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# ITERATION OF FIXED POINTS ON HYPERSPACES

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**Abstract**

Let $X$ be a compact, convex subset of a Banach space $E$ and $CC(X)$ be the collection of all non-empty compact, coonvex subset of $X$ equipped with the Hausdorff metric $h$. Suppose $\mathcal{K}$ is a compact, convex subset of $CC(X)$ and $T : (\mathcal{K}, h) \to (\mathcal{K}, h)$ is a nonexpansive mapping. Then for any $A_0 \in \mathcal{K}$, the sequence $\{A_n\}$ defined by $A_{n+1} = (A_n + TA_n)/2$ converges to a fixed point of $T$. The special case that $\mathcal{K}$ consists of singletons only yields results previously obtained by H. Schaefer, M. Edelstein and S. Ishikawa respectively.

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## §1. Introduction

Suppose $X$ is a compact convex subset of a Banach space $E$ and $2^X$ is the collection of all non-empty compact, subset of $X$ equipped with Hausdorff metric $h$. It is well-known that $(2^X, h)$ is compact. Let $CC(X)$ be the collection of all non-empty compact, convex subsets of $X$ and $CS(X)$ the collection of all non-empty compact, star-shaped subsets of $X$ respectively. We shall deduce that both $(CC(X), h)$ and $(CS(X), h)$ are compact subsets of $(2^X, h)$. If the underlying Banach space is of finite dimension, the former is Blaschke's Convergence Theorem[1] and the latter is Valentine's Conjecture[7]. On the other hand, suppose $T : X \to X$ is a nonexpansive mapping, it has been proved by Ishikawa[4] that for any initial $x_0$ in $X$ the sequence $\{x_n\}$, where

$$x_n = (x_{n-1} + Tx_{n-1})/2 \quad \text{for} \quad n = 1, 2, 3, \cdots,$$

converges to a fixed point of $T$. If the underlying Banach space $E$ is assumed to be uniformly convex or strictly convex, Ishikawa's result has been previously proved by Schaefer[5] and Edelstein[2] respectively. We shall prove that Ishikawa's result can be extended to $(CC(X), h)$. Also, we call attention to the readers that the structure of the hyperspaces can be very different from the underlying set $X$ since for $X = [0, 1]$, the clsoed unit interval, it has been proved by Schori and West[6] that $(2^{[0,1]}, h)$ is homeomorphic to the Hilbert cube.

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## §2. Basic Definitions and Notations

Let $(\mathbf{M}, d)$ be a metric space, and $CB(M)$ be the collection of all non-empty closed, bounded subsets of $M$. If $A$ is a subset of $M$ and $\varepsilon > 0$, let $N(A;\varepsilon) = \{x \in M : d(x,a) < \varepsilon$ for some $a \in A\}$. Suppose $A, B \in CB(X)$. The Hausdorff metric $h$ induced by $d$ is defined as

$$h(A,B) = \inf\{\varepsilon > 0 : A \subset N(B;\varepsilon) \text{ and } B \subset N(A;\varepsilon)\}.$$

Equivalently,

$$h(A,B) = \max\{\sup_{x\in A} d(x,B),\ \sup_{x\in B} d(x,A)\},$$

where $d(x,A) = \inf\limits_{a\in A} d(x,a)$. Suppose now $E$ is a Banach space. A subset $S \subseteq E$ is said to be star-shaped if and only if there exists an element $p \in S$ such that $tp + (1-t)x \in S$ for $t \in [0,1]$ and $x \in S$; such an element $p$ is called a star-point of $S$.

## §3. Main Resuults

We shall begin with the following lemma which has been noted in [3], and we shall reproduce it for the sake of completenees.

**Lemma 3.1.** *Suppose $(M,d)$ is a metric space, $A, A_n \in CB(\mathbf{M})$ for $n = 1,2,3,\cdots$. If $h(A_n, A) \to 0$ as $n \to \infty$, then $A = \{a \in \mathbf{M} : a = \lim\limits_{k\to\infty} a_{n_k}, a_{n_k} \in A_{n_k}$ and $\{A_{n_k}\}$ is a subsequence of $A_n\}$.*

**Proof.** Let $a \in A$. Since $h(A_n, A) \to 0$ as $n \to \infty$, we may choose $n_1 < n_2 < \cdots$ such that $n \geq n_k$ implies $h(A_n, A) < 1/k$ and hence $h(A_{n_k}, A) < 1/k$. Thus for $a \in A$, there exists $a_{n_k} \in A_{n_k}$ such taht $d(a_{n_k}, a) < 1/k$. Consequently, $\lim\limits_{k\to\infty} a_{n_k} = a$. On the other hand, let $x = \lim\limits_{k\to\infty} a_{n_k}$, where $a_{n_k} \in A_{n_k}$. Supoose $r = d(x,A) > 0$. Since $h(A_n, A) \to 0$, we also have $h(A_{n_k}, A) \to 0$. Thus there exists some $k$ such that $d(a_{n_k}, x) < r/2$ and $h(A_{n_k}, A) < r/2$. Hence there exists $a \in A$ with $d(a_{n_k}, a) < r/2$. Therefore

$$d(a,x) \leq d(a, a_{n_k}) + d(a_{n_k}, x) < r/2 + r/2 = r,$$

which implies that $d(x,A) < r$. That is a comtradiction and the proof is complete.

