Chin. Ann. of Math.
18B: 4(1997),405-422.

AN IMPROVEMENT OF A RESULT OF IVOCHKINA AND LADYZHENSKAYA ON A TYPE OF PARABOLIC MONGE-AMPÈRE EQUATION

Wang Rouhuai* Wang Guangli*

Abstract

For the initial-boundary value problem about a type of parabolic Monge-Ampére equation of the form (IBVP): $\{-D_t u + (\det D_x^2 u)^{1/n} = f(x,t),\ (x,t) \in Q = \Omega \times (0,T],\ u(x,t) = \phi(x,t)\ (x,t) \in \partial_p Q\}$, where Ω is a bounded convex domain in $\mathbf{R}^n$, the result in [4] by Ivochkina and Ladyzheskaya is improved in the sense that, under assumptions that the data of the problem possess lower regularity and satisfy lower order compatibility conditions than those in [4], the existence of classical solution to (IBVP) is still established (see Theorem 1.1 below). This can not be realized by only using the method in [4]. The main additional effort the authors have done is a kind of nonlinear perturbation.

Keywords Nonlinear perturbation, Less regularity about data, Interior regularity of viscosity solutions

1991 MR Subject Classification 35K20, 35K55, 35Q99

Chinese Library Classification O175.26, O175.29

§1. Introduction

In a recent paper by Ivochkina and Ladyzhenskaya[4], they discussed the initial-boundary value problem for a type of parabolic Monge-Ampére equation:

$$\begin{cases} \mathcal{P}u := -D_t u + (\det D_x^2 u)^{1/n} = f(x,t), & (x,t) \in Q = \Omega \times (0,T], \\ u(x,t) = \phi(x,t), & (x,t) \in \partial_p Q, \end{cases} \tag{1.1}$$

where Ω is a bounded convex domain in $\mathbf{R}^n$, $D_t u = \frac{\partial u}{\partial t}$, $D_x^2 u$ is the Hessian of the function $u(x,t)$, i.e., $D_x^2 u = (u_{ij}) = (\frac{\partial^2 u}{\partial x_i \partial x_j}), i,j = 1,2,\cdots,n$, $\partial_p Q$ denotes the parabolic boundary of Q. As a solution of (1.1), $u(x,t)$ should be a strictly convex function in $x \in \Omega$ for any fixed $t \in [0,T]$, so the differential equation in (1.1) is a non-uniformly parabolic equation.

Under the structure conditions that either

$$\begin{cases} \min\limits_Q f + \min\limits_{(x,t)\in\partial_p Q} D_t\phi(x,t) - \dfrac{1}{2} a d^2 \equiv \nu_1 > 0, \\ d \text{ is the radius of the minimal ball containing } \Omega, \\ a = \max\{0; \max\limits_Q D_t f\}, \end{cases} \tag{1.2}$$

Manuscript received January 24, 1997. Revised April 2, 1997.

*Institute of Mathematics, Jilin University, Changchun 130023, China.

**Department of Mathematics, Jilin University, Changchun 130023 China.

***Project supported by the National Natural Science Foundation of China.

or

$$\begin{cases} \min\limits_{\partial_p Q}(D_t\phi + f) \equiv \tilde{\nu}_1 > 0, \\ f(x,t) \text{ is a concave function in } x \in \overline{\Omega} \text{ for any fixed } t \in [0,T], \\ (\det D_x^2\phi(x,0))^{1/n} \text{ is a concave function in } x \in \overline{\Omega}, \end{cases} \tag{1.2'}$$

as well as

$$\mathcal{P}\phi(x,t) = f(x,t), \quad \forall x \in \overline{\Omega} \ \ t = 0, \tag{1.3}$$

they essentially established the following existence result of solution with global regularity in [4]:

If Ω is a bounded and strictly convex domain in $\mathbf{R}^n$, $\partial\Omega \in C^4, f(x,t) \in C^{2,1}(\overline{Q}), \phi(x,t) \in C^{4,2}(\overline{Q})$, $(D_x^2\phi(x,0)) > 0$ on $\overline{\Omega}$; f and ϕ satisfy the compatibility conditions up to the first order, and moreover either both (1.2) *and* (1.3) *or both* (1.2') *and* (1.3) *are valid, then problem* (1.1) *has a unique solution $u(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$, $\forall \alpha \in (0,1)$.*

As mentioned by Ivochkina and Ladyzhenskaya in [4], our early work [6] studied another kind of parabolic Monge-Ampére equation, i.e.,

$$\begin{cases} -D_t u \det D_x^2 u = f(x,t), \quad (x,t) \in Q = \Omega \times (0,T], \\ u(x,t) = \phi(x,t), \quad (x,t) \in \partial_p Q, \end{cases} \tag{1.1'}$$

where $u(x,t)$ is strictly "convex-monotone" i.e., strictly convex in x and strictly decreasing in t on $\overline{Q}$. And, as can be seen, the results in both [6] and [4] have one thing in common: under higher regularity and compatibility assumptions they obtained solutions of higher regularity; but limited by the method used there (which is a kind of "linear perturbation"), the higher regularity assumptions can not be relaxed in those papers, even at the expense of only lower (or even only lower and interior) regularity of the solutions is required.

By further extending the idea and techniques used in [8], which so relaxed the regularity and compatibility conditions of the data in [6] that the existence of classical solution with interior regularity was still obtained, we improve their result in [4] mentiened above.

Besides $f(x,t)$ being only required to be Lipschitz continuous, the condition (1.3), which is a little stronger than the first order compatibility condition, can also be relaxed. When (1.2) holds, we replace (1.3) with something like the "one-side first order compatibility condition" as follows

$$f(x,0) - [-D_t\phi(x,0) + (\det(D_x^2\phi(x,0)))^{\frac{1}{n}}] \le 0, \quad \forall\, x \in \partial\Omega. \tag{1.4}$$

When (1.2') is valid, there is not any kind of compatibility conditions required, but, to replace (1.3), we need other restrictions on $\phi(x,t)$, i.e.

$$\begin{cases} D_t\phi(x,t) \ \text{ is increasing in } t \text{ near } 0 \text{ for } x \in \partial\Omega, \\ -D_t\phi(x,0) \ \text{ is concave in } x \in \overline{\Omega}. \end{cases} \tag{1.4'}$$

The main result in this note is the following

Theorem 1.1. *Assume that Ω is a bounded and strictly convex domain in $\mathbf{R}^n$, $\partial\Omega \in C^2$, $f(x,t) \in C^{0+1,0+1}(\overline{Q})$ (i.e., $f(x,t)$ is Lipschitz continuous with respect to x and t), $\phi(x,t) \in C^{2,1}(\overline{Q})$, and $(D_x^2\phi(x,0)) > 0$. If, moreover, either conditions both* (1.2) *and* (1.4), *or conditions both* (1.2') *and* (1.4'), *are valid, then the problem* (1.1) *has a unique solution $u(x,t)$ which is Lipschitz continuous on $\overline{Q}$ and belongs to $C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(Q)$.*

Remark. Here and in the sequel the notation of the subscript *loc* means locally in the parabolic sense, e.g. $f(x,t) \in C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(Q)$ means $f(x,t) \in C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{D})$ for any domain D with $\overline{D} \subset Q$, which is also denoted by $D \subset\subset Q$ (note $Q = \Omega \times (0,T]$).

Theorem 1.1 will be proved by an approach consisting of establishing the uniqueness, existence and $C_{loc}^{2+\alpha,1+\frac{\alpha}{2}}(Q)$-regularity of the viscosity solution to (1.1), which is, of course, also a classical solution to (1.1).

In §2 we show the uniqueness and existence of the viscosity solution to (1.1). This viscosity solution is specially constructed as the limit of certain approximation solutions, their properties are also studied there.

The regularity is established via nonlinear perturbation introduced by Caffarelli in [1] and [2]. To realize the procedure one needs both the existence of the solution to the "frozen problem" and suitable interior estimates of its higher order derivatives. These are given in §3.

The regularity result is stated in §4.

