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CLASSICAL SOLUTION OF VERIGIN PROBLEM WITH SURFACE TENSION

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Abstract

The author considers Verigin problem with surface tension. Under natural conditions the existence of classical solution locally in time is proved by Schauder fixed point theorem.

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§1. Introduction

We shall consider "pushing oil by water around it" and assume that the displacement of water and oil is piston-like. The displacement is called piston-like, that means during the process of displacement porous media always can be divided in two parts: one of them contains oil only and the other contains no oil but water. We neglect the gravity and consider a two-dimensional "horizontal" field $\widetilde{\Omega} \in I\!\!R^2$, from which the oil is produced through the well D in $\widetilde{\Omega}$. Let $\Omega_1(t), \Omega_2(t)$ be the water field and the oil field respectively, Γ_t be the interface of two fluids, $\Omega = \Omega_1(t) \cup \Gamma_t \cup \Omega_2(t) \equiv \widetilde{\Omega} - \overline{D}$.

When the porous media is compressible, Verigin put forth the mathematical model of this kind of problem in 1957 as Muskat problem[9]. We call it Verigin problem[14]. In fact, from the law of conservation of mass and Darcy's law it follows that

$$\frac{\partial p_i}{\partial t} - \nabla \cdot \left(\frac{k}{\mu_i} \nabla p_i\right) = 0 \qquad \text{in } Q_i \equiv \bigcup_{t>0} (\Omega_i(t) \times \{t\}) \quad (i = 1, 2), \tag{1.1}$$

$$p_1 - p_2 = 0 \qquad \text{on} \quad \Gamma \equiv \bigcup_{t>0} \Gamma_t \times \{t\}, \tag{1.2}$$

$$-\frac{k}{\mu_1}\frac{\partial p_1}{\partial n_t} = -\frac{k}{\mu_2}\frac{\partial p_2}{\partial n_t} = \phi V_n \quad \text{on } \Gamma, \tag{1.3}$$

where $\Omega_1(t), \Omega_2(t)$ are regions of water and oil respectively, Γ_t is a free boundary between $\Omega_1(t)$ and $\Omega_2(t)$. Let $\Omega = \Omega_1(t) \cup \Gamma_t \cup \Omega_2(t)$ be a bounded annular domain in $I\!\!R^2$, $\Omega_2(t)$ is inside. n_t is a normal of Γ_t, pointing inside $\Omega_2(t)$, p_1 and p_2 are pressures of water and oil respectively, μ_1 and μ_2 are viscosities of water and oil respectively, k is the permeability, ϕ is the porosity, V_n is the normal velocity of Γ_t.

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When two immiscible fluids are in contact in the interstices of a porous medium, a discontinuity in pressure exists across the interface separating them. Its magnitude depends on the interface mean curvature at the point. The difference in pressure is called capillary pressure p_c (see [1]):

$$p_{nw} - p_w = p_c,$$

where p_{nw}, p_w are the pressures in nonwetting and wetting phase respectively. And from Laplace equation for capillary pressure $p_c = \sigma K$, where σ is the surface tension and K is the mean curvature.

In the piston-like displacement two immiscible fluids are in contact only at the interface which separates two fluids into two parts.

Thus Verigin problem with surface tension is the problem (1.1), (1.3) and

$$p_1 - p_2 = \sigma K \quad \text{on} \quad \Gamma. \tag{1.4}$$

In the physical fact the capillary force may affect the stability of the front (see [1]). For the stability of the interface it is a good term, it always tends to stabilize the displacement front.

So we expect that the above Verigin problem with surface tension is well-posed under natural conditions.

In one-dimensional case, there are many results about Verigin problem (see [3], [8], etc.). In multidimensional case, E. V. Radkerich has considered the problem (1.1)–(1.3) in [10].

This paper is devoted to the study of the problem (1.1), (1.3) and (1.4). It is organized as follows. In Section 2, we shall parameterize the free boundary by the distance function ρ from initial position of free boundary Γ_0 (in R^2) and reformulate the problem. In Section 3, we construct an initial approximation and introduce two key Lemmas 3.1 and 3.2, which the main result is based on. In Sections 4–5, we prove Lemma 3.2. In Section 4 we derive the model problem. In Section 5 we study the model problem by establishing some estimates of the kernels of it.

§2. Formulation of the Problem

For simplicity, we assume without loss of generality that $\phi = 1$ (which corresponds to the hypothesis that the porous media is homogeneous) and $\frac{k}{\mu_1} = 1$. Then

$$\frac{k}{\mu_2} = \frac{\mu_1}{\mu_2} \equiv \alpha^{(2)} > 1.$$

Remark 2.1. In piston-like displacement of oil and water, the viscosity of viscositied water is grater than the viscosity of oil, i.e., $\mu_2 < \mu_1$ (see [6]).

Introduce ω as the local coordinates of points on the surface Γ_0. We also use $x = X_0(\omega)$ to denote the points on Γ_0 in R^2. Let $n_0(\omega)$ be the unit normal to Γ_0 which is outer with respect to $\Omega_1(0)$. Let $\rho(\omega, t)$ be a function of class $C^{2,1}(\Gamma_0 \times [0, T])$ such that $\rho(\omega, 0) = 0$.

Let $T > 0$ be small and let

$$\Gamma_t = \{x = X_0(\omega) + \rho(\omega, t) n_0(\omega),\ t \in [0, T]\}$$

denote the free boundary.

