y_+}\Big)$,

in Line 11, $\Omega_2=\bigcup_{k=0}^{m_2}\Big(\frac{2k\pi}{y_-},\tau_++\frac{2k\pi}{y_+}\Big)$, *in Line* 12, $\Omega_{02}=\boldsymbol{R}^+\backslash\Big\{\frac{2k\pi}{\sqrt{a_0}}\Big|k\in N^+\Big\}$,

in Line 17, $\Omega_3=\bigcup_{k=0}^{m_2}\Big(\tau_-+\frac{2k\pi}{y_-},\tau_++\frac{2k\pi}{y_+}\Big)$.

Its proof will be given in Section 6.

Theorem 5.1 can be regarded as an improved version of Capyrin's Theorem[3] as Proposition 5.2 of Hayes's Theorem[10]. It plays an important role in the study of stabilization of two-dimensional linear systems by time-delay feedback controls[11,12]. Readers who are interested in this topic may also see [22, 23].

§6. Appendix

Proof of Proposition 2.1. First, denote

$$H_s(z,\tau) \equiv H(z,\tau) \equiv \sum_{k=0}^{s} p_k^{(s)} e^{k\tau z}, \tag{6.1}$$

where $p_k^{(s)}(z)$ $(k=0,1,\cdots,s)$ is a real polynomial of z. Next, define successively

$$H_j(z,\tau) \equiv e^{-\tau z}(p_0^{(j+1)}(z)e^{(j+1)\tau z}H_{j+1}(-z,\tau) - p_{j+1}^{(j+1)}(-z)H_{j+1}(z,\tau)) \quad (0 \le j \le s-1). \tag{6.2}$$

Let

$$p_k^{(j)}(z) \equiv \det\begin{pmatrix} p_{j-k}^{(j+1)}(-z) & p_{k+1}^{(j+1)}(z) \\ p_{j+1}^{(j+1)}(-z) & p_0^{(j+1)}(z) \end{pmatrix} \quad (0 \le k \le j \le s-1). \tag{6.3}$$

We have

$$H_j(z,\tau) = \sum_{k=0}^{j} p_k^{(j)}(z) e^{k\tau z} \quad (0 \le j \le s-1). \tag{6.4}$$

Define

$$h_j(z,t) \equiv \sum_{k=0}^{j} p_k^{(j)}(z) t^k \quad (0 \le j \le s), \tag{6.5}$$

and denote for $H_j(z,\tau)$

$$\widetilde{Y_j} \equiv \bigcup_{\tau \ge 0} \{y > 0 | H_j(yi,\tau) = 0\}. \tag{6.6}$$

We have the following

Lemma 6.1.1. *$\widetilde{Y}_{j+1} \subset \widetilde{Y}_j$ $(0 \le j \le s-1)$. For $y \in \widetilde{Y_j}$, if $|p_0^{(j+1)}(iy)| \ne |p_{j+1}^{(j+1)}(iy)|$, then $y \in \widetilde{Y}_{j+1}$.*

Proof. If $y \in \tilde{y}_{j+1}$, then $\exists \tau \ge 0$ such that $H_{j+1}(yi,\tau) = 0$. Taking the conjugate form, we have $H_{j+1}(-yi,\tau) = 0$. From (6.2), we obtain $H_j(yi,\tau) = 0$. Thus, $y \in \widetilde{Y_j}$. This proves $\widetilde{Y}_{j+1} \subseteq \widetilde{Y}_j$.

If $y \in \widetilde{Y_j}$, there exists $\tau \ge 0$ such that $H_j(yi,\tau) = 0$. From (6.2), we know

$$p_0^{(j+1)}(yi)e^{(j+1)\tau yi}H_{j+1}(-yi,\tau) = p_{j+1}^{(j+1)}(-yi)H_{j+1}(yi,\tau). \tag{6.7}$$

Taking modular forms on both sides of (6.7), and noticing that

$$|H_{j+1}(yi,\tau)| = |H_{j+1}(-yi,\tau)| \quad \text{and} \quad |p_0^{(j+1)}(yi)| \ne |p_{j+1}^{(j+1)}(-yi)|,$$

we obtain $H_{j+1}(yi,\tau) = 0$. Thus $y \in \widetilde{Y}_{j+1}$.