Those approximation solutions in §2 are of global regularity. Because we assume that neither $f, \phi, \partial\Omega$ are smooth enough, nor the first order compatibility condition is satisfied, in order to use the result in [4] (more precisely, what we use is the approach to establish the result) to obtain the existence of such approximation solutions, we need, after smoothing f, ϕ, $\partial\Omega$ and so on, to modify the value of the smoothed $f(x,t)$ in a neighborhood of the base of the smoothed cylindrical domain, so that the compatibility conditions up to the first order be satisfied. Of course, the range where the smoothed $f(x,t)$ is modified must be getting smaller and smaller and eventually tending to an empty set. Obviously this requirement certainly brings some trouble to the task of establishing uniform (independent of the range of modification) estimate of the derivatives with respect to t of the approximation solutions, which is of crucial importance in our approach. We confront with the same kind of difficulties in dealing with the "frozen problem" as can be seen in §3, where we show how to overcome these difficulties.

Owing to the form of the equation in (1.1) treated here, which is different from the corresponding parabolic Monge-Ampère equations in [8], in proving the existence of approximation solutions or of the solutions to the frozen problem, in the present case we need to verify the necessary condition of preventing them from blow up, i.e. "the sum of the derivative of the solution with respect to t and the function on the right hand side of the equation must be strictly positive", which is showed in §5.

By the way, comparing the result here with an early one on the existence of solutions in $C_{loc}^{2+\alpha,1+\alpha/2}(Q)$ to (1.1') in [5], one can see that $f(x,t)$ needs to be in $C^{2,1}(\overline{Q})$ in [5] but only needs to be Lipschitz continuous on $\overline{Q}$ in [8] and in this note. Actually, in [5] the author did not give the proof (even the precise formulation) of a theorem of the existence of solutions with global regularity, as it was needed to get his conclusion, so the problem of how to deal with less regularity of $f(x,t)$ and the lack of compatibility conditions as we do in this note and in [8] were ignored. Mainly the interior estimate of second order spatial derivatives (like Theorem 3.5 in this note) and some lower order estimates for the solutions to (1.1') are obtained in [5]. Moreover, as can be seen in [8] and in this note, to realize the nonlinear perturbation, we have more and crucial work to do. Of course, owing to the special structure of the equation in (1.1'), there is no problem of "preventing the solution from blow up" in [5] to be treated, but we need to treat such problem for (1.1).

§2. The Special Viscosity Solution

Assumptions.

(A1) *Ω is a bounded and strictly convex domain $\subset \mathbf{R}^n$ with C^2 boundary, i.e. there is a strictly convex function $\Psi(x) \in C^2(\mathbf{R}^n)$ such that $\Omega = \{x \in \mathbf{R}^n | \Psi(x) < 0\}$ with $\partial\Omega = \{x \in \mathbf{R}^n | \Psi(x) = 0\}$ and that $|D_x\Psi(x)| \neq 0, \ \forall\, x \in \partial\Omega$.*

(**A2**) $f(x,t) \in C^{0+1,0+1}(\overline{Q})$.

(**A3**) *$\phi(x,t) \in C^{2,1}(\overline{Q})$ with $(D_x^2\phi(x,0)) > 0, \forall x \in \overline{\Omega}$. (Hence, by* (A1)*, one may assume that $\phi(x,t)$ is strictly convex in x for any fixed $t \in [0,T]$.)*

(**A4**) *Either* (1.2) *and* (1.4)*, or* (1.2') *and* (1.4')*, are valid.*

Convention.

We will say that "a constant C is under control" or "a controllable constant C", if the constant C depends only on the date in (A1)— (A4)*, e.g. the C^2 norm of $\partial\Omega$, the Lipschitz constant of $f(x,t)$, the $C^{2,1}$ norm of ϕ, bounds of the eigenvalues of $(D_x^2\phi(x,t))$ as well as n—the dimension of $\mathbf{R}^n$, etc.*

Definition. *We call $u(x,t)$ a viscosity subsolution (supersolution) of the equation*

$$\mathcal{P}u(x,t) = f(x,t) \ \text{ in } Q, \tag{2.1}$$

if $u(x,t) \in C(\overline{Q})$ is convex in x and there exists a constant $C > 0$ such that $u(x,t) - Ct$ is strictly decreasing in t. Moreover for any $\psi \in C^{2,1}(\overline{Q})$, whenever

$$u(x,t) - \psi(x,t) \leq (\geq) u(x_0,t_0) - \psi(x_0,t_0), \ \ \forall\,(x,t) \in Q \cap \{t \leq t_0\},$$

we must have

$$\mathcal{P}\psi(x_0,t_0) \geq (\leq) f(x_0,t_0)$$

(for supersolution, we also require that $(D_{ij}\psi(x_0,t_0)) > 0$ in matrix sense). If $u(x,t)$ is both a viscosity subsolution and supersolution of (2.1)*, then we call $u(x,t)$ a viscosity solution of* (2.1)*. By a viscosity solution $u(x,t)$ to the problem* (1.1) *we mean that $\mathcal{P}u = f$ in Q in the sense of viscosity solution with $u(x,t) = \phi(x,t)$ holding point wise everywhere on ∂_pQ.*

As a standard consequence of this definition we have:

Let $u_k(x,t)$, $u(x,t)$, $f_k(x,t)$ and $f(x,t)$ be continuous functions on $\overline{Q}$. Assume that $u_k(x,t)$ and $f_k(x,t)$ uniformly converge to $u(x,t)$ and $f(x,t)$ on $\overline{Q}$ respectively. If there exist constants $C_k > 0$ such that $u_k(x,t) - C_kt$ is strictly decreasing in t with C_k uniformly bounded, then, in the sense of viscosity solution, "$\mathcal{P}u_k = f_k$ in Q" implies "$\mathcal{P}u = f$ in Q".

For the special viscosity solution and its approximations we have

Theorem 2.1. *If* (A1)—(A4) *hold, then the problem* (1.1) *has a unique viscosity solution $u(x,t) \in C(\overline{Q})$. Moreover there exists an approximation sequence $\{u_k(x,t)\}_{k=1}^{\infty} \subset C^{\infty}(Q) \cap C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q})$ such that*

$$\sup_Q |u_k(x,t) - u(x,t)| \to 0 \ \text{ as } k \to \infty$$

and that

$$\begin{cases} |u_k(x,t)| \le M_0, \ \ \forall\,(x,t) \in \overline{Q}, \\ |D_t u_k(x,t)| \le M_T, \ \ \forall\,(x,t) \in \overline{Q}, \\ |D_x u_k(x,t)| \le M_1, \ \ \forall\,(x,t) \in \overline{Q}, \\ D_t u_k + f_k(x,t) \ge \dfrac{1}{2}\nu_1 > 0, \quad \forall\,(x,t) \in \overline{Q} \end{cases} \tag{2.2}$$

as well as that

$$\begin{cases} \mathcal{P}u_k(x,t) = f_k(x,t), \\ \sup\limits_Q |f_k(x,t)| \le \sup\limits_Q |f(x,t)| + 1, \\ \sup\limits_Q |f(x,t)| + \sup\limits_Q |-D_t\phi + (\det D_x^2\phi)^{1/n}| \le M_2, \end{cases} \tag{2.3}$$

where the constants M_0, M_T, M_1 and M_2 are all under control.

The uniqueness can be obtained in the same way as in Proposition 2.3 in [9]. And the proof of existence can be realized by almost the same approximation procedure as in [8], with the $f_k(x,t)$ constructed by the procedure stated in the opening part of §5, i.e. set $\epsilon = \epsilon_k > 0$ small enough in (5.6) or (5.6'). But, to prove the existence of the approximation solutions is different from [8] in directly using the existence theorem of classical solution in [6]. In the present case one can not use directly the corresponding result in [4] mentioned in Section 1 of this note since the required structure conditions may not be satisfied. What we have to do is to follow the approach for deriving this theorem in [4] to establish the existence again. In so doing there are some points that we need to deal with carefully. For instance, in proving

$$|D_t u_k(x,t)| \le M_T, \ \ \forall\,(x,t) \in \overline{Q},$$

there is a little difference from [8], which can be seen from, and can be treated by the method in the proof of Proposition 3.4 below; moreover, for establishing the fact that

$$D_t u_k + f_k(x,t) \ge \frac{1}{2}\nu_1 > 0, \quad \forall\,(x,t) \in \overline{Q},$$

we give a derivation in Lemma 5.1 and Lemma 5.1' in §5. The rest part of the proof for the existence of approximation solution is omitted here.