Straighten the free boundary (see [5]).

$$\Gamma_t \xrightarrow{\text{transformation}} \Gamma_0,$$

$$\bigcup_t \Omega_i(t) \xrightarrow{\text{transformation}} Q_{iT} = \Omega_i(0) \times [0,T] \quad (i=1,2),$$

$$p_1(x,t), p_2(x,t) \xrightarrow{\text{transformation}} V_i = V_i(y,t) \quad (i=1,2).$$

The Verigin problem with surface tension becomes

$$L_\rho^{(j)} v_\rho^{(j)} = 0, \quad j=1,2 \qquad \text{in } \ Q_{jT}, \tag{2.1}$$

$$v_\rho^{(1)} - v_\rho^{(2)} = \sigma\mathcal{K}(\rho, \partial_\omega\rho, \partial_\omega(\partial_\omega\rho)) \qquad \text{on } \ \Gamma_{0T}, \tag{2.2}$$

$$\begin{aligned}\partial_t\rho &= -\alpha^{(2)} S_\rho \partial_\eta v_\rho^{(2)} + \alpha^{(2)} k_{\rho,\omega}\partial_\omega v_\rho^{(2)} \\ &= -S_\rho\partial_\eta v_\rho^{(1)} + k_{\rho,\omega}\partial_\omega v_\rho^{(1)} \qquad \text{on } \ \Gamma_{0T},\end{aligned} \tag{2.3}$$

$$v_\rho^{(j)} = g^{(j)}(y,t) \qquad \text{on } \ \Gamma_{jT}, \tag{2.4}$$

$$v_\rho^{(j)} = \phi^{(j)}(y) \qquad \text{at } \ t=0, \tag{2.5}$$

where (see [10, 11])

$$a_{\rho kl}^{(j)}(y,t) = a_{kl}^{(j)}(\rho, \partial_\omega\rho), \quad j=1,2, \quad 1\le k,l\le 2,$$

$$a_{\rho k}^{(j)}(y,t) = a_k^{(j)}(\rho, \partial_\omega\rho, \partial_t\rho, \partial_\omega(\partial_\omega\rho)), \quad k=1,2,$$

$$\sigma_0|\xi|^2 \le a_{\rho kl}^{(j)}\xi_k\xi_l \le \sigma_1|\xi|^2, \quad \text{for } \xi\in I\!R^2,$$

$$\sigma_0, \sigma_1 > 0, \quad \text{depends on } \ \rho, T,$$

$$S_\rho = S(\rho, \partial_\omega\rho), \quad k_{\rho,\omega} = k(\rho, \partial_\omega\rho).$$

$$\Gamma_1 = \partial\Omega_1(0)\setminus\Gamma_0, \quad \Gamma_2 = \partial\Omega_2(0)\setminus\Gamma_2, \quad \Gamma_{iT} = \Gamma_i\times[0,T] \quad (i=0,1,2),$$

$$L_\rho^{(j)} = \partial_t - \sum_{k,l=1}^2 a_{\rho kl}^{(j)}\partial^2_{y_k y_l} - \sum_{k=1}^2 a_{\rho k}^{(j)}\partial_{y_k},$$

$g^{(j)}(y,t)$ and $\phi^{(j)}(y,t)$ are known functions defined on Γ_{jT} and $\Omega_j(0)$.

Let G_T be an open set in $I\!R^n \times (0,\infty)$, $n=1,2$. Define

$$\widehat{C}^{k+\alpha,(k+\alpha)/2}(\overline{G}_T) = \left\{v\in C^{k-1+\alpha,(k-1+\alpha)/2}(\overline{G}_T), \partial_x^2 v \in C^{k-2+\alpha,(k-2+\alpha)/2}(\overline{G}_T)\right\},$$

$$0<\alpha<1, \ k=4,5,\cdots;$$

$$\begin{aligned}\|v\|_{\widehat{C}^{k+\alpha,(k+\alpha)/2}(\overline{G}_T)} &= \|v\|_{C^{k-2+\alpha,(k-2+\alpha)/2}(\overline{G}_T)} + \|\partial_t v\|_{C^{k-3+\alpha,(k-3+\alpha)/2}(\overline{G}_T)} \\ &\quad + \|\partial_x^2 v\|_{C^{k-2+\alpha,(k-2+\alpha)/2}(\overline{G}_T)}.\end{aligned}$$

Denote

$$\underset{\circ}{C}^{k+\alpha,(k+\alpha)/2}(\overline{G}_T) = \left\{v\in C^{k+\alpha,(k+\alpha)/2}(\overline{G}_T); \ v(\cdot,0)=0\right\};$$

$$\underset{\circ}{\widehat{C}}^{k+\alpha,(k+\alpha)/2}(\overline{G}_T) = \left\{v\in \widehat{C}^{k+\alpha,(k+\alpha)/2}(\overline{G}_T); \ v(\cdot,0)=\partial_t v(\cdot,0)=0\right\}.$$

Now we can state our main result as follows:

Theorem 2.1. *If* $\Gamma_j\in C^{6+\alpha}$ $(j=0,1,2)$, dist $(\Gamma_0, \partial\Omega)\ge L>0$, *and* $\sigma>0$. *Then for any initial boundary data*

$$g^{(j)}(y,t)\in C^{6+\alpha,(6+\alpha)/2}(\Gamma_{jT}), \quad \phi^{(j)}(y)\in C^{6+\alpha}(\Omega_j(0))$$

satisfying the consistency conditions to order 3, *on a sufficiently small time interval* $[0,T]$ *there exists a classical solution of problem* (2.1)–(2.5) *such that*

$$v_\rho^{(j)} \in C^{2+\alpha,(2+\alpha)/2}(Q_{jT}),\ \ \rho \in \widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T}).$$

Moreover,

$$\sum_{j=1}^{2} \|v_\rho^{(j)}\|_{C^{2+\alpha,(2+\alpha)/2}} + \|\rho\|_{\widehat{C}^{4+\alpha,(4+\alpha)/2}}$$
$$\le C\Big[\sum_{j=1}^{2} \|\phi^{(j)}\|_{C^{6+\alpha}(\Omega_j(0))} + \|g^{(j)}\|_{C^{6+\alpha,(6+\alpha)/2}(\Gamma_{jT})}\Big].$$

We now consider the nonlinear operator:

$$\mathcal{F}(\rho) = \partial_t\rho + S_\rho\partial_\omega v_\rho^{(1)} - k_{\rho,\omega}\partial_\omega v_\rho^{(1)}$$

on function $\rho \in \widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})$ with small T, where $v_\rho^{(1)}$ is a solution of the parabolic diffraction problem in (2.1)–(2.5) for given $\rho \in \widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})$ (here the first equality is canceled in (2.3)). We have

$$\mathcal{F} : \widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T}) \to C^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T}).$$

Obviously, that (2.1)–(2.5) has solution $(\rho, v_\rho^{(j)})$ $(j=1,2)$ is equivalent to the existence of a root of $\mathcal{F}(\rho)=0$. So Theorem 2.1 of this section is reformulated as follows.