If we denote in $h(z,t)$

$$\chi_*^{(s)}(t) \equiv a_s^{(s)} t^s + a_{s-1}^{(s)} t^{s-1} + \cdots + a_1^{(s)} t + a_0^{(s)} \tag{6.8}$$

and let

$$a_k^{(j)} \equiv (-1)^{r2^{s-j-1}} \det\begin{pmatrix} a_{j-k}^{(j+1)} & a_{k+1}^{(j+1)} \\ a_{j+1}^{(j+1)} & a_0^{(j+1)} \end{pmatrix} \quad (0 \le k \le j \le s-1), \tag{6.9}$$

then we have the coefficient term of $z^{r2^{s-j}}$ in $h_j(z,t)$ as follows:

$$\chi_*^{(j)}(t) \equiv \sum_{k=0}^{j} a_k^{(j)} t^k. \tag{6.10}$$

Definition 6.1.1. *Polynomial* $f(t) = c_s t^s + c_{s-1}t^{s-1} + \cdots + c_1 t + c_0$ $(c_i \in C, 0 \le i \le s)$ *is called Schur polynomial, if it has all its roots lying strictly in the unit disk. For simplicity, we denote* $f \in$ Schur.

For $f_s(t) = c_s t^s + c_{s-1}t^{s-1} + \cdots + c_1 t + c_0$ $(c_s \neq 0)$, we define

$$\hat{f}_s(t) = \bar{c_0} t^s + \bar{c_1} t^{s-1} + \cdots + \bar{c_s},$$

where $\bar{c_i}$ $(0 \le i \le s)$ is the conjugate number of c_i.

Let

$$f_j(t) \equiv \frac{1}{t}(\hat{f}_{j+1}(0) f_{j+1}(t) - f_{j+1}(0)\hat{f}_{j+1}(t)) \quad (1 \le j \le s-1).$$

We have the following

Lemma 6.1.2. *If* $f_{j+1} \in$ Schur, *then* $|f_{j+1}(0)| < |\hat{f}_{j+1}(0)|$. *If* $|f_{j+1}(0)| < |\hat{f}_{j+1}(0)|$, *then* $f_{j+1} \in$ Schur *iff* $f_j \in$ Schur.

Proof. From Viète's Theorem, it is easy to verify that $|f_{j+1}(0)| < |\hat{f}_{j+1}(0)|$ is a necessary condition for $f_{j+1} \in$ Schur. Now we come to prove the equivalence of $f_{j+1} \in$ Schur and $f_j \in$ Schur under the condition $|f_{j+1}(0)| < |\hat{f}_{j+1}(0)|$. Obviously, if $f_{j+1} \in$ Schur (or $f_j \in$ Schur), for each $t \in C_1 \equiv \{z \in C||z| = 1\}$, the following inequality

$$|\hat{f}_{j+1}(0) f_{j+1}(t) - t f_j(t)| = |f_{j+1}(0)\hat{f}_{j+1}(t)| < |\hat{f}_{j+1}(0) f_{j+1}(t)|$$

holds. From Rouché's theorem, we assert that $f_{j+1}(t)$ and $f_j(t)$ have the same number of roots in the open unit disk. Thus, our lemma follows.

Proof of Proposition 2.1. First, $Y \subset \widetilde{Y_0}$ is directly obtained from Lemma 6.1.1. From assumptions (A1)–(A2), we know $\chi_*^{(s)} \in$ Schur. Also from (6.9)–(6.10), we have

$$\chi_*^{(j)}(t) = (-1)^{r2^{s-j-1}+1} \frac{1}{t}(\hat{\chi}_{j+1}(0)\chi_{j+1}(t) - \chi_{j+1}(0)\hat{\chi}_{j+1}(t)) \quad (1 \le j \le s-1).$$

Therefore, $H_1(z,\tau)$ has the form of Lemma 5.4. Thus, $\widetilde{Y_0} = \widetilde{Y_1}$ is finite.