Then obviously we have

Corollary 2.1. *Let $u(x,t)$ and $u_k(x,t)$ be those from Theorem 2.1. Then, for any constant $C > M_T + M_2$ with M_2 from (2.3), the functions*

$$v_k(x,t;C) \equiv u_k(x,t) - Ct \tag{2.4}$$

and

$$v(x,t;C) \equiv u(x,t) - Ct \tag{2.5}$$

are strictly "convex–monotone", i.e., strictly convex in x and strictly decreasing in t on $\overline{Q}$ with

$$\mathcal{P}v_k(x,t;C) = f_k(x,t) + C \ \ in\ Q, \tag{2.6}$$

$$\begin{cases} -M_T - C \le D_t v_k(x,t;C) \le M_T - C < -M_2 < 0, \ \ \forall\ k \in \mathbf{N}, \\ D_t v_k(x,t;C) + f_k(x,t) + C \ge \nu/2 > 0, \ \ \forall\ k \in \mathbf{N}, \end{cases} \tag{2.7}$$

$$\mathcal{P}v(x,t;C) = f(x,t) + C \quad in\, Q. \tag{2.8}$$

It is easily seen that the regularity of the viscosity solution $u(x,t)$ follows from the same regularity of $v(x,t;C)$ given by (2.5).

§3. Auxiliary Results for Regularity

In this section some auxiliary results for establishing the regularity of $v(x,t;C)$ given by (2.5) are formulated, most of which can be derived in similar ways as in [8] except one fact (Proposition 3.4 below).

Theorem 3.1 below tells us that, for any fixed point $(x_0,t_0) \in D'$ with $D' \subset\subset Q$, one can find a sequence of domains D_H with "desirable" boundaries, which eventually shrink to this point. Theorems 3.2 and 3.6 are corresponding to Theorem 3 in [2], which guarantee the existence of solutions to the frozen problems and the "desirable" interior estimates for the derivatives of them.

Since the proof of Theorem 3.1 in [8] is only based on the "convex–monotone" property, by Corollary 2.1 we have

Theorem 3.1. *Assume that* (A1)—(A4) *hold. If* $u(x,t)$ *is the viscosity solution to* (1.1) *obtained from Theorem* 2.1, $v(x,t;C)$ *is given by* (2.5), *then for every domain* $D' \subset\subset Q$ *there exists a constant* $H_0 > 0$ *possessing the following property: for* $(x_0,t_0) \in D'$ *and* $p = D_x v(x_0,t_0;C)$ *which is the only one among all* $p \in \mathbf{R}^n$ *that satisfies*

$$p\cdot(x-x_0) + v(x_0,t_0;C) \le v(x,t_0;C), \quad \forall\, x \in \Omega,$$

if we set , for $v(x,t;C)$ *given by* (2.5),

$$\begin{aligned} &v_H \equiv v_H(x,t;C) \equiv v(x,t;C) - p\cdot(x-x_0) - v(x_0,t_0;C) - H,\\ &D_H \equiv \{(x,t)\in Q \mid v_H(x,t;C) < 0, t \le t_0\}, \end{aligned} \tag{3.1}$$

then we have

$$\operatorname{dist}\{D_H, \partial_p Q\} > 0, \quad \forall\, H \in (0, H_0]. \tag{3.2}$$

Moreover, $\forall\, \eta > 0$, $\exists\, H(\eta) > 0$ *such that, whenever* $\forall\, H \le H(\eta)$,

$$\operatorname{diam} D_H \le \eta, \quad \forall\, (x_0,t_0) \in D'. \tag{3.3}$$

Now we are in a position to formulate the frozen problem and establish the existence of the smooth solution to it, thus building up a step stone to do nonlinear perturbation.

Theorem 3.2. *Under the conditions and notations of Theorem* 3.1, *there exists a controllable constant* $C > M_T + M_2$ *such that the problem*

$$\begin{cases} \mathcal{P}w_H(x,t;C) = f(x_0,t_0) + C \quad in\;\; D_H,\\ w_H(x,t;C) = 0 \quad on\, \partial_p D_H \end{cases} \tag{3.4}$$

has a unique solution $w_H(x,t;C) \in C^\infty(D_H) \cap C(\overline{D_H})$, *and there exists a controllable constant* $C_1 > 0$ *such that*

$$0 < C_1^{-1} \le -D_t w_H(x,t;C) \le C_1 \quad in\, D_H, \tag{3.5}$$

$$|D_x w_H(x,t;C)| \le C_1 \quad in\, D_H. \tag{3.6}$$

To prove this theorem we use twofold approximations again as in [8]. For $v_k(x,t;C)$ given by (2.4) we set

$$\begin{cases} v_{k,H} \equiv v_{k,H}(x,t;C) \\ \qquad \equiv v_k(x,t;C) - D_x v_k(x_0,t_0;C)\cdot(x-x_0) - v_k(x_0,t_0;C) - H, \\ D_{k,H} \equiv \{(x,t)\in Q \mid v_{k,H}(x,t;C)<0,\ t\le t_0\} \end{cases} \tag{3.7}$$

and

$$\begin{cases} D_{k,H;l} = D_{k,H}\cap\{t>t_{k,o}+l^{-1}\} \text{ for sufficient large } l\in\mathbf{N}, \\ t_{k,o} = \inf\{t \mid (x,t)\in D_{k,H}\} \end{cases} \tag{3.8}$$

and consider problems

$$\begin{cases} \mathcal{P}\widetilde{u}(x,t) = \widetilde{f}(x,t) \ \text{ in } G, \\ \widetilde{u}(x,t) = g(x,t) \ \text{ on } \partial_p G, \end{cases} \tag{3.9}$$

where, in case of $G = D_{k,H}$, we set

$$\begin{cases} \widetilde{f}(x,t) = f(x_0,t_0) + C, \\ g(x,t) = 0, \end{cases} \tag{3.10}$$

but in order to satisfy the compatibility conditions up to the first order, in case of $G = D_{k,H;l}$, we set

$$\begin{aligned} \widetilde{f}(x,t) &= f(x_0,t_0) + C \\ &\qquad - h\Big(\frac{t-t_{k,o}-l^{-1}}{l^{-1}}\Big)\{f(x_0,t_0)+C-\mathcal{P}v_{k,H}(x,t_{k,o}+l^{-1};C)\}, \\ g(x,t) &= 0(=v_{k,H}(x,t;C)) \ \text{ in } \partial_p D_{k,H;l}\cap\{t>t_{k,o}+l^{-1}\}, \\ g(x,t) &= v_{k,H}(x,t_{k,o};C) \ \text{ on } \partial_p D_{k,H;l}\cap\{t=t_{k,o}+l^{-1}\}, \end{aligned} \tag{3.11}$$

where

$$\begin{cases} h(s)\in C^\infty(\mathbf{R}^1) \ \text{ with } 0\le h(s)\le 1, \\ h(s)\equiv 1, \ \ \forall s\le \dfrac14, \ \ h(s)\equiv 0, \ \ \forall s\ge\dfrac12, \\ \Big|\dfrac{d}{ds}h(s)\Big| \le 8. \end{cases}$$

Now we go further along a line as in [8].

The first step is to get a simple lemma:

Lemma 3.1. *Let $D_{k,H}$ and $D_{k,H;l}$ be given by* (3.7) *and* (3.8) *respectively, $G = D_{k,H}$ or $D_{k,H;l}$. If $\widetilde{u}(x,t)\in C^{2,1}(G)\cap C(\overline{G})$ satisfies* (3.9) *with $\widetilde{f}(x,t)$ and $g(x,t)$ given by* (3.10) *or* (3.11) *respectively, then there is a controllable constant $M>0$ such that*

$$0 \ge \widetilde{u}(x,t) \ge -M \quad on\ \overline{G}.$$

The second step is to prove the key result in this note:

Proposition 3.1. *Under the conditions of Lemma* 3.1, *assume further that $\widetilde{u}(x,t)\in C^{4,2}(G)\cap C^{1,1}(\overline{G})$. Then we have*

$$\widetilde{f}(x,t) + D_t\widetilde{u}(x,t) \ge \frac14\nu_1, \quad \forall\,(x,t)\in D_{k,H;l}, \tag{3.12}$$

where the constant ν_1 *comes from* (1.2). *Moreover, there exists such a controllable constant* $C > M_T + M_2$ *that one can find a positive constant* C_1 *under control such that*

$$0 < C_1^{-1} \le -D_t\widetilde{u}(x,t) \le C_1, \ \ \forall\,(x,t)\in G, \tag{3.13}$$

$$|D_x\widetilde{u}(x,t)| \le C_1, \quad \forall\,(x,t)\in G. \tag{3.14}$$

Proof. Noticing the facts that $f \in C^{0+1,0+1}(\overline{Q})$ and $f_k \to f$ in $C(\overline{Q})$ with their first order derivatives uniformly bounded, as well as the second inequality in (2.7), one can easily see that, for k large enough, the conditions (5.9) and (5.10) of Lemma 5.2 in § 5 are all satisfied. Then by that lemma (or repeating its proof) we can obtain (3.12).