Theorem 2.2. *Under the same assumptions as in Theorem* 2.1, *for a sufficiently small* T, *there exists* $\rho \in \widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})$ *satisfying* $\mathcal{F}(\rho)=0$. *Here we suppose* ρ *is the only unknown of the problem because* $v_\rho^{(j)}$ *are obtained once* ρ *is determined.*

§3. An Initial Approximation and Two Key Lemmas

From the compatibility condition which $\{g^{(j)}(y,t), \phi^{(j)}(y)\}$ satisfies up to order 3 at $t=0$, and using Theorem 4.3 in [7, p. 298], we know that there exists $\rho_0(\omega,t) \in C^{6+\alpha,(6+\alpha)/2}(\Gamma_{oT})$ which satisfies

$$\|\rho_0\|_{C^{6+\alpha\,(6+\alpha)/2}(\Gamma_{oT})} \le C,$$

where C only depends on $\|\phi^{(j)}(y)\|_{C^{6+\alpha\,(6+\alpha)/2}}$.

Let $v_{\rho_o}^{(j)}$ be a solution of parabolic diffraction problem (2.1)–(2.5) for $\rho=\rho_o$ (here the first equality is canceled in (2.3)). We have $v_{\rho_o}^{(j)}(y,t) \in C^{4+\alpha,(4+\alpha)/2}(\overline{Q}_{jT})$, and

$$\|v_{\rho_o}^{(j)}(y,t)\|_{C^{4+\alpha,(4+\alpha)/2}} \le C,$$

where C is a constant, which only depends on initial-boundary data.

The proof of Theorem 2.2 is based on the following two key Lemmas.

Lemma 3.1. *For any* $\delta\rho \in \underset{\circ}{\widehat{C}}{}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})$, $\|\delta\rho\|_{\widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})} \le N$, *where* N *is to be determined later on, we define* $\rho=\rho_0+\delta\rho$ *and*

$$m = \mathcal{B}_1(\delta\rho) \equiv \mathcal{F}(\rho) - \mathcal{F}(\rho_0) - D\mathcal{F}(\rho_0)\delta\rho. \tag{3.1}$$

Then $m \in \underset{\circ}{C}{}^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})$ *and*

$$\|m\|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})} \le C\|\delta\rho\|^2_{\widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})}, \tag{3.2}$$

where C *only depends on* σ, $g^{(j)}$ $\phi^{(j)}$.

Lemma 3.2. *For any* $m \in \mathop{C}\limits_{\circ}^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})$, *we define* $\overline{\delta\rho} = \mathcal{B}_2(m)$ *as the solution of*

$$-\mathcal{F}(\rho_0) - \mathcal{DF}(\rho_0)\overline{\delta\rho} = m. \tag{3.3}$$

If T *is small enough, then there exists a unique* $\overline{\delta\rho} \in \mathop{\widehat{C}}\limits_{\circ}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})$ *satisfying* (3.3), *and*

$$\|\overline{\delta\rho}\|_{\widehat{C}^{4+\alpha,(4+\alpha)/2}(\Gamma_{0T})} \le C\left(\|m\|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})} + \|\mathcal{F}(\rho_0)\|_{C^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})}\right), \tag{3.4}$$

where C *only depends on* σ, $g^{(j)}$, $\phi^{(j)}$.

The proof of Lemma 3.1 is straight forward with tedious calculations. We omit the detail. The main difficulty is to prove Lemma 3.2, which is left in Sections 4–5. The proof of Theorem 2.2 can be done by Schuader fixed point theorem, the reader can be referred to paper [11]. We shall skip the proof here.

§4. Model Problem

In order to study the invertibility of $\mathcal{F}'(\rho_0)$ and prove Lemma 3.2, we will solve the equation:

$$\begin{cases} -\mathcal{DF}(\rho_0)\overline{\delta\rho} = m + \mathcal{F}(\rho_0) \equiv \overline{m} & \text{in } \Gamma_{0T}, \\ \overline{\delta\rho}\big|_{t=0} = 0 \end{cases}$$

for any $m \in \mathop{C}\limits_{\circ}^{1+\alpha,(1+\alpha)/2}(\Gamma_{0T})$, where $\overline{\delta\rho}$ is unknown.

Using partition of unity (localization), freezing the coefficient at $t = 0$, neglecting the lower order terms and using the continuity methods (see [4, Theorem 5.2]), we only need to consider the simplest model problem as follows (see [10, 11]):

$$\partial_t W^{(j)} - \alpha^{(j)}\nabla_z^2 W^{(j)} = 0 \qquad \text{in } \mathbb{R}_j^2 \times (0,\infty), \tag{4.1}$$

$$W^{(j)}\big|_{t=0} = 0, \tag{4.2}$$

$$\partial_{z_2} W^{(1)} = \alpha^{(2)}\partial_{z_2} W^{(2)} \qquad \text{on } \{z_2 = 0\} \times (0,\infty), \tag{4.3}$$

$$W^{(1)} - W^{(2)} = -\sigma\partial_{z_1}^2\delta\rho + \beta\delta\rho \qquad \text{on } \{z_2 = 0\} \times (0,\infty), \tag{4.4}$$

$$\partial_t\delta\rho + \partial_{z_2} W^{(1)} = G \qquad \text{on } \{z_2 = 0\} \times (0,\infty), \tag{4.5}$$

where $G \in \mathop{C}\limits_{\circ}^{1+\alpha,(1+\alpha)/2}$ on $\{z_2 = 0\} \times (0,\infty)$ is a known compactly supported function, $\beta = (\partial_n\phi^{(2)}(p) - \partial_n\phi^{(1)}(p)) > 0$ for any $p \in \Gamma_0$, $R_j^2 = \{(z_1, z_2) : (-1)^j z_2 > 0\}$. We take the Fourier transform with respect to z_1 and the Laplace transform with respect to t in (4.1)–(4.5) and then solve the resulting problem (4.1)–(4.5) on the half line. We obtain

$$\widetilde{\delta\rho} = \widetilde{G}/g(S,\xi),$$

where $\mathcal{G}(S,\xi) = S + (\sigma\xi^2 + \beta)br_1r_2(r_1 + br_2)^{-1}$, $r_1(s,\xi) = (S + \xi^2)^{1/2}$, $r_2(s,\xi) = (S + b^2\xi^2)^{1/2}$, $S = a + i\xi_0$, $a > 0$, $(\xi_0,\xi) \in R^2$, $b = \sqrt{\alpha^{(2)}} > 1$, $\tilde{f}$ denotes the Fourier-Laplace transform of the function f in the variable z_1 and t respectively.