6.2. Proof of Lemma 2.4. From the given conditions, we know that when $\tau \in [\alpha,\beta], H(z,\tau)$ has root $iy(\tau)$, where $y(\tau)$ is a real analytic function. Clearly, $y(\tau)$ is continuous of τ. Assumption (A3) guarantees that $y(\tau) \neq 0$ for each $\tau \in [\alpha,\beta]$. Thus, without loss of generality, we assume $y(\tau) > 0$ for $\tau \in [\alpha,\beta]$.

Now, we are sure to find two numbers $\alpha', \beta' (\alpha \le \alpha' < \beta' \le \beta)$ such that relation $\frac{\tau y(\tau)}{2\pi} \in N$ does not hold for all $\tau \in [\alpha',\beta']$. For if it is not the case, there must be $\tau y(\tau) \equiv 2k_0\pi$, where k_0 is a fixed integer. Thus, $H(iy(\tau),\tau) = 0$ is equivalent to $\sum\limits_{m,n} a_{mn}(iy(\tau))^m = 0$. If for $\tau \in [\alpha',\beta'], y(\tau)$ takes at least two different values, then from the continuity of $y(\tau)$, we know that equation $\sum\limits_{m,n} a_{mn} z^m = 0$ has an infinite number of different roots. This is impossible. If for $\tau \in [\alpha',\beta']$, $y(\tau) \equiv c_1$ (c_1 is a constant), then it also contradicts $\tau y(\tau) \equiv 2k_0\pi$. Therefore, we may assume that for $\tau \in [\alpha',\beta'], \theta(\tau) = \frac{1}{y(\tau)} \tan \frac{\tau y(\tau)}{4}$ is a continuous function of τ. From Lemma 2.1, $\Delta_*(\theta(\tau)) \equiv 0$. If $\theta(\tau)$ takes at least two different values, then $\Delta_*(\theta) = 0$ necessarily has an infinite number of roots. Since $\Delta_*(\theta)$ is defined as a polynomial of θ, it is necessary that $\Delta_*(\theta) \equiv 0$ for $\theta > 0$. If $\theta(\tau) \equiv c_2$ (c_2 is a constant) for $\tau \in [\alpha', \beta']$, then we have $G(iy(\tau), c_2) \equiv 0$. Since $G(z, c_2) = 0$ has only a finite number of roots, it is necessary that $y(\tau) \equiv c_3$ (c_3 is a constant and $G(ic_3, c_2) = 0$). So, we get $c_2 \equiv \frac{1}{c_3} \tan \frac{\tau c_3}{4}$ for $\tau \in [\alpha', \beta']$, which is also impossible.

Combining all the above, we obtain $\Delta_*(\theta) \equiv 0 (\theta > 0)$.

6.3. Proof of Theorem 5.1. First, from (5.22)-(5.25), we have

$$F'(y_\pm) = \pm 2y_\pm \sqrt{\Delta}, \tag{6.11}$$

$$F''(y_\pm) = 4\Lambda \pm 6\sqrt{\Delta}. \tag{6.12}$$

Also from Lemma 5.4, we get (5.28). From Lemma 5.3, we obtain

$$\tau_\pm = \frac{1}{y_\pm}(\pi + 2\tan^{-1} K(y_\pm)), \tag{6.13}$$

where

$$K(y_\pm) = \frac{p(iy_\pm) - q(iy_\pm)}{p(iy_\pm) + q(iy_\pm)} i. \tag{6.14}$$

Realificating the denominator of $K(y_\pm)$ and utilizing $F(y_\pm) = 0$, we get another expression of (6.14), i.e., (5.28).

Now, we can prove our theorem through Theorem 3.1, Lemmas 5.1–5.5, Proposition 5.1 and Corollary 3.3.