(3.14) can be proved easily (in the same way as in [8]), so we only prove (3.13) here.

Case 1. $G = D_{k,H}$.

By definition $\widetilde{u}(x,t)$ satisfies

$$\begin{cases} \mathcal{P}\widetilde{u}(x,t) = f(x_0,t_0) + C \ \text{ in } \ D_{k,H}, \\ \widetilde{u}(x,t) = 0 \quad \text{on } \ \partial_p D_{k,H}. \end{cases}$$

We thus have

$$\begin{cases} -D_t(D_t\widetilde{u}(x,t)) + \Theta(x,t)\widetilde{u}^{ij}(x,t)D_{ij}(D_t\widetilde{u}(x,t)) = 0 \ \text{ in } D_{k,H}, \\ (\widetilde{u}^{ij}(x,t)) \equiv (D_{ij}\widetilde{u}(x,t))^{-1}, \\ \Theta(x,t) \equiv \dfrac{1}{n}\det^{\frac{1}{n}}(D_{i,j}\widetilde{u}(x,t)). \end{cases}$$

By maximum principle we have

$$\inf_{\partial_p D_{k,H}}(D_t\widetilde{u}(x,t)) \le D_t\widetilde{u}(x,t) \le \sup_{\partial_p D_{k,H}}(D_t\widetilde{u}(x,t)), \ \ \forall\,(x,t)\in \ D_{k,H}.$$

To obtain a bound of $D_t\widetilde{u}(x,t)$ on $\partial_p D_{k,H}$, we take $Kv_{k,H}(x,t;C)$ as a barrier with $K>0$ being a constant to be determined. Noting (3.7) and (2.6), we have

$$\begin{aligned} \mathcal{P}[Kv_{k,H}(x,t;C)] &= -KD_tv_{k,H} + K\det^{\frac{1}{n}}(D_{ij}v_{k,H}) \\ &= K[f_k(x,t)+C]. \end{aligned}$$

Then, by (2.3), we have

$$\begin{cases} \mathcal{P}[Kv_{k,H}(x,t:C)] > f(x_0,t_0) + C = \mathcal{P}\widetilde{u}(x,t) \ \text{ in } D_{k,H} \\ \text{for } \ K=3, \ \ C \ge 4\{\sup|f|+1\}, \end{cases}$$

and

$$\begin{cases} \mathcal{P}[Kv_{k,H}(x,t;C)] < f(x_0,t_0) + C = \mathcal{P}\widetilde{u}(x,t) \ \text{ in } D_{k,H} \\ \text{for } \ K=\dfrac{1}{2}, \ \ C \ge 4[\sup|f|+1], \end{cases}$$

as well as

$$v_{k,H}(x,t;C) = \widetilde{u}(x,t) = 0 \ \text{ on } \partial_p D_{k,H}.$$

Therefore we have

$$\begin{cases} \dfrac{1}{2}D_tv_{k,H}(x,t;C) \ge D_t\widetilde{u}(x,t), \\ D_t\widetilde{u}(x,t) \ge 3D_tv_{k,H}(x,t;C), \\ \text{for } \ C \ge 4[\sup|f|+1], \ \ \forall\,(x,t)\in\partial_p D_{k,H}, \end{cases}$$

which shows that (3.13) is true for any $C > M_T + 4M_2$ in case $G = D_k$.

Case 2. $G = D_{k,H;l}$.

We divide G into three parts:

$$G_1 = D_{k,H;l} \cap \Big\{ t \le t_{k,o} + l^{-1} + \frac{1}{4} l^{-1} \Big\},$$
$$G_2 = D_{k,H;l} \setminus (G_1 \cup G_3),$$
$$G_3 = D_{k,H;l} \cap \Big\{ t > t_{k,o} + l^{-1} + \frac{1}{2} l^{-1} \Big\},$$

and consider the problem in each of them separately.

Firstly note that when $(x,t) \in G_1$ we have

$$\widetilde{f}(x,t) = f(x, t_{k,o} + l^{-1}) + C,$$
$$D_t \widetilde{f}(x,t) \equiv 0.$$

So by the same reasoning as in Case 1 we have

$$\inf_{\partial_p G_1} D_t \widetilde{u}(x,t) \le D_t \widetilde{u}(x,t) \le \sup_{\partial_p G_1} D_t \widetilde{u}(x,t), \ \ \forall\ (x,t) \in \overline{G}_1. \tag{3.15}$$

To estimate the extreme sides we use barrier again and, as in Case 1, we get

$$\begin{cases} \dfrac{1}{2} D_t v_{k,H}(x,t;C) \ge D_t \widetilde{u}(x,t) \ge 3 D_t v_{k,H}(x,t;C) \\ \text{for } \ C \ge 4[\sup|f|+1], \forall\ \ (x,t) \in \partial_p G_1 \cap \{t > t_{k,o} + l^{-1}\} \end{cases} \tag{3.16}$$

and then, by the compatibility condition of the first order, we have

$$\begin{cases} D_t \widetilde{u}(x, t_{k,o} + l^{-1}) = D_t v_{k,H}(x, t_{k,o} + l^{-1}; C), \\ \ \forall (x, t_{k,o} + l^{-1}) \in \partial_p G_1 \cap \{t = t_{k,o} + l^{-1}\}. \end{cases} \tag{3.17}$$

Therefore, from (3.15)—(3.17) we have

$$\begin{cases} \dfrac{1}{2} \sup\limits_{\partial_p G_1} D_t v_{k,H}(x,t;C) \ge D_t \widetilde{u}(x,t) \ge 3 \inf\limits_{\partial_p G_1} D_t v_{k,H}(x,t;C) \\ \text{for } \ C \ge 4[\sup|f|+1], \ \ \forall\ \ (x,t) \in G_1. \end{cases} \tag{3.18}$$

Then from (3.7) and (2.7) we conclude

$$\begin{cases} -\dfrac{1}{2}(C - M_T) \ge D_t \widetilde{u}(x,t) \ge -3(C + M_T) \\ \text{for } \ C \ge 4[\sup|f|+1], \ \ \forall (x,t) \in G_1, \end{cases} \tag{3.19}$$

which means that (3.13) is true for any $(x,t) \in G_1$.

Secondly note that when $(x,t) \in G_2$ we have

$$\begin{aligned} \widetilde{f}(x,t) = {} & f(x_0,t_0) + C \\ & - h\Big(\frac{t - t_{k,o} - l^{-1}}{l^{-1}}\Big)\{f(x_0,t_0) + C - \mathcal{P} v_{k,H}(x, t_{k,o} + l^{-1}; C)\}. \end{aligned}$$

By definitions of $h(s)$ and $v_{k,H}$ as well as (2.6) and (2.3) we have

$$|[t - t_{k,o} - l^{-1}] D_t \widetilde{f}(x,t)| \le 8(2\sup|f| + 1), \ \ \forall\ (x,t) \in G_2. \tag{3.20}$$

Denote

$$L \equiv -D_t \ + \Theta(x,t) \widetilde{u}^{ij}(x,t) D_{ij},$$
$$(\widetilde{u}^{ij}(x,t)) \equiv (D_{ij} \widetilde{u}(x,t))^{-1},$$
$$\Theta(x,t) \equiv \frac{1}{n} \det{}^{\frac{1}{n}} (D_{i,j} \widetilde{u}(x,t)).$$

Then from (3.9) and (3.11) we have

$$\begin{aligned} L\widetilde{u}(x,t) &= -D_t(D_t\widetilde{u}(x,t)) + \Theta(x,t)\widetilde{u}^{ij}(x,t)D_{ij}(D_t\widetilde{u}(x,t)) \\ &= D_t\widetilde{f}(x,t), \quad \forall\ (x,t)\in G_2. \end{aligned}$$