We now proceed to investigate the kernel $U_{\sigma,\beta}(z_1,t)$ of the symbol $\mathcal{G}^{-1}(S,\xi)$, i.e., of the fundamental solution

$$U_{\sigma,\beta}(z_1,t) = (2\pi i \cdot 2\pi)^{-1}\int_{a-i\infty}^{a+i\infty} e^{St}ds\int_{\mathbb{R}^1} e^{iz_1\xi}\frac{d\xi}{\mathcal{G}(S,\xi)}, \quad a \ge 0.$$

§5. Estimates of the Kernels of the Model Problems

To estimate the kernels of the noncoercive model problems in this section we apply a very useful generalization of Mikhlin's theorem to Hölder spaces.

The following theorem can be found in [10].

Theorem 5.1. *Suppose that* $\mathcal{K}(x,t),\ x\in I\!R^m,\ t\in I\!R$, *vanishes at* $t=0$ *and its Fourier-Laplace transform* $\check{\mathcal{K}}(S,\xi)$ *satisfies the conditions*

$$M^{(k)}_{j,h}[\check{\mathcal{K}}]\equiv\int_0^\infty\frac{d\tau_0}{\tau_0^{3/2}}\cdots\int_0^\infty\frac{d\tau_m}{\tau_m^{3/2}}\times\left\|\triangle_{\xi_0}(\tau_0)\cdots\triangle_{\xi_m}(\tau_m)\left[\xi_j^{\nu_j}\check{\phi}\left(\xi h^k,Sh^{k_0}\right)\check{K}(\xi,S)\right]\right\|_{L_2(I\!R^{m+1})}\le C_1h^{l-\nu_jk_j},$$

and

$$M^{(k)}_{0,h}[\check{\mathcal{K}}]\equiv\int_0^\infty\frac{d\tau_0}{\tau_0^{3/2}}\cdots\int_0^\infty\frac{d\tau_m}{\tau_m^{3/2}}\times\left\|\triangle_{\xi_0}(\tau_0)\cdots\triangle_{\xi_m}(\tau_m)\left[S^{\nu_0}\check{\phi}\left(\xi h^k,Sh^{k_0}\right)\check{K}(\xi,S)\right]\right\|_{L_2(I\!R^{m+1})}\le C_1h^{l-\nu_0k_0}$$

for sufficiently large $\nu_j,\ j=0,1,\cdots,n-1$, *where* $\xi=(\xi_1,\cdots,\xi_m)$ *and* $S=a+i\xi_0,\ a\ge 0$. *Then the convolution* $u=\mathcal{K}*f$ *satisfies*

$$\sum_{j=0}\mathcal{J}\left[h^{-r}\big|\|\triangle_j^{m_j}(-h^{k_j})u\|\big|_{(a)}\right]\le C_3\sum_{j=0}^m\mathcal{J}\left[h^{-r+l}\big|\|\triangle_j^{m_j}(-h^{k_j})f\|\big|_{(a)}\right],$$

where $\big|\|v\|\big|_{(a)}=\big|\|ve^{-at}\|\big|,\ u=0,\ f=0$ *for* $t<0$, $\triangle_0^{m_0}(-h^{k_0})$ *is the finite difference of order* m_0 *in the variable* t *with step size* $h^{k_0},\triangle_j^{m_j}(-h^{k_j})$ *are finite differences of order* m_j *in the variable* x_j *with step size* h^{k_j}, $\|v\|_p=\|v;L_p(I\!R^m\times I\!R)\|$, $\big|\|\cdot\|\big|$ *is any monotonic, transitionally semi-invariant norm defined on functions in* $I\!R^{m+1}$, *and* $v=0$ *for all* $t<0$.

Here $k_0=2,k_1=\cdots k_m=1$, *the* ν_j *are positive integers,* $m_jk_j>r>0$, *and* $m_jk_j>r-l>0$.

$$\phi(x,t)=\prod_{j=1}^{n-1}\phi(x_j)\phi(t),$$

where

$$\phi(z)=\sum_{k=1}^N\frac{(-1)^{k+1}N!}{k!(N-k)!}\cdot\frac{1}{k}\zeta\left(\frac{z}{k}\right).$$

Here $\zeta\in C_0^\infty((0,1)),\ \int_0^1\zeta(z)dz=1,\ \zeta\ge 0$.

Semi-invariance is connected with the validity of the estimate

$$\big|\,\|u(x,t-h)\|\,\big|_{(a)}\le\big|\,\|u(x,t)\|\,\big|,\quad\forall h\ge 0,\quad a\ge 0.$$

This latter is valid, for example, for the norms $\big|\|\cdot\|\big|=\|\cdot\|_p$ *and the Hölder norms. For* $a>0$ *we have*

$$\int_h^\infty dt\int|u(x,t-h)|^pe^{-a(t-h)p}dx\le\int_h^\infty dt\int|u(x,t-h)|^pdx=\|u\|_p^p.$$

Below we denote by $\mathcal{J}$ a "functional of maximum type", i.e., a monotonically dimensionless norm defined on functions of positive argument. In the case where $\big|\|\cdot\|\big| = \|\cdot\|_p$ and

$$I[\cdot] = I_Q[\cdot] = \left(\int_0^\infty \|\cdot\|_p^Q \frac{dh}{h}\right)^{1/Q}, \quad p, Q > 1,$$

the expression

$$\sum_{j=1}^m I_Q\left[h^{-l_j}\triangle_j^{k_j}(h)f\right] + \|f\|_p$$

is a norm in the Besov space $\mathcal{B}_{p,Q}^l(I\!R_x^m)$. If $p = Q = 1$, this norm becomes the Hölder norm.