**Theorem 3.1.** *Let $X$ be a compact, convex subset of a Banach space $E$. Then $(CC(X), h) \subseteq (CS(X), h)$ and both are compact subsets of $(2^X, h)$.*

**Proof.** We shall establish that $(CS(X), h)$ is sequentially compact. For that purpose, let $\{S_n\} \subseteq (CS(X), h) \subseteq (2^X, h)$. Since $(2^X, h)$ is compact, $S_n$ has a convergent subsequence and by relabelling if necessary, we may assume $S_n \to S \in 2^X$. It remains to show that $S$ is star-shaped. Since each $S_n$ is star-shaped, each $S_n$ contains a star-point, say $x_n$. Compactness of $X$ implies that $x_n$ has a convergent subsequence $\{x_n\}$ such taht $x_{n_i} \to x \in X$. Lemma 3.1 implies that $x \in S$. We claim that $x$ is a star-point for $S$. For that purpose, let $y \in S$ and $t \in [0,1]$. Since $y \in S = \lim\limits_{i\to\infty} S_{n_i}$, again Lemma 3.1 implies the existence of $y_{n_{i(j)}} \in S_{n_{i(j)}}$ such that $\lim\limits_{j\to\infty} y_{n_{i(j)}} = y$. That each $x_{n_{i(j)}}$ is a star-point of $S_{n_{i(j)}}$ now yields

$$tx_{n_{i(j)}} + (1-t)y_{n_{i(j)}} \in S_{n_{i(j)}}.$$

Also continuity of vector addition and scalar multiplication implies

$$tx_{n_{i(j)}} + (1-t)y_{n_{i(j)}} \to tx + (1-t)y.$$

Thus Lemma 3.1 implies that $tx+(1-t)y \in S$ and consequently $(CS(X), h)$ is sequentially compact. In the case of $CC(X)$, let $A_n \in CC(X)$ and $A_n \to A \in 2^X$. That $A$ is convex can be proved by observing that every $x \in A$ is a star-point of $A$ as in the previous case and the proof is complete.

**Remark.** If the underlying Banach space $E$ is finite dimensional, Theorem 3.1 yields Blaschke's Convergence Theorem as well as a solution to Valentine's Conjecture which has been noted by T. Hu[3]. Also, we remark that since the intersection of star-shaped sets is not necessarily star-shaped, the usual techniques for treating convex sets can hardly carry over to star-shaped sets.

Next, we shall establish the following results which, together with Theorem 3.1, motivate the formulation of Theorem 3.2 and they are also the basic tools for its proof.

**Lemma 3.2.** *Let $E$ be a Banach space. Suppose $A, B, C, D$ are compact, convex subsets of $E$. Then we have the following:*

(a) *$\alpha A+\beta B$ is a compact, convex subset of $E$.*

(b) *$A$ is convex if and only if $A=\sum\limits_{i=1}^{n}\alpha_i A$, where $\alpha_i \geq 0$, $\sum\limits_{i=1}^{n}\alpha_i=1$.*

(c) *$h(\alpha A, \alpha B)=|\alpha| h(A, B)$.*

(d) *$h(A+C, B+D) \leq h(A, B)+h(C, D)$.*

(e) *$h(A+C, B+C)=h(A, B)$.*

**Proof.** The results (a), (b), (c), (d) are either well-known or easily verifiable and are thus omitted. To prove (e), first, observe that

$$h(A+C, B+C) \leq h(A, B)+h(C, C)=h(A, B).$$

To establish the reverse inequality, we let $h(A+C, B+C)=r$ and $a_0 \in A$. Claim that there exists $b_0 \in B$ such that $\|a_0-b_0\| \leq r$. For that purpose fixing $n$ and choosing $c_0 \in C$, we have $a_0+c_0 \in A+C \subseteq N(B+C; r+1/n)$. Thus there exists $b_1 \in B, c_1 \in C$ with