Set

$$\begin{cases} U(x,t) = [t-t_{k,o}-l^{-1}]^K \exp\{D_t\widetilde{u}(x,t)\}, \\ \widetilde{L}U(x,t) \equiv L(x,t) - \sum\limits_i \Theta(x,t)(\sum\limits_i \widetilde{u}^{ij}(x,t)D_iD_t\widetilde{u}(x,t))D_jU(x,t) \end{cases}$$

with K being a constant to be determined. Then we have

$$\begin{aligned} \widetilde{L}U(x,t) &= -K(t-t_{k,o}-l^{-1})^{K-1}\exp\{D_t\widetilde{u}(x,t)\} \\ &\quad - (t-t_{k,o}-l^{-1})^K \exp\{D_t\widetilde{u}(x,t)\}D_t(D_t\widetilde{u}(x,t)) \\ &\quad + (t-t_{k,o}-l^{-1})^K \exp\{D_t\widetilde{u}(x,t)\}\Theta(x,t)\widetilde{u}^{ij}(x,t)D_{ij}(D_t\widetilde{u}(x,t)) \\ &= (t-t_{k,o}-l^{-1})^{K-1}\exp\{D_t\widetilde{u}(x,t)\}[-K+(t-t_{k,o}-l^{-1})D_t\widetilde{f}(x,t)]. \end{aligned}$$

Hence by (3.20) we have

$$\begin{cases} \widetilde{L}U(x,t) \le (t-t_{k,o}-l^{-1})^{K-1}\exp\{D_t\widetilde{u}(x,t)\}[-K+8(2\sup|f|+1)] \le 0 \\ \quad \text{for } K = K_0 \equiv 8(\sup|f|+1), \ \ \forall\,(x,t)\in G_2 \end{cases}$$

and

$$\begin{cases} \widetilde{L}U(x,t) \ge (t-t_{k,o}-l^{-1})^{K-1}\exp\{D_t\widetilde{u}(x,t)\}[-K-8(2\sup|f|+1)] \ge 0 \\ \quad \text{for } K = -K_0 \equiv -8(\sup|f|+1), \ \ \forall\,(x,t)\in G_2. \end{cases}$$

By maximum principle we thus have

$$\begin{cases} [t-t_{k,o}-l^{-1}]^{K_0}e^{D_t\widetilde{u}(x,t)} \ge \inf\limits_{\partial_pG_2}[t-t_{k,o}-l^{-1}]^{K_0}e^{D_t\widetilde{u}(x,t)}, \\ [t-t_{k,o}-l^{-1}]^{-K_0}e^{D_t\widetilde{u}(x,t)} \le \sup\limits_{\partial_pG_2}[t-t_{k,o}-l^{-1}]^{-K_0}e^{D_t\widetilde{u}(x,t)}, \\ \quad K_0 = 8(\sup|f|+1), \ \ \forall\,(x,t)\in G_2. \end{cases} \tag{3.21}$$

Note that

$$\frac{1}{4l} \le (t-t_{k,o}-l^{-1}) \le \frac{1}{2l}, \ \ \forall\,(x,t)\in G_2, \tag{3.22}$$

so we have, from (3.21) and (3.22), that

$$\begin{cases} \left(\dfrac{1}{2l}\right)^{K_0} e^{D_t\widetilde{u}(x,t)} \ge \left(\dfrac{1}{4l}\right)^{K_0} \exp\{\inf\limits_{\partial_pD_2} D_t\widetilde{u}(x,t)\}, \\ \left(\dfrac{1}{2l}\right)^{-K_0} e^{D_t\widetilde{u}(x,t)} \le \left(\dfrac{1}{4l}\right)^{-K_0} \exp\{\sup\limits_{\partial_pD_2} D_t\widetilde{u}(x,t)\}, \\ K_0 = 8(\sup|f|+1), \ \ \forall\,(x,t)\in G_2. \end{cases}$$

Hence it holds that

$$\begin{cases} D_t\widetilde{u}(x,t) \ge -K_1 + \inf\limits_{\partial_pG_2} D_t\widetilde{u}(x,t), \\ D_t\widetilde{u}(x,t) \le K_1 + \sup\limits_{\partial_pG_2} D_t\widetilde{u}(x,t), \\ K_1 = 16(\sup|f|+1), \ \ \forall\,(x,t)\in G_2. \end{cases} \tag{3.23}$$

From (3.19) we have

$$\begin{cases} \inf\limits_{\partial_p G_2 \cap \{t=t_{k,o}+l^{-1}+\frac{1}{4}l^{-1}\}} D_t\widetilde{u}(x,t) \geq -3(C+M_T), \\ \sup\limits_{\partial_p G_2 \cap \{t=t_{k,o}+l^{-1}+\frac{1}{4}l^{-1}\}} D_t\widetilde{u}(x,t) \leq -\dfrac{1}{2}(C-M_T), \\ \text{for } \ C \geq 4[\sup|f|+1]. \end{cases} \tag{3.24}$$

And a barrier argument as used for G_1 gives

$$\begin{cases} \inf\limits_{\partial_p G_2 \cap \{t>t_{k,o}+l^{-1}+\frac{1}{4}l^{-1}\}} D_t\widetilde{u}(x,t) \geq -3(C+M_T), \\ \sup\limits_{\partial_p G_2 \cap \{t>t_{k,o}+l^{-1}+\frac{1}{4}l^{-1}\}} D_t\widetilde{u}(x,t) \leq -\dfrac{1}{2}(C-M_T), \\ \text{for } \ C \geq 4[\sup|f|+1]. \end{cases} \tag{3.25}$$

From (3.23)—(3.25) we then have

$$\begin{cases} K_1 - \dfrac{1}{2}(C-M_T) \geq D_t\widetilde{u}(x,t) \geq -K_1 - 3(C+M_T),, \\ K_1 = 16(\sup|f|+1) \ \ \forall\,(x,t) \in G_2, \\ \text{for}, \ \ C \geq 4[\sup|f|+1]. \end{cases}$$

So we may choose, e.g.,

$$\begin{cases} C = M_T + 32(\sup|f|+1)+1, \\ C_1 = 120[\sup|f|+1] + 8M_T. \end{cases} \tag{3.26}$$

Then we have

$$-C_1 \leq D_t\widetilde{u}(x,t) \leq -{C_1}^{-1} < 0, \ \ \forall\,(x,t) \in G_2, \tag{3.27}$$

which means that (3.13) is true for any $(x,t) \in G_2$.

Finally note that when $(x,t) \in G_3$ we have

$$\widetilde{f}(x,t) \equiv f(x_0,t_0) + C, \quad D_t\widetilde{f}(x,t) \equiv 0,$$

so by the same reasoning as for G_1, we have

$$\inf_{\partial_p G_3} D_t\widetilde{u}(x,t) \leq D_t\widetilde{u}(x,t) \leq \sup_{\partial_p G_3} D_t\widetilde{u}(x,t), \ \ \forall\,(x,t) \in \overline{G}_3 \tag{3.28}$$

and

$$\begin{cases} \dfrac{1}{2}D_t v_{k,H}(x,t;C) \geq D_t\widetilde{u}(x,t) \geq 3D_t v_{k,H}(x,t;C), \\ \forall\,(x,t) \in \partial_p G_3 \cap \{t > t_{k,o} + l^{-1} + \dfrac{1}{2}l^{-1}\} \\ \text{for } \ C \geq 4[\sup|f|+1], \end{cases} \tag{3.29}$$

as well as

$$\begin{cases} D_t\widetilde{u}(x,t) = D_t\widetilde{u}(x, t_{k,o} + l^{-1} + \dfrac{1}{2}l^{-1}), \\ \ \forall\,(x,t) \in \partial_p G_3 \cap \{t = t_{k,o} + l^{-1} + \dfrac{1}{2}l^{-1}\}. \end{cases} \tag{3.30}$$

From (3.28)—(3.30) and (3.27) we conclude that (3.13) is true for any $(x,t) \in G_3$, the combination of which with (3.19) and (3.27) shows that (3.13) is true in the case $G = D_{k,H;l}$ with C and C_1 determined by (3.26). And the proof of Proposition 3.1 is thus completed.