First, it is easy to prove the following lemmas.

Lemma 5.1. *For $\phi(x,t)$, we have*

$$\big|\check{\Phi}(\xi,S)\big| \le \frac{C(N)}{(1+|\xi|^N)(1+|S|^N)}, \quad \textit{for any positive integer } N.$$

Lemma 5.2.

$$\mathrm{Re}[r_1r_2(r_1+br_2)^{-1}] \ge C_1(|S|+\xi^2)^{1/2} \ge C_1'\big|\mathrm{Im}[r_1r_2(r_1+br_2)^{-1}]\big|.$$

Lemma 5.3. *For any symbol $r(S,\xi)$, if $r(S,\xi)$ satisfies $\mathrm{Re}r \ge C_1|\mathrm{Im}r|$, then*

$$|S^1 + r(S,\xi)| \ge C_2(|S^1|+|r|), \quad \forall \mathrm{Re}S^1 \ge 0,\ \mathrm{Re}S \ge 0,\ \xi \in I\!R^1.$$

Proof. See [12].

We shall now show that for the symbol $\mathcal{G}(S,\xi)$ the following lemma holds.

Lemma 5.4. *Suppose $\sigma, \beta > 0$. Then the symbol $\mathcal{G}(S,\xi)$ satisfies*

$$M_{j,h}^{(k)}[\alpha G^{-1}] \le Ch^{1-\nu_j k_j}, \quad j = 0, 1, \tag{5.1}$$

if the ν_j are sufficiently large.

Proof. To shorten our computation we assume that $a = 0$. By a change of variables in the integrals it is easy to show that $M_{j,h}^{(k)}[\mathcal{G}^{-1}] \le h^{1-\nu_j k_j} M_j(h)$, where

$$M_1(h) = \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \times \big\|\triangle_{\xi_0}(\tau_0)\triangle_\xi(\tau_1) \times [\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]\big\|_{L_2(I\!R^2)},$$

$$M_0(h) = \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \times \big\|\triangle_{\xi_0}(\tau_0)\triangle_\xi(\tau_1) \times [S^{\nu_0}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]\big\|_{L_2(I\!R^2)}.$$

Here $\mathcal{G}(S,\xi;h) = h^{-1}S + (\beta + \sigma h^{-2}\xi^2)r(S,\xi)$, where $r(S,\xi) = r_1r_2(r_1+br_2)^{-1}$.

The estimates below make use of the ideas and methods of the estimates of [10], which are very closely related to Hólder spaces.

We proceed to the proof. We set

$$\begin{aligned}\triangle_\xi(\tau_1)[\mathcal{G}^{-1}\xi^{\nu_1}\check{\phi}] = &\Big\{(\xi+\tau_1)^{\nu_1}\check{\phi}(S,\xi+\tau_1) \times \big[h^{-1}S + (1+h^{-2}(\xi+\tau_1)^2)r(S,\xi+\tau_1)\big]^{-1} \\ &-(\xi)^{\nu_1}\check{\phi}(S,\xi) \times \big[h^{-1}S + (1+h^{-2}(\xi+\tau_1)^2)r(S,\xi)\big]^{-1}\Big\} \\ &+(\xi)^{\nu_1}\check{\phi}(S,\xi)\Big\{\big[h^{-1}S + (1+h^{-2}(\xi+\tau_1)^2)r(S,\xi)\big]^{-1} \\ &-\big[h^{-1}S + (1+h^{-2}\xi^2)r(S,\xi)\big]^{-1}\Big\} \\ \equiv\ &\triangle_1'[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}] + \triangle_1''[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}].\end{aligned}$$

In the case (5.2) ξ^{ν_1} is replaced by S^{ν_0}. The definition of the operators $\triangle'_{\xi_0}$ and $\triangle''_{\xi_0}$ is the same as above.

It is obvious that

$$\begin{aligned}
M_1(h) &= \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \times \left\|(\triangle'_0+\triangle''_0)(\triangle'_1+\triangle''_1)[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]\right\|_{L_2(I\!R^2)} \\
&\le \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \left\|\triangle'_0\triangle'_1[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]\right\|_{L_2(I\!R^2)} \\
&\quad + \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\triangle'_0\triangle''_1[\cdot]\|_{L_2(I\!R^2)} + \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\triangle''_0\triangle'_1[\cdot]\|_{L_2(I\!R^2)} \\
&\quad + \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\triangle''_0\triangle''_1[\cdot]\|_{L_2(I\!R^2)} \\
&\equiv \text{(I)+(II)+(III)+(IV)}.
\end{aligned}$$

We first consider the integral (I). It is obvious that

$$\begin{aligned}
\text{(I)} &= \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \times \left\|\triangle'_0\triangle'_1[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]\right\|_{L_2(I\!R^2)} \\
&= \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} + \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \int_{Q_1(h)}^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} \\
&\quad + \int_{Q_0(h)}^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} + \int_{Q_0(h)}^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_{Q_1(h)}^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} \\
&\equiv \mathrm{I}_{1,1} + \mathrm{I}_{1,2} + \mathrm{I}_{1,3} + \mathrm{I}_{1,4},
\end{aligned}$$

where the functions $Q_j(h)$ are defined below. For the integral $\mathrm{I}_{1,1}$, we have

$$\begin{aligned}
\mathrm{I}_{1,1} &= \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} \\
&\le \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|^\alpha_{L_q(I\!R^2)} \|\cdot\|^{1-\alpha}_{L_p(I\!R^2)}, \qquad (5.3)
\end{aligned}$$

where $\alpha/q + (1-\alpha)/p = \frac{1}{2}$, $\forall\alpha \in \left(0, \frac{1}{2}\right)$, $1 < q < 2$. We estimate the finite differences over ξ_0, ξ in the norms $\|\cdot\|_{L_q}$ $\|\cdot\|_{L_p}$ in terms of the corresponding derivatives. After simple computations we obtain

$$\begin{aligned}
\|\cdot\|^\alpha_{L_q(I\!R^2)} \|\cdot\|^{1-\alpha}_{L_p(I\!R^2)} \le \text{const.}\ (\tau_0\tau_1) &\times \left\|D'_{\xi_0}D'_\xi[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)]\right\|^\alpha_{L_q(I\!R^2)} \\
&\times \left\|D'_{\xi_0}D'_\xi[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)]\right\|^{1-\alpha}_{L_p(I\!R^2)}. \qquad (5.4)
\end{aligned}$$