$$\|(a_0+c_0)-(b_1+c_1)\|<r+1/n.$$

Similarly, there exists $b_2 \in B, c_2 \in C$ with

$$\|(a_0+c_1)-(b_2+c_2)\|<r+1/n.$$

Inductively, we get $b_n \in B, c_n \in C$ with

$$\|(a_0+c_{n-1})-(b_n+c_n)\|<r+1/n.$$

Summing up the inequalities, we obtain

$$\|na_0-(b_1+b_2+\cdots+b_n)+(c_0-c_n)\|<nr+1,$$

or

$$\|a_0-(b_1+b_2+\cdots+b_n)/n\|<r+(1+\delta(C))/n,$$

where $\delta(C)=\operatorname{diam}(C)$. Putting $\bar{b}_n=(b_1+b_2+\cdots+b_n)/n \in B$, we get

$$\|a_0-\bar{b}_n\|<r+(1+\delta(C))/n.$$

$B$ compact implies $\{\bar{b}_n\}$ has convergent subsequence $\bar{b}_{n_j}$, with $\bar{b}_{n_j} \to b_0$. Comsequently

$$\|a_0-b_0\|=\lim_{j \to \infty}\|a_0-b_{n_j}\| \leq r$$

and the claim is proved. Similarly, for any $b_0 \in B$, there exists $a_0 \in A$ such that $\|a_0 - b_0\| \le r$. Thus $h(A, B) \le r = h(A + C, B + C)$ and the proof is complete.

Now, observe that $CC(X)$ has "convexity" structure since for any $A, B \in CC(X)$, we have $\alpha A + (1-\alpha)B \in CC(X)$ by Lemma 3.2. Also suppose

$$\overline{X} = \{\bar{x} = \{x\} : x \in X\}.$$

We have $(\overline{X}, h) \subseteq (CC(X), h)$ and $(\overline{X}, h)$ is isometric to $(X, \| \ \ \|)$. Thus, it is natural to ask if Ishikawas's result is extendable to compact, convex subsets of $CC(X)$. We shall prove that is true in the following theorem.

**Theorem 3.2.** *Let $X$ be a non-empty, compact, convex subset of a Banach space and $\mathcal{K}$ a nonempty compact convex subset of $CC(X)$. Suppose $T : (\mathcal{K}, h) \to (\mathcal{K}, h)$ is nonexpansive. Then for any $A_0 \in \mathcal{K}$, the sequence defined by*

$$A_n = (A_{n-1} + TA_{n-1})/2 \quad \text{for} \quad n = 1, 2, 3, \cdots,$$

*converges to a fixed point of $T$.*

**Proof.** For simplicity, the results of Lemma 3.2 shall be applied extensively without explicitly mentioning them. First, note that

$$h(A_{n+1}, A_n) = h((A_n + TA_n)/2, (A_n + A_n)/2)) = h(TA_n, A_n)/2,$$

and similarly

$$h(A_{n+1}, TA_n) = h((A_n + TA_n)/2, (TA_n + TA_n)/2) = h(A_n, TA_n)/2.$$

Thus we have

$$h(A_n, TA_n) = 2h(A_n, A_{n+1}) = 2h(TA_n, A_{n+1}) \quad \text{for } n = 0, 1, 2, \cdots. \tag{3.1}$$

Now

$$\begin{aligned} h(A_{n+1}, TA_{n+1}) &\le h((A_{n+1} + TA_n) + h(TA_n + TA_{n+1}) \\ &\le h(A_n, TA_n)/2 + h(A_n, A_{n+1}) \\ &= h(A_n, TA_n)/2 + h(A_n, TA_n)/2 \\ &= h(A_n, TA_n) \end{aligned}$$

by (3.1) and nonexpansiveness of $T$. Thus $\{h(A_n, TA_n)\}_{n=1}^{\infty}$ is a decreasing sequence of non-negative numbers and hence $\lim\limits_{n\to\infty} h(A_n, TA_n)$ exists. Suppose

$$\lim_{n\to\infty} (A_n, TA_n) = 2r.$$

Then for $\varepsilon > 0$ there exists $N > 0$ such that $n \ge N$ implies

$$2r \le h(A_n, TA_n) \le 2(r + \varepsilon)$$

and (3.1) yields $r \le h(A_n, A_{n+1}) \le r + \varepsilon$. To simplify notation, put $B_k = A_{N+k}$ for $k = 0, 1, 2. \cdots$ Hence we have $2r \le h(B_k, TB_k) \le 2(r + \varepsilon)$ and

$$r \le h(B_k, B_{k+1}) \le r + \varepsilon \quad \text{for} \quad k = 0, 1, 2, \cdots.$$