Having established the above Proposition we can prove or derive the following two theorems in the same ways as in [8].

Theorem 3.3. *Let* $D_H, D_{k,H}$ *and* $D_{k,H;l}$ *be given by* (3.1), (3.7) *and* (3.8) *respectively. Assume that* $w(x,t) \in C^\infty(G) \cap C(\overline{G})$ *with its first order derivatives bounded in* G *for* $G = D_H$ *or* $D_{k,H}$, *and* $w(x,t) \in C^{4,2}(G) \cap C^{2,1}(\overline{G})$ *for* $G = D_{k,H;l}$, *as well as that* $w(x,t)$ *is the solution of* (3.4) *or* $w(x,t) = \widetilde{u}(x,t)$ *is the solution of* (3.9) *with* $\widetilde{f}(x,t)$ *and* $g(x,t)$ *given by* (3.10) *and* (3.11). *Then for any* $G_1 \subset\subset G$ (*in the conventional parabolic sense*) *it holds that*

$$\sup_{G_1} |D_x^2 w(x,t)| \le \frac{C_2 \mathrm{diam}_x G}{\mathrm{dist}_x\{G_1, \partial_p G\} \cdot \mathrm{dist}_t^2\{G_1, \partial_p G\}} \tag{3.31}$$

where

$$\begin{cases} \mathrm{diam}_x G = \sup\{|x_1 - x_2| \ | \ (x_1,t),(x_2,t) \in G\}, \\ \mathrm{dist}_x\{G_1, \partial_p G\} = \inf\{|x_1 - x_2| \ | \ (x_1,t) \in G_1, \ (x_2,t) \in \partial_p G\}, \\ \mathrm{dist}_t\{G_1, \partial_p G\} = \inf\{|t_1 - t_2| \ | \ (x,t_1) \in G_1, (x,t_2) \in \partial_p G\}, \\ C_2 = C_2(n, T, \sup|w|, \sup|D_x w|, \inf|D_t w|^{-1}, \sup|\widetilde{f}|, \sup|D_x \widetilde{f}|, \sup|D_x^2 \widetilde{f}|). \end{cases}$$

Note that, for frozen problem (3.4) or (3.9) with $\widetilde{f}$ given by (3.10), the right hand side of the equation is a constant, so the constant C_2 depends only on the bounds of f and is independent of the derivatives of f.

Theorem 3.4. *Under the conditions and notations of Theorem* 3.3, *for any domain* $G_2 \subset\subset G_1$ (*in the conventional parabolic sense*), *there exist constant* $C_3 > 0$ *and* $\beta \in (0,1)$ *depending only on the data from* (A1)—(A4) *as well as* $\mathrm{dist}(G_2, \partial_p G_1)$ *such that*

$$|w(x,t)|_{C^{2+\beta, 1+\frac{1}{2}\beta}(\overline{G}_2)} \le C_3. \tag{3.32}$$

Note that, for $G = D_H$ or $G = D_{k,H}$, what Theorem 3.4 gives is just the "desirable" interior estimates for the higher order derivatives of the solution to the frozen problems.

The last step is the proof of Theorem 3.2 itself, which can be realized by using Lemma 3.1, Proposition 3.14, Theorems 3.3 and 3.4 in the same way as in the proof of Lemma 3.1 of [8]. About this procedure we should add the following crucial remark, which was also used implicitly there.

Remark 3.1. Note that the defining functions $\Phi_k(x,t) \equiv v_{k,H}(x,t;C)$ of $D_{k,H}$ in (3.7), as well as in Lemma 3.1 of [8], are strictly convex-monotone and smooth in a neighborhood of $\overline{D}_{k,H}$. So it follows easily that, as in the beginning part of the proof of Proposition 1.1 in [9], each of the Legendre transformations

$$\mathcal{L}_k : (x,t) \in \overline{D}_{k,H} \to (p = D_x \Phi_k(x,t), h = D_x \Phi_k(x,t) \cdot x - \Phi_k(x,t))$$

is an injection. Therefore, because at the lowest point $(x_{k,0}, t_{k,0})$ of $\partial_p D_{k,H}$ we have $D_x \Phi = 0$, and $\Phi_k(x,t) = 0$ everywhere on $\partial_p D_{k,H}$, we conclude that, for all large $l \in \mathbf{N}$,

$$D_x \Phi_k(x,t) \neq 0 \quad \text{everywhere on the lateral part of} \quad \partial_p D_{k,H}.$$

This property of $D_{k,H;l}$ just meets the needs of the method of continuity in employing the known result about linear equation (see Theorem 7 on page 65 in [3]).

Remark 3.2. If $f(x_0,t_0)$ in (3.10) is replaced by $f_k(x_0,t_0)$, then (3.9) becomes the frozen problem related to $v_{k,H}$. From above one can see that, for these frozen problems, the existence of solutions and the kind of "desirable" interior estimates for the derivatives of solutions are all valid, which are uniform for k.

§4. Regularity

We establish the regularity of the function $v(x,t;C)$ given by (2.5) with C fixed by (3.26), from which the regularity of the viscosity solution $u(x,t)$ obtained from Theorem 2.1 follows at once.

The regularity result of the note is the following

Theorem 4.1. *If* (A1)—(A4) *hold, then the viscosity solution* $u(x,t)$ *to problem* (1.1) *belongs to* $C^{2+\alpha,1+\frac{\alpha}{2}}_{\rm loc}(Q)$, $\alpha\in(0,1)$.

To prove this theorem, by using the results of [6] as in [8], it is enough to establish the locally uniform bounds of the second order derivatives (or the second order deference quotient) with respect to x for the approximation solutions. But, since the derivation of this kind of bounds depends only on the existence of solutions to the frozen problems and on the "desirable" interior estimates of these solutions, in view of Remark 3.2, one can use the same procedure to derive this kind of bounds for v or for v_k. And this can be realized by the same procedure as in §4 of [8], so we omit it.

§5. Appendix

In this section we show the strict positiveness of the sum of the derivative of the solutions with respect to t and the function on the right hand side of the equation. Lemma 5.1 and Lemma 5.1' deal with the case of the approximation problem, the case of frozen problem is dealt with in Lemma 5.2.

In order to use the result like those in [4] to construct approximation solutions, we need to make the data of the problem smooth and satisfy the compatibility conditions. To this end, we extend the data to the outside of $\overline{\Omega}\times[0,T]$ and then modify them suitably to make them not only smooth enough but also satisfy certain inequalities. Precisely speaking, suppose, in doing so, f,ϕ,Ω and $[0,T]$ become g,ψ,Γ and $(\check{t},T]$ respectively with

$$\begin{cases}\overline{Q}=\overline{\Omega}\times[0,T]\subset\Gamma\times(\check{t},T]=:K,\\ \psi(x,t)\text{ is strictly convex in }x,\\ \Gamma\text{ is a strictly convex domain,}\end{cases}\tag{5.1}$$

then the following inequalities are satisfied : When it is supposed that the conditions (1.2) and (1.4) hold, we have

$$\begin{cases}\min\limits_{\overline{K}}g+\min\limits_{(x,t)\in\partial_pK}D_t\psi(x,t)-\dfrac{1}{2}\bar{a}\bar{d}^2\geq\nu_1/2>0,\\ \bar{d}\ \text{ is the radius of the minimal ball containing }\Gamma,\\ \bar{a}=\max\{0;\max\limits_{K}D_tg\},\\ \min\limits_{\overline{\Gamma}}(\det D_x^2\psi(x,\check{t}))^{1/n}\geq\tilde{\tilde{\nu}}_1\end{cases}\tag{5.2}$$

and

$$\begin{cases} g(x,t) - [-D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{\frac{1}{n}}] < 0 \\ \text{for } (x,t) \text{ near } \partial\Gamma \times \{t = \breve{t}\}; \end{cases} \tag{5.3}$$

and if (1.2') and (1.4') hold, we have

$$\begin{cases} \min\limits_{\partial_p K}(D_t\psi + g) \geq \tilde{\nu}_1/2 > 0, \\ g(x,t) \text{ is a concave function in } x \in \overline{\Gamma} \text{ for any fixed } t \in [\breve{t}, T], \\ (\det D_x^2\psi(x,\breve{t}))^{1/n} \text{ is a concave function in } x \in \overline{\Gamma}, \\ \min\limits_{\overline{\Gamma}}(\det D_x^2\psi(x,\breve{t}))^{1/n} \geq \tilde{\tilde{\nu}}_1 \end{cases} \tag{5.2'}$$

and

$$\begin{cases} -D_t\psi(x,t) \ \text{ is decreasing in } t \text{ near } \breve{t} \text{ for } x \in \overline{\Gamma}, \\ -D_t\psi(x,\breve{t}) \ \text{ is concave in } x \in \overline{\Gamma}. \end{cases} \tag{5.3'}$$