If $a_1 + b_1 > 0$, then $p_2(z, 0)$ is stable. From (5.26), we know $Y = \emptyset$ for Lines 1, 3, 5, 7 respectively. Thus from Corollary 3.1, we get $T = \mathbb{R}^+$. For Line 2 and Line 6, since $Y = \{y_\pm\}$ and $F'(y_+) > 0$, we know from Lemma 5.5 that iy_+ is a simple root. Then, from Theorem 3.1 and Lemma 5.5 again, we have $T = [0, \tau_+)$. While for Line 4, $Y = \{y_+, y_-\}$, since $F'(y_-) < 0, F'(y_+) > 0$ and $N(p_2(z, 0)) \subset C^-$, from the continuity of the roots of $p_2(z, \tau)$ concerning τ, definitely we have $0 < \tau_+ < \tau_-$. Now turn to Steps 2-3 of Theorem 3.1. If inequality

$$\tau_- + \frac{2(k_0 - 1)\pi}{y_-} < \tau_+ + \frac{2k_0\pi}{y_+} \quad (k_0 \geq 1) \tag{6.15}$$

holds, then the following inequality chain also holds:

$$\tau_+ + \frac{2(k-1)\pi}{y_+} < \tau_- + \frac{2(k-1)\pi}{y_-} < \tau_+ + \frac{2k\pi}{y_+} \quad (1 \leq k \leq k_0). \tag{6.16}$$

The greatest integer such that (6.15) holds is $k_0 = m_1$. Thus, from Theorem 3.1 and Proposition 5.1, $N(p_2(z, \tau)) \subset C^-$ iff

$$\tau \in T = [0, \tau) \bigcup \Big(\bigcup_{k=1}^{m_1} \Big(\tau_- + \frac{2(k-1)\pi}{y_-}, \tau_+ + \frac{2k\pi}{y_+} \Big) \Big). \tag{6.17}$$

For Line 8, e.g.,

$$p_2(z, \tau) = z^2 + 2z + \frac{15}{2} + (\frac{3}{5} z^2 + z) e^{\tau z},$$

we know $F(y) = 0$ has double root $y_* = y_+ = y_-$, which satisfies $F'(y_*) = 0$. By (5.17), for the case $y_+ = y_-$ (i.e., $\Delta = 0$), we have

$$w(iy_*) = \frac{(a_0 - b_1 c) y_*^2 + a_1 a_0 - b_1 b_0}{p_1^2(y_*) + p_2^2(y_*)}. \tag{6.18}$$

Since

$$\Delta = \Lambda^2 - 4(1-c^2)(a_0^2 - b_0^2) = (b_1^2 - a_1^2)^2 + 4(a_0 - b_0 c)(b_1^2 - a_1^2) + 4(b_0 - a_0 c)^2 = 0$$

and $a_0 + b_0 > 0$, $a_1 + b_1 > 0$, $|c| < 1$, we can easily obtain $a_0 > |b_0| \geq 0, a_1 \geq |b_1| \geq 0$. Therefore, we get $w(iy_*) > 0$. From Proposition 5.1, we know iy_* is a simple root. Also, from Proposition 5.1, for

$$\tau_k = \tau_+ + \frac{2k\pi}{y_+} (= \tau_- + \frac{2k\pi}{y_-}, k \in N^+),$$

we have

$$\frac{d\mathrm{Re}\, z(\tau)}{d\tau}\Big|_{\tau=\tau_k} = 0, \quad \frac{d^2\mathrm{Re}\, z(\tau)}{d\tau^2}\Big|_{\tau=\tau_k} < 0.$$

Thus, from Theorem 3.1, we assert that $N(p_2(z,\tau)) \subset C^-$ iff

$$\tau \in \boldsymbol{R}^+ \Big\backslash \Big\{ \tau_+ + \frac{2k\pi}{y_+} \Big| k \in N^+ \Big\}.$$