We also have

$$\begin{aligned}
&|D'_1[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]| \\
&\le C_1\Big[\Big|\frac{\partial}{\partial\xi}\left(\xi^{\nu_1}\check{\phi}(S,\xi)\right)\Big||\mathcal{G}|^{-1} + |\xi^{\nu_1}\check{\phi}(S,\xi)|\ |r|^{-1}|\mathcal{G}|^{-1}\Big], \qquad (5.5) \\
&|D'_0[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]| \\
&\le C_0\Big[\Big|\frac{\partial}{\partial S}\left(\xi^{\nu_1}\check{\phi}(S,\xi)\right)\Big||\mathcal{G}|^{-1} + |\xi^{\nu_1}\check{\phi}(S,\xi)|\ |r|^{-2}|\mathcal{G}|^{-1}\Big], \qquad (5.6)
\end{aligned}$$

$$
\begin{aligned}
&|D_0'D_1'[\xi^{\nu_1}\check{\phi}(S,\xi)\mathcal{G}^{-1}(S,\xi;h)]| \\
&\le C_2\Big[\Big|\frac{\partial}{\partial S}\left(\xi^{\nu_1}\check{\phi}(S,\xi)\right)\Big||\mathcal{G}|^{-1}+|\xi^{\nu_1}\check{\phi}(S,\xi)|\ |r|^{-2}|\mathcal{G}|^{-1}\Big].
\end{aligned}\tag{5.7}
$$

It is not hard to see that

$$
\begin{aligned}
M_{1,1}&=\Big\||\mathcal{G}|^{-1}\Big|\frac{\partial}{\partial S}\left(\xi^{\nu_1}\check{\phi}(S,\xi)\right)\Big|\Big\|^q_{L_q}\\
&\le \big\||\mathcal{G}|^{-1}|\xi|^{\nu_1}|S||\check{\phi}(S,\xi)|\big\|^q_{L_q}\\
&\le \big\||\xi|^{\nu_1-1}|S||\check{\phi}(S,\xi)|\big\|^q_{L_q}\quad (\text{by } |\mathcal{G}|\ge|r(S,\xi)|\ge|\xi|)\\
&\le \text{const.}\int|\xi|^{(\nu_1-1)q}(1+|\xi|^N)^{-1}d\xi\int\frac{|S|^q}{1+|S|^N}dS\\
&\le C_3,
\end{aligned}\tag{5.8}
$$

if $(\nu_1-1)q>-1,\ N-(\nu_1-1)q>1$ and $N-q>1$.

$$
\begin{aligned}
M_{1,2}&=\big\||\xi|^{\nu_1}\check{\phi}(S,\xi)||r|^{-2}|\mathcal{G}|^{-1}\big\|^q_{L_q}\\
&\le \big\||\xi|^{\nu_1-3}|\check{\phi}(S,\xi)|\big\|^q_{L_q}\\
&\le \text{const.}\int|\xi|^{\nu_1-3}(1+|\xi|^N)^{-1}d\xi\int(1+|S|^N)^{-1}dS\\
&\le C_4,
\end{aligned}\tag{5.9}
$$

if $\nu_1-3>-1,\ N-(\nu_1-3)>1$ and $N>1$.

Similarly, we have

$$
M_{1,3}=\Big\||\mathcal{G}|^{-1}\Big|\frac{\partial}{\partial S}(\xi^{\nu_1}\check{\phi}(S,\xi))\Big|\Big\|^p_{L_p}\le\text{const.},\tag{5.10}
$$

$$
M_{1,4}=\big\||\xi|^{\nu_1}|\check{\phi}(S,\xi)||r|^{-2}\mathcal{G}|^{-1}\big\|^p_{L_p}\le\text{const.}\tag{5.11}
$$

We set $Q_0(h)=Q_1(h)=1$. Thus, from (5.3)–(5.11), we obtain

$$
\mathrm{I}_{1,1}\le C_5\int_0^1\frac{d\tau_0}{\tau_0^{1/2}}\int_0^1\frac{d\tau_1}{\tau_1^{1/2}}\le C_6.\tag{5.12}
$$

For the integral $I_{1,2}$, we estimate the finite differences over ξ_0,ξ in the norm $\|\cdot\|_{L_q}$ and over ξ_0 in the norm $\|\cdot\|_{L_p}$ in terms of corresponding derivatives. After simple computations we obtain

$$
\begin{aligned}
\|\cdot\|^{\alpha}_{L_q}\|\cdot\|^{1-\alpha}_{L_p}\le\text{const.}\ (\tau_0\tau_1^{\alpha})\times\big\|D'_{\xi_0}D'_{\xi}[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)]\big\|^{\alpha}_{L_q}\\
\times\big\|D'_{\xi_0}[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)]\big\|^{1-\alpha}_{L_p}.
\end{aligned}\tag{5.13}
$$

Noticing (5.6)–(5.9) and (5.13), we have

$$
\mathrm{I}_{1,2}\le C_7\int_0^1\frac{d\tau_0}{\tau_0^{1/2}}\int_1^{\infty}\tau_1^{-\frac{3}{2}+\alpha}\le C_5.\tag{5.14}
$$