In the case when either (1.4) or (1.4') holds (hence either (5.3) or (5.3') is valid), we need further modification of the data. It is easy to see that we can construct cutoff functions $\eta(s)$ and $\zeta(x)$ such that they not only satisfy the following conditions

$$\begin{cases} \eta(s) \in C^\infty(R^1) \text{ with } 0 \leq \eta(s) \leq 1, \eta'(s) \leq 0, \\ \eta(s) \equiv 1 \text{ for } s \leq \dfrac{1}{4}; \eta(s) \equiv 0 \ \text{ for } \ s \geq \dfrac{1}{2} \end{cases} \tag{5.4}$$

and

$$\begin{cases} \zeta(x) \in C^\infty(R^n),\ 0 \leq \zeta(x) \leq 1, \\ \zeta(x) \equiv 1 \text{ for } x \text{ near } \partial\Gamma;\ \zeta(x) \equiv 0 \text{ for } x \text{ a little far from } \partial\Gamma \end{cases} \tag{5.5}$$

but also possess the following properties: when (5.3) holds we have, for small $\epsilon > 0$,

$$\begin{cases} -\eta\Big(\dfrac{t-\breve{t}}{\epsilon}\Big)\zeta(x)\{g(x,t) - [-D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{\frac{1}{n}}]\} \geq 0, \\ -\dfrac{1}{\epsilon}\eta'\Big(\dfrac{t-\breve{t}}{\epsilon}\Big)\zeta(x)\{g(x,t) - [-D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{\frac{1}{n}}]\} \leq 0, \\ \forall (x,t) \in K; \end{cases} \tag{5.3-1}$$

when (5.3') holds we then have

$$\begin{cases} \eta\Big(\dfrac{t-\breve{t}}{\epsilon}\Big)\{D_t\psi(x,t) - D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{1/n}\} \geq 0, \\ \forall\, (x,t) \in \partial\Gamma \times [\breve{t}, T], \\ -D_t\psi(x,\breve{t}) \ \text{ is concave in } x \in \overline{\Gamma}. \end{cases} \tag{5.3'-1}$$

Now if we set

$$\begin{cases} \tilde{f}(x,t) = g(x,t) - \eta\Big(\dfrac{t-\breve{t}}{\epsilon}\Big)\zeta(x)\{g(x,t) \\ \quad - [-D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{\frac{1}{n}}]\} \end{cases} \tag{5.6}$$

or

$$\tilde{f}(x,t) = g(x,t) - \eta\Big(\frac{t-\breve{t}}{\epsilon}\Big)\{g(x,t) - [-D_t\psi(x,\breve{t}) + (\det(D_x^2\psi(x,\breve{t})))^{\frac{1}{n}}]\} \tag{5.6'}$$

(Note that the notation $\tilde{f}$ in this section is different from the same one in §3), then the problem

$$\begin{cases} \mathcal{P}u := -D_t u + (\det D_x^2 u)^{\frac{1}{n}} = \tilde{f}(x,t), \quad (x,t) \in K, \\ u(x,t) = \psi(x,t) \quad (x,t) \in \partial_p K \end{cases} \tag{5.7}$$

obviously satisfies the compatibility conditions up to the first order.

In the case when both (1.2) and (1.4) (hence (5.2) and (5.3-1)) hold, we need to show

Lemma 5.1. *Let* Γ *be a strictly convex domain in* $\mathbf{R}^n$, $K = \Gamma \times (\check{t}, T]$. *If* $u(x,t) \in C^{2,1}(\overline{K})$ *is the solution to the problem* (5.7) *with* $\tilde{f}(x,t)$ *defined in* (5.6), *and* $g, \psi, \partial\Gamma$ *are all smooth,* (5.2), (5.3-1) *are valid, then it holds that*

$$D_t u + \tilde{f}(x,t) \geq \frac{\nu_1}{2}, \quad \forall\, (x,t) \in K \tag{5.8}$$

with ν_1 *from* (5.2).

And in the case when (1.2') and (1.4') are satisfied (hence (5.2') and (5.3'-1) hold), we do not need to do further modification. Based on them we can prove the "strictly positiveness", i.e.,

Lemma 5.1'. *Under the assumptions of Lemma* 5.1 *with* (5.3-1) *and* (5.2) *replaced by* (5.3'-1) *and* (5.2') *as well as* $\tilde{f}(x,t)$ *defined in* (5.6'), *it holds that*

$$D_t u + \tilde{f}(x,t) \geq \min\left\{\frac{\tilde{\nu}_1}{2}, \frac{\tilde{\tilde{\nu}}_1}{2}\right\} \tag{5.8'}$$

with $\tilde{\nu}_1$, $\tilde{\tilde{\nu}}_1$ *from* (5.2') *respectively* .

Finally we consider an analogy of the frozen problems discussed before, which can still be written in the form as in (5.7) with $\tilde{f}(x,t)$ defined by (5.6'), but with the $g(x,t)$ =const. in it, and moreover with the structure conditions being both

$$\begin{cases} \inf\limits_K g(x,t) + \inf\limits_{\partial_p K} D_t\psi \geq \nu > 0, \\ g(x,t) \equiv \text{const.} \end{cases} \tag{5.9}$$

and

$$\begin{cases} |g(x,t) - [-D_t\psi(x,\check{t}) + (\det(D_x^2\psi(x,\check{t})))^{\frac{1}{n}}]| \leq \mu, \quad \forall\, x \in \Gamma \\ \text{with} \quad 0 < \mu \leq \dfrac{\nu}{2(1 + ||\eta'||_{L^\infty})}. \end{cases} \tag{5.10}$$

The motivation of doing this is to illustrate the method of proving (3.12) of Proposition 3.1.

We have

Lemma 5.2. *Under the assumptions of Lemma* 5.1' *with* (5.2') *and* (5.3'-1) *replaced by* (5.9) *and* (5.10), *it holds that*

$$D_t u + \tilde{f}(x,t) \geq \frac{\nu}{2} \tag{5.11}$$

with ν *from* (5.9).

Proof of Lemmas 5.1, 5.1'and 5.2. Let us consider the linear parabolic operator, which is the linearization of $\mathcal{P}$ around u, acting on v, and is given by

$$\mathcal{L}_u(v) = -D_t v + F_{ij}(D_x^2 u) D_{ij} v, \tag{5.12}$$

where $F_{ij}(D_x^2 u) = \frac{\partial(\det(u_{ij}))^{1/n}}{\partial u_{ij}}$ satisfying the inequality

$$\text{trace}(F_{ij}(D_x^2 u)) \equiv \sum_{i=1}^n F_{ii}(D_x^2 u) \geq 1. \tag{5.13}$$

Firstly we prove Lemma 5.2.