If $a_1 + b_1 = 0$, then $p_2(z, 0)$ has only pure imaginary roots. Now, $\Lambda = 2(a_0 - b_0 c)$ and $\sqrt{\Delta} = 2|b_0 - a_0 c|$. In Line 9, $|a_0| - |b_0| \leq 0$, since $a_0 + b_0 > 0$, we get $b_0 \geq |a_0|$. At the present case,

$$\Lambda - \sqrt{\Delta} = (1+c)(a_0 - b_0) \leq 0.$$

So, $Y = \{y_+\}$. Consider that $p_2(z, 0)$ is not stable. From Theorem 3.1, we conclude that $p_2(z,\tau)$ is not stable for $\tau \geq 0$. For Line 10, since $|a_0| - |b_0| > 0$ and $b_0 - a_0 c > 0$, we have

$$y_+ = \sqrt{\frac{a_0 + b_0}{1+c}}, \quad \tau_+ = 0;$$
$$y_- = \sqrt{\frac{a_0 - b_0}{1-c}}, \quad \tau_- = \frac{1}{y_-}\Big(\pi - 2\tan^{-1}\frac{a_1(1-c)y_-}{b_0 - a_0 c}\Big).$$

Through an analogous discussion as in Line 4, the conclusion is drawn. In Line 11, as $|a_0| - |b_0| > 0, b_0 - a_0 c < 0$, we have

$$y_+ = \sqrt{\frac{a_0 - b_0}{1-c}}, \quad \tau_+ = \frac{1}{y_+}\Big(\pi - 2\tan^{-1}\frac{a_1(1-c)y_+}{b_0 - a_0 c}\Big);$$
$$y_- = \sqrt{\frac{a_0 + b_0}{1+c}}, \quad \tau_- = 0.$$

Consider all the open intervals of the form $(\frac{2k\pi}{y_-}, \tau_+ + \frac{2k\pi}{y_+})$, similar to Line 4, we get our result. In Line 12 , when $|a_0| - |b_0| > 0, b_0 - a_0 c = 0$ and $a_1 > 0$, we have

$$Y = \{y_+\} = \{y_-\} = \{\sqrt{a_0}\}, \quad \tau_+ = \tau_- = 0.$$

From (6.18), we get

$$w(\sqrt{a_0}i) = \frac{(a_1 - b_1 c)a_0 + a_1 a_0 - b_1 b_0}{p_1^2(\sqrt{a_0}) + p_2^2(\sqrt{a_0})} = \frac{2(1+c)}{a_1} > 0.$$

Analogous to Line 8, we can prove $p_2(z,\tau)$ is stable iff

$$\tau \in \boldsymbol{R}^+ \Big\backslash \Big\{ \frac{2k\pi}{\sqrt{a_0}} \Big| k \in N^+ \Big\}.$$

For Line 13, when $|a_0| - |b_0| > 0$, $b_0 - a_0 c = 0$, $a_1 \leq 0$, we have

$$Y = \{\sqrt{a_0}\}, \quad \tau_+ = \tau_- = 0.$$

If $a_1 = 0$, then $p_2(z,\tau) = (z^2 + a_0)(1 + ce^{-\tau z})$ is obviously not stable. When $a_1 < 0$, we have

$$w(\sqrt{a_0}i) = \frac{2(1+c)}{a_1}, \quad \tau_k = \frac{2k\pi}{\sqrt{a_0}} \quad (k \in N^+).$$

Therefore, if $\frac{2(1+c)}{|a_1|} \notin \{\frac{2k\pi}{\sqrt{a_0}}|k \in N\}$, then $\tau_k + w(iy) \neq 0$; thus $\sqrt{a_0}i$ is a simple root. Since

$$\frac{d\mathrm{Re}\, z(\tau)}{d\tau}\Big|_{\tau=\tau_k} = 0, \quad \frac{d^2\mathrm{Re}\, z(\tau)}{d\tau^2}\Big|_{\tau=\tau_k} \sim -\Big(\tau_k + \frac{2(1+c)}{a_1}\Big)F''(\sqrt{a_0}) \neq 0,$$