The integral $\mathrm{I}_{1,3}$ is similar to $\mathrm{I}_{1,2}$, so we have $\mathrm{I}_{1,3}\le C_9$. For the integral $I_{1,4}$, noticing that

$$
\|\cdot\|^{\alpha}_{L_q}\|\cdot\|^{1-\alpha}_{L_p}\le\text{const.}\ (\tau_0^{\alpha}\tau_1^{\alpha})\times\big\|D'_{\xi_0}D'_{\xi}[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)]\big\|^{\alpha}_{L_q}\times\big\|\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)\big\|^{1-\alpha}_{L_p},
$$

and (5.7)–(5.9) and

$$
M_{1,5}=\|\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}(S,\xi;h)\|^p_{L_p}\le\ \text{const.},\tag{5.15}
$$

we obtain

$$\mathrm{I}_{1,4} \le \text{ const.} \int_1^\infty \tau_0^{-\frac{3}{2}+\alpha} d\tau_0 \int_1^\infty \tau_1^{-\frac{3}{2}+\alpha} d\tau_1 \le C_{10}.$$

Hence, we have

$$(\mathrm{I}) = \mathrm{I}_{1,1} + \mathrm{I}_{1,2} + \mathrm{I}_{1,3} + \mathrm{I}_{1,4} \le \text{const.}$$

Next, we will study the integral (IV):

$$\begin{aligned}(\mathrm{IV}) &= \int_0^\infty \frac{d\tau_0}{\tau_0^{3/2}} \cdot \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} \times \left\| \triangle_0'' \triangle_1'' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}(S,\xi;h)] \right\|_{L_2(I\!R^2)} \\ &= \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \cdot \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} + \int_0^{Q_0(h)} \frac{d\tau_0}{\tau_0^{3/2}} \int_{Q_1(h)}^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} \\ &\quad + \int_{Q_0(h)}^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_0^{Q_1(h)} \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} + \int_{Q_0(h)}^\infty \frac{d\tau_0}{\tau_0^{3/2}} \int_{Q_1(h)}^\infty \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(I\!R^2)} \\ &\equiv \mathrm{I}_{4,1} + \mathrm{I}_{4,2} + \mathrm{I}_{4,3} + \mathrm{I}_{4,4}.\end{aligned}$$

Noticing that

$$\begin{aligned}&\triangle_1' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}(S,\xi;h)] \\ &= \left\{ [h^{-1}S + (1 + h^{-2}(\xi+\tau_1)^2) r(S,\xi)]^{-1} - [h^{-1}S + (1 + h^{-2}\xi^2) r(S,\xi)]^{-1} \right\} \xi^{\nu_1} \check{\phi}(S,\xi) \\ &= \left\{ [h^{-1}S + (1 + h^{-2}\eta^2) r(S,\xi)]^{-1} \Big|_{\eta=\xi}^{\xi+\tau_1} \right\} \xi^{\nu_1} \check{\phi}(S,\xi) \\ &\triangleq F(S,\xi;\eta) \Big|_{\eta=\xi}^{\xi+\tau_1} \cdot \xi^{\nu_1} \check{\phi}(S,\xi) \\ &= F_\eta'(S,\xi;\xi+\lambda_1)\tau_1 \cdot \xi^{\nu_1} \check{\phi}(S,\xi), \quad 0 < \lambda_1 < \tau_1 \\ &= -[h^{-1}S + (1 + h^{-2}(\xi+\lambda_1)^2) r(S,\xi)]^{-2} \cdot 2h^{-2} r(S,\xi)(\xi+\lambda_1) \xi^{\nu_1} \check{\phi}(S,\xi) \cdot \tau_1,\end{aligned}$$

we have

$$\begin{aligned}&D_\xi'' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}] \\ &= -[h^{-1}S + (1 + h^{-2}(\xi+\lambda_1)^2) r(S,\xi)]^{-2} \cdot 2h^{-2} r(S,\xi)(\xi+\lambda_1) \xi^{\nu_1} \check{\phi}(S,\xi).\end{aligned} \tag{5.17}$$

Similarly, we have

$$\begin{aligned}D_{\xi_0}'' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}] &= -[h^{-1}(S + i\lambda_0) + (1 + h^{-2}\xi^2) r(S,\xi)]^{-2} \\ &\quad \cdot h^{-1} i \xi^{\nu_1} \check{\phi}(S,\xi), \quad 0 < \lambda_0 < \tau_0,\end{aligned} \tag{5.18}$$

$$\begin{aligned}D_{\xi_0}'' D_\xi'' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}(S,\xi;h)] &= [h^{-1}(S + i\lambda_0) + (1 + h^{-2}(\xi+\lambda_1)^2) r(S,\xi)]^{-3} \\ &\quad \cdot 4h^{-3} i r(S,\xi)(\xi+\lambda_1) \xi^{\nu_1} \check{\phi}(S,\xi).\end{aligned} \tag{5.19}$$

So, we have

$$\begin{aligned}M_{4,1} &= \| D_{\xi_0}'' D_\xi'' [\xi^{\nu_1} \check{\phi}(S,\xi) \mathcal{G}^{-1}(S,\xi;h)] \|_{L_q}^q \\ &\le \text{const.} \int \frac{|\xi|^{\nu_1 q}}{1 + |\xi|^N} d\xi \\ &\quad \cdot \int_{-\infty}^\infty \frac{|\xi+\lambda_1|^q (h^{-3}|r|)^q}{|h^{-1}(S + i\lambda_0) + (1 + h^{-2}(\xi+\lambda_1)^2) r(S,\xi)|^{3q} \cdot (1 + |S|^N)} dS\end{aligned}$$

$$\leq \text{const.} \int \frac{|\xi|^{\nu_1 q}}{1+|\xi|^N} d\xi$$

$$\cdot \int_{-\infty}^{\infty} \frac{|\xi+\lambda_1|^q h^{-3q}}{|h^{-1}(S+i\lambda_0)+(1+h^{-2}(\xi+\lambda_1)^2)r(S,\xi)|^{2q}\cdot(1+|S|^N)} dS$$