We divide the domain K into three parts:

$$K_1 = K \cap \{t - \check{t} \le \epsilon/4\},$$
$$K_2 = K \cap \{\check{t} + \epsilon/4 < t \le \check{t} + \epsilon/2\},$$
$$K_3 = K \setminus (K_1 \cup K_2).$$

On the domain K_1, we have

$$\mathcal{L}_u(D_t u) = D_t \tilde{f}(x,t) \equiv 0.$$

Then, by comparison theorem, it holds that

$$D_t u(x,t) \ge \min_{\partial_p K_1} D_t \psi, \quad \forall\, (x,t) \in K_1. \tag{5.14}$$

Hence, for $(x,t) \in K_1$, we have, by (5.9) and (5.10),

$$\begin{aligned} \tilde{f}(x,t) + D_t u &= [-D_t \psi(x,\check{t}) + (\det(D_x^2 \psi(x,\check{t})))^{1/n}] - g(x,t) + g(x,t) + D_t u \\ &\ge -\mu + g(x,t) + \min_{\partial_p K} D_t \psi \\ &\ge -\mu + \inf_K g(x,t) + \min_{\partial_p K} D_t \psi \\ &\ge \nu/2 \quad \text{for } \mu \le \nu/2. \end{aligned} \tag{5.15}$$

On the domain K_2, let

$$v = D_t u + b(t - t_0), \quad \text{with } t_0 = \check{t} + \epsilon/4, \quad b = \mu ||\eta'||_{L^\infty}/\epsilon.$$

Then we have

$$\mathcal{L}_u(v) \le D_t \tilde{f}(x,t) - b \le 0 \quad \text{for } (x,t) \in K_2.$$

By comparison, it follows that

$$D_t u + b(t - t_0) \ge \min_{\partial_p K_2} D_t u \ \ \forall\, (x,t) \in K_2.$$

Hence for $(x,t) \in K_2$ we have, in view of (5.14),

$$\begin{aligned} \tilde{f}(x,t) + D_t u(x,t) &\ge \tilde{f}(x,t) + \min_{\partial_p K} D_t \psi - b\epsilon/4 \\ &\ge g(x,t) - \mu + \min_{\partial_p K} D_t \psi - \mu ||\eta'||_{L^\infty}/4 \\ &\ge \frac{1}{2}\nu \quad \text{for } \mu \le \frac{\nu}{2(1 + ||\eta'||_{L^\infty})}. \end{aligned} \tag{5.16}$$

On the domain K_3, we have

$$\mathcal{L}_u(D_t u) = D_t \tilde{f}(x,t) = D_t g(x,t) = 0 \quad \text{for } (x,t) \in K_3.$$

By comparison, it holds that

$$D_t u(x,t) \ge \min_{\partial_p K_3} D_t u, \quad \forall\, (x,t) \in K_3.$$

Hence, for $(x,t) \in K_3$, we have

$$\tilde{f}(x,t) + D_t u(x,t) \ge \tilde{f}(x,t) + \min_{\partial_p K_3} D_t u = g(x,t) + \min_{\partial_p K_3} D_t u. \tag{5.17}$$

And combining this with (5.16) we get

$$\tilde{f}(x,t) + D_t u(x,t) \ge g(x,t) + \min_{\partial_p K_2} D_t \psi \ge \nu/2. \tag{5.18}$$

Thus Lemma 5.2 is proved.

Next we prove Lemma 5.1, which is much simple.

From (5.6) and (5.3-1) we have

$$\begin{aligned}D_t\tilde{f}(x,t) &= \Big[1-\eta\Big(\frac{t-\breve{t}}{\epsilon}\Big)\zeta(x)\Big]D_tg(x,t)\\ &\quad -\frac{1}{\epsilon}\eta'\Big(\frac{t-\breve{t}}{\epsilon}\Big)\zeta(x)\{g(x,t)-[-D_t\psi(x,\breve{t})+(\det(D_x^2\psi(x,\breve{t})))^{1/n}\}\\ &\le(1-\eta\zeta)D_tg(x,t),\end{aligned}$$

therefore, with the constant $\bar{a}$ from (5.2),

$$D_t\tilde{f}(x,t)-\bar{a}\le(1-\eta\zeta)D_tg(x,t)-\bar{a}\le 0\quad\text{in } K.$$

Hence for

$$v=D_tu-\frac{1}{2}\bar{a}(x-x_0)^2$$

we obtain, in virtue of (5.13),

$$\mathcal{L}_u(v)\le D_t\tilde{f}(x,t)-\bar{a}\le 0\quad\text{in } K.$$

By comparison, it holds that

$$D_tu-\frac{1}{2}\bar{a}(x-x_0)^2\ge\min_{\partial_pK}D_t\psi-\frac{1}{2}\bar{a}\bar{d}^2.$$

Noticing the definition of $\tilde{f}(x,t)$, (5.3-1) and (5.2), we thus have

$$\begin{aligned}\tilde{f}(x,t)+D_tu &\ge\tilde{f}(x,t)+D_tu-\frac{1}{2}\bar{a}(x-x_0)^2\\ &\ge\tilde{f}(x,t)+\min_{\partial_pK}D_t\psi-\frac{1}{2}\bar{a}\bar{d}^2\\ &\ge g(x,t)+\min_{\partial_pK}D_t\psi-\frac{1}{2}\bar{a}\bar{d}^2\\ &\ge\nu_1/2\quad\text{for } (x,t)\in K.\end{aligned}$$

Thus Lemma 5.1 is proved.

Finally let us go to prove Lemma 5.1', which is also simple.

From

$$\begin{aligned}\mathcal{L}_u(D_tu+\tilde{f}(x,t)) &= D_t\tilde{f}-D_t\tilde{f}+(1-\eta)F_{ij}D_{ij}g\\ &\quad+\eta F_{ij}D_{ij}\{-D_t\psi(x,\breve{t})+(\det(D_x^2\psi(x,\breve{t})))^{1/n}\}\\ &\le\eta F_{ij}D_{ij}\{-D_t\psi(x,\breve{t})+(\det(D_x^2\psi(x,\breve{t})))^{1/n}\}\le 0\quad\text{in } K,\end{aligned}$$

where we used the concaveness assumption, then by comparison we have

$$D_tu(x,t)+\tilde{f}(x,t)\ge\min_{\partial_pK}\{D_tu+\tilde{f}\}. \tag{5.19}$$

When $(x,t)\in\partial\Gamma\times[\breve{t},T]$, from (5.2') and (5.3'-1) we have

$$\begin{aligned}D_tu(x,t)+\tilde{f} &= D_tu(x,t)+g(x,t)-\eta\Big(\frac{t-\breve{t}}{\epsilon}\Big)\{g(x,t)\\ &\quad-[-D_t\psi(x,\breve{t})+(\det(D_x^2\psi(x,\breve{t})))^{1/n}]\}\\ &\ge(1-\eta)\tilde{\nu}_1/2+\eta(\det(D_x^2\psi(x,\breve{t})))^{1/n}\\ &\ge\min\{\tilde{\nu}_1,\tilde{\tilde{\nu}}_1\}/2.\end{aligned}$$

When $(x,t) \in \Gamma \times \{t = \breve{t}\}$, from (5.7) and (5.2'), we have

$$\begin{aligned} D_t u(x,\breve{t}) + \tilde{f}(x,\breve{t}) &= (\det(D_x^2 u(x,t)))^{1/n}|_{t=\breve{t}} \\ &= (\det(D_x^2 \psi(x,\breve{t})))^{1/n} \geq \tilde{\tilde{\nu}}_1. \end{aligned}$$

Combining the above two inequalities with (5.19), we complete the proof of Lemma 5.2.

References

[1] Caffarelli, L. A., A localization property of viscosity solutions to the Monge-Ampère equation, *Annals of Math.,* **131** (1990), 129–134.

[2] Caffarelli, L. A., Interior $W^{2,p}$ estimates for the Monge-Ampère equations, *Annals of Math.,* **131** (1990), 135–150.

[3] Friedman, A., Partial differential equations of parabolic type, Printice-Hall, Inc., Englewood Cliffs, N.J., (1964), 65.

[4] Ivochkina, N. M. & Ladyzhenskaya, O. A., On parabolic equations generated by symmetric functions of the principal curvatures of the evolving surface, or of the eigenvalues of the Hessian, Part I: Monge-Ampére equations, *St. Petersburg Math. J.,* **6** (1995), 575–594.

[5] Reye, S. J., Fully nonlinear parabolic partial differential equations of second order, §4. The parabolic Monge-Ampere equation, Ph.D Thesis, Austral National Univ., 1985.

[6] Wang Guanglie,, The first boundary value problem for parabolic Monge-Ampère equation, *Northeastern Math. J.,* **3** (1987), 463–478.

[7] Wang Lihe, On the regularity theory of fully nonlinear parabolic equations, I, II, *Comm. Pure Appl. Math.,* **45** (1992), 27–76, 141–178.

[8] Wang Rouhuai & Wang Guanglie, On the existence, uniqueness and regularity of viscosity solution for the first boundary value problem to parabolic Monge-Ampère equations, *Northeastern Math. J.,* **8** (1992), 417–446.

[9] Wang Rouhuai & Wang Guanglie, The geometric measure theoretical characterization of viscosity solutions to parabolic Monge-Ampère equation, *J. Partial Diff. Eqs.,* **3** (1993), 237–254.