from Theorem 3.1, we assert that $p_2(z,\tau)$ is not stable for all $\tau \geq 0$. If $\frac{2(1+c)}{|a_1|} \in \{\frac{2k\pi}{\sqrt{a_0}}|k \in N\}$, we set

$$a_0(\epsilon) = a_0 + \epsilon, \quad b_0(\epsilon) = (a_0 + \epsilon)c \ (\epsilon > 0),$$

where ϵ is sufficiently small such that

$$\frac{2(1+c)}{|a_1|} \notin \Big\{\frac{2k\pi}{\sqrt{a_0+\epsilon}}|k \in N\Big\}.$$

Now, from the discussion above, we know

$$p_2(z,\tau,\epsilon) = z^2 + a_1 z + a_0(\epsilon) + (cz^2 + b_1 z + b_0(\epsilon))e^{-\tau z}$$

is not stable for $\tau \geq 0$. Let $\epsilon \to 0^+$. From Corollary 3.3, we conclude that $p_2(z,\tau)$ is not stable for all $\tau \in \boldsymbol{R}^+$.

If $a_1 + b_1 < 0$, then $p_2(z,0)$ has two roots with positive real parts. In Line 14, if $\Delta < 0$, then $Y = \emptyset$. It is easy to verify that $p_2(z,\tau)$ $(\tau \geq 0)$ is not stable. If $\Delta = 0$ and $\Lambda \leq 0$, we also have $Y = \emptyset$. Therefore, $p_2(z,\tau)$ $(\tau \geq 0)$ is not stable. If $\Delta = 0$ but $\Lambda > 0$, then $y_+ = y_- = y, \tau_+ = \tau_-$. Similar to the proof of Line 8, we have

$$a_0 > |b_0| > 0, \quad a_1 \leq -|b_1| \leq 0.$$

From (6.18), we know $w(iy) < 0$. If $-w(iy) \notin \{\tau_+ + \frac{2k\pi}{y_+}|k \in N^+\}$, as in Line 13, we can prove that $p_2(z,\tau)$ $(\tau \geq 0)$ is not stable. If $-w(iy) \in \{\tau_+ + \frac{2k\pi}{y_+}|k \in N^+\}$, set

$$a_1(\epsilon) = a_1 + \epsilon, b_1(\epsilon) = \sqrt{b_1^2 + 2a_1\epsilon + \epsilon^2} \ (\epsilon > 0),$$

where ϵ is small enough such that y_ϵ and τ_ϵ corresponding to

$$p_2(z,\tau,\epsilon) = z^2 + a_1(\epsilon)z + a_0 + (cz^2 + b_1(\epsilon)z + b_0)e^{-\tau z}$$

staisfy

$$-w(iy_\epsilon) \notin \Big\{\tau_\epsilon + \frac{2k\pi}{y_\epsilon}\Big|k \in N^+\Big\}.$$

Since in the present case,

$$\Delta(\epsilon) = (b_1^2(\epsilon) - a_1^2(\epsilon))^2 + 4(a_0 - b_0 c)(b_1^2(\epsilon) - a_1^2(\epsilon)) + 4(b_0 - a_0 c)^2 = \Delta = 0,$$

from the above results, we know $p_2(z,\tau,\epsilon)(\tau \geq 0)$ is not stable. Now letting $\epsilon \to 0^+$, we conclude from Corollary 3.3 that $p_2(z,\tau)(\tau \geq 0)$ is not stable. For Lines 15-16, $Y = \{y_+\}$ and $\emptyset$ respectively. We can easily verify that $T = \emptyset$. For Line 17, e.g.,

$$p_2(z,\tau) = z^2 + 1 - \frac{\sqrt{2}}{2}ze^{-\tau z}$$

($p_2(z,\tau)$ is stable iff $\tau \in (\frac{\sqrt{2}}{2}\pi, \frac{3\sqrt{2}}{4}\pi)$), results are gained as in Line 13.

Combining all the above, we thus prove our theorem.

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