(by Lemma 5.3)

$$\leq \text{const.} \int \frac{|\xi|^{\nu_1 q}}{1+|\xi|^N} d\xi \cdot \int_{-\infty}^{\infty} \frac{h^{-3q}|\widehat{\xi}|^q}{(1+h^{-2}|\widehat{\xi}|^2)^q |\xi|^q \cdot (h^{-1}|\widehat{S}|+|\xi|)^q} d\widehat{S}$$

(where $\widehat{S} = S + i\lambda_0,\ \widehat{\xi} = \xi + \lambda_1,\ 0 < \lambda_0 < \tau_0,\ 0 < \lambda_1 < \tau_1$)

$$\leq \text{const.} \int \frac{h^{-3q+1}|\widehat{\xi}|^q}{(1+h^{-2}|\widehat{\xi}|^2)^q} d\widehat{\xi} \leq \text{const. } h^{-2q+2}. \tag{5.20}$$

So, we have (let $Q_0(h) = Q_1(h) = h$)

$$\begin{aligned} \mathrm{I}_{4,1} &= \int_0^h \frac{d\tau_0}{\tau_0^{3/2}} \int_0^h \frac{d\tau_1}{\tau_1^{3/2}} \|\cdot\|_{L_2(\mathbb{R}^2)} \\ &\leq \text{const.} \int_0^h \frac{d\tau_0}{\tau_0^{3/2}} \int_0^h \frac{d\tau_1}{\tau_1^{3/2}} (\tau_0\tau_1) \times \left\|D''_{\xi_0} D''_{\xi}[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}]\right\|_{L_q}^{\alpha} \times \left\|D''_{\xi_0} D''_{\xi}[\xi^{\nu_1}\check{\phi}\mathcal{G}^{-1}]\right\|_{L_p}^{1-\alpha} \\ &\leq \text{const.} \int_0^h \frac{d\tau_0}{\tau_0^{1/2}} \int_0^h \frac{d\tau_1}{\tau_1^{1/2}} \cdot h^{\alpha(-2q+2)/q+(1-\alpha)(-2p+2)/p} \quad \text{by (5.20)} \\ &\leq \text{const.} \int_0^h \frac{d\tau_0}{\tau_0^{1/2}} \int_0^h \frac{d\tau_1}{\tau_1^{1/2}} \cdot h^{-1} \leq \text{const.} \end{aligned} \tag{5.21}$$

The integrals $\mathrm{I}_{4,2}, \mathrm{I}_{4,3}, \mathrm{I}_{4,4}$ can be estimated in the same way as above. Thus, we have

$$\text{(IV)} \leq \text{ const.}$$

In the same way as above, we also obtain

$$\text{(II)},\ \ \text{(III)} \leq \text{const.}$$

Here we take $Q_0(h) = 1,\ Q_1(h) = h$ and $Q_0(h) = h,\ Q_1(h) = 1$ for the integrals (II) and (III) respectively.

Finally, we have

$$M_1(h) = \text{(I)+(II)+(III)+(IV)} \leq \text{const.}$$

The estimate about $M_0(h)$ is the same as above. It has thus been proved that (5.1) holds for $a = 0$. Since the constant in Lemma 5.3 does not depend on Re $S \geq 0$, (5.6) holds for any $a \geq 0$ with a constant not depending on a. Lemma 5.4 is proved.

Lemma 5.5. *The symbol $\mathcal{G}_1 = S\mathcal{G}^{-1}(S,\xi)$ satisfies*

$$M_{j,h}^{(k)}[\mathcal{G}_1] \leq \text{const. } h^{-\nu_j k_j}, \quad j = 0, 1, \tag{5.22}$$

if the ν_j are sufficiently large.

The proof proceeds in the same way as the proof of Lemma 5.4. In this case

$$\mathcal{G}_1(S,\xi;h) = h^{-1}S[h^{-1}S + (\beta + h^{-2}\xi^2)r(S,\xi)]^{-1}.$$

It is easy to prove the following

Lemma 5.6.

$$M_{j,h}^{(k)}[r(S,\xi)^{-1}] \leq \text{const. } h^{1-\nu_j k_j}, \tag{5.23}$$

$$M_{j,h}^{(k)}[(r_1+br_2)^{-1}(S,\xi)] \leq \text{const. } h^{1-\nu_j k_j},\ j=0,1, \tag{5.24}$$

if the ν_j are sufficiently large.

Noticing that

$$\sigma\xi^2\delta\tilde{\rho} = s\delta\rho\cdot r(S,\xi)^{-1}\delta\tilde{\rho} - \tilde{G}\cdot r(S,\xi)^{-1}$$

and Theorem 5.1, Lemmas 5.4–5.6, we have

Theorem 5.2. *Suppose $\beta>0$, $\sigma>0$. For any compactly supported $\check{\Phi}$, $\check{G}\in \underset{\circ}{C}^{1+\alpha,(1+\alpha)/2}(I\!R^1_{z_1}\times(0,\infty))$, there exists a unique solution $W^{(j)}\in \underset{\circ}{C}^{2+\alpha,(2+\alpha)/2}(I\!R^2_j\times(0,\infty))$, $j=1,2$, $\delta\rho\in \underset{\circ}{\widehat{C}}^{4+\alpha,(4+\alpha)/2}(I\!R^1_{z_1}\times(0,\infty))$ of problem (7.21)–(7.25). Moreover,*

$$\sum_{j=1}^{2}\|W^{(j)}\|_{C^{2+\alpha,(2+\alpha)/2}(I\!R^2_j\times(0,\infty))} + \|\delta\rho\|_{\widehat{C}^{4+\alpha,(4+\alpha)/2}(I\!R^1_{z_1}\times(0,\infty))}$$
$$\leq C\left[\|\Phi\|_{C^{1+\alpha,(1+\alpha)/2}(I\!R^1_{z_1}\times(0,\infty))} + \|G\|_{C^{1+\alpha,(1+\alpha)/2}(I\!R^1_{z_1}\times(0,\infty))}\right]$$

Finally, we remove the technical condition $\beta>0$ by Schauder fixed point theorem, the reader can be referred to paper [11].

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