Next, we claim that

$$h(B_k, TB_{k+n}) \ge (n+2)(r+\varepsilon) - 2^{n+1}\varepsilon \quad \text{for } k = 0, 1, 2, \cdots, \quad \text{and for } n = 1, 2, \cdots. \tag{3.2}$$

Note

$$h(B_{k+1}, C) = h((B_k + TB_k)/2, (C + C)/2) \le h(B_k, C)/2 + h(TB_k, C)/2$$

which implies

$$h(B_k, C) \geq 2h(B_{k+1}, C) - h(TB_k, C) \quad \text{for any} \quad C \in \mathcal{K}. \tag{3.3}$$

We shall now prove our claim by induction on $n$. For $n = 1$, we have

$$\begin{aligned} h(B_k, TB_{k+1}) &\geq 2h(B_{k+1}, TB_{k+1}) - h(TB_k, TB_{k+1}) \geq 2(2r) - h(B_k, B_{k+1}) \\ &\geq 4r - (r + \varepsilon) = 3r - \varepsilon \quad \text{for all } k = 0, 1, 2, \cdots \end{aligned}$$

by applying (3.3), (3.1) and nonexpansiveness of $T$. Suppose

$$h(B_k, TB_{k+n}) \geq (n+2)(r+\varepsilon) - 2^{n+1}\varepsilon \quad \text{for all} \quad k = 0, 1, 2, \cdots .$$

Then

$$\begin{aligned} h(B_k, TB_{k+n+1}) &\geq 2h(B_{k+1}, TB_{k+n+1}) - h(TB_k, TB_{k+n+1}) \\ &\geq 2h(B_{k+1}, B_{k+n+1}) - h(B_k, B_{k+n+1}) \\ &\geq 2h(B_{k+1}, TB_{k+n+1}) - \sum_{i=0}^{n} h(B_{k+i}, B_{k+i+1}) \\ &\geq 2\{(n+2)(r+\varepsilon) - 2^{n+1}\varepsilon\} - (n+1)(r+\varepsilon) \\ &= (n+3)(r+\varepsilon) - 2^{n+2}\varepsilon \end{aligned}$$

by applying (3.3), nonexpansiveness of $T$, triangular inequality and induction hypothesis successively. Now that (3.2) is proved, we shall deduce that $r = 0$. Assuming the contrary, we have $r > 0$. Putting $\varepsilon = r/2^{n+1}$ we get

$$\begin{aligned} h(B_0, TB_n) &\geq (n+2)(r + r/2^{n+1}) - 2^{n+1}(r/2^{n+1}) \\ &\geq (n+2)r - r = (n+1)r. \end{aligned}$$

That is a contradiction to compactness of $\mathcal{K}$. Thus

$$\lim_{n\to\infty} h(A_n, TA_n) = 0.$$

$\mathcal{K}$ compact implies $\{A_n\}$ has a convergent subsequence $\{A_{n_k}\}$ with $A_{n_k} \to A_\infty \in \mathcal{K}$. Then

$$\begin{aligned} h(A_\infty, TA_\infty) &\leq h(A_\infty, A_{n_k}) + h(A_{n_k}, TA_{n_k}) + h(TA_{n_k}, TA_\infty) \\ &\leq 2h(A_\infty, A_{n_k}) + h(A_{n_k}, TA_{n_k}) \to 0 \quad \text{as } k \to \infty \end{aligned}$$

and hence $A_\infty = TA_\infty$. Also

$$\begin{aligned} h(A_{n+1}, A_\infty) &= h((A_n + TA_n)/2, (A_\infty + A_\infty)/2) \\ &\leq h(A_n, A_\infty)/2 + h(TA_n, TA_\infty)/2 \\ &\leq h(A_n, A_\infty) \end{aligned}$$

and thus $\lim\limits_{n\to\infty} h(A_n, A_\infty)$ exists. But

$$\lim_{k\to\infty} h(A_{n_k}, A_\infty) = 0$$

and consequently

$$\lim_{n\to\infty} h(A_n, A_\infty) = 0$$

and the proof is complete.

**Remark.** The special case that $\mathcal{K} = \overline{X}$ in Theorem 3.2 yields Ishikawa's result. As noted in the introduction, the structure of hyperspaces can be rather complicated; however the mapping $[x, y] \to ((y+x)/2, (y-x)/2)$ is affine isometry of $CC([0,1])$ onto a triangular region in $(R^2, \|\ \|_1)$ as noted by T. Hu[3] and consequently, this affine embedding provides many interesting and non-trivial examples of our results in this paper. For instance, there are infinitely many distinct compact convex $\mathcal{K} \subseteq CC(X)$ other than the two special cases $\mathcal{K} = \overline{X}$ and $\mathcal{K} = CC(X)$.

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