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ON TWO CONJECTURES OF THE QUADRATIC DIFFERENTIAL SYSTEMS

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Abstract

Two conjectures in the qualitative theory of quadratic differential systems are proved under certain conditions.

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Ye Yanqian[1] have investigated limit cycle (LC) bifurcation of the quadratic differential system

$$\dot{x} = -y + lx^2 + mxy + ny^2 \triangleq P(x, y), \qquad \dot{y} = x(1 + ax + by) \triangleq Q(x, y), \tag{1}$$

under the conditions: $mb \neq 0, \quad a < 0, \quad b + 2l > 0, \quad n + l < 0, \quad n = 1$. He proposes the following

Conjecture 1. *Under the condition*

$$(n + b)(n + l)^2 - a^2(n + b + 2l) = 0, \tag{2}$$

when $m \neq m^* = a(b + 2l)/(n + l) > 0$*, the system* (1) *has no LC around O.*

With the help of Dulac function, we will prove this conjecture when $m < 0$.

Take a Dulac function

$$B(x,\, y) = L_+^{\alpha} L_-^{\beta} (1 + by)^{\frac{\gamma}{b}}, \tag{3}$$

where

$$\alpha = \frac{-am(m + \sigma)}{2\sigma[am - (n + b)(n + l)]} + \frac{m}{\sigma}, \quad \beta = \frac{am(m - \sigma)}{2\sigma[am - (n + b)(n + l)]} - \frac{m}{\sigma},$$
$$\gamma = -\frac{(n + b)(n + l)W_1}{a[am - (n + b)(n + l)]}, \qquad W_1 = m(n + l) - a(b + 2l),$$
$$L_+ = (m + \sigma)(ny - 1) - 2n(n + b)x, \quad L_- = (m - \sigma)(ny - 1) - 2n(n + b)x,$$
$$\sigma = \sqrt{m^2 + 4n(n + b)}.$$

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Then we have

$$
\begin{aligned}
&\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}\\
&=-2n(n+b)L_+^{\alpha-1}L_-^{\beta-1}(1+by)^{\frac{\gamma}{b}}\{2n(n+b)(\alpha+\beta)(xy-lx^3-mx^2y-nxy^2)\\
&\quad+[\alpha(m-\sigma)+\beta(m+\sigma)](1-ny)(y-lx^2-mxy-ny^2)\\
&\quad+n[\alpha(m+\sigma)+\beta(m-\sigma)](x^2+ax^3+bx^2y)\\
&\quad-2n(\alpha+\beta)[x(1-ny)+ax^2(1-ny)+bxy(1-ny)]\\
&\quad-2[n(n+b)(b+2l)x^3+mn(n+b)x^2y+m(b+2l)x^2(1-ny)\\
&\quad+m^2xy(1-ny)-(b+2l)x(1-ny)^2-my(1-ny)^2]\\
&\quad-2\gamma[n(n+b)x^3+mx^2(1-ny)-x(1-ny)^2]\}+a\gamma x^2L_+^{\alpha}L_-^{\beta}(1+by)^{\frac{\gamma}{b}-1}\\
&=-2n(n+b)L_+^{\alpha-1}L_-^{\beta-1}(1+by)^{\frac{\gamma}{b}}\{y(1-ny)^2[\alpha(m-\sigma)+\beta(m+\sigma)+2m]\\
&\quad-\frac{m}{n}x(1-ny)[\alpha(m-\sigma)+\beta(m+\sigma)+2m]\\
&\quad+x^2[-2m(n+b)(\alpha+\beta+1)+\alpha(n+b)(m+\sigma)+\beta(n+b)(m-\sigma)]\\
&\quad-2x(1-ny)^2[n(\alpha+\beta)-(b+2l+\gamma)]\\
&\quad+x^2(1-ny)[2m(n+b)(\alpha+\beta)-2an(\alpha+\beta)+2m(n-l)\\
&\quad-b[\alpha(m+\sigma)+\beta(m-\sigma)]-2m\gamma]\\
&\quad+x^3[-2nl(n+b)(\alpha+\beta)-2n(n+b)(b+2l)+an[\alpha(m+\sigma)+\beta(m-\sigma)]\\
&\quad-2\gamma n(n+b)]\}+\gamma ax^2L_+^{\alpha}L_-^{\beta}(1+by)^{\frac{\gamma}{b}-1}\\
&=-\frac{4n(n+b)W_1}{am-(n+b)(n+l)}x^2\{am(1-ny)+(n+b)(n+l)[n(n+b)x^2\\
&\quad+mx(1-ny)-(1-ny)^2](1+by)^{-1}\}L_+^{\alpha-1}L_-^{\beta-1}(1+by)^{\frac{\gamma}{b}}. \qquad (4)
\end{aligned}
$$

Under the conditions in [1], L_+ and L_- are either invariant straight lines or tangents of the two separatrices at the saddle $N(0,\frac{1}{n})$; in the latter case, trajectories intersect $L_+(L_-)$ all from one side to the other side. Since $1+by=0$ is a straight line with the same property, and $O(0,0)$ locates in the triangular domain Ω constructed by the lines $L_+=0$, $L_-=0$ and $1+by=0$ (see Fig.1), if system (1) has closed orbit or singular closed orbit around O, it must locate in Ω. When $W_1=0$, O is a center; when $W_1\neq 0$, since $n(n+b)x^2+mx(1-ny)-(1-ny)^2<0$ in Ω, $am\cdot(n+b)(n+l)<0$, (4) does not change its sign in Ω, and $\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}\not\equiv 0$ in any subdomain of Ω. We know from the Dulac theorem that system (1) has no closed orbit or singular closed orbit for $m\leq 0$.

Fig.1 Fig.2

Similarly, if in (1) we have $n+l>0$ and $m>0$, then the same conclusion holds.

Remark 1. If we let

$$\alpha=\frac{(b+2l)}{2n\sigma}(\sigma+m)+\frac{m}{\sigma},\quad \beta=\frac{(b+2l)}{2n\sigma}(\sigma-m)-\frac{m}{\sigma},\quad \gamma=0,$$

in the Dulac function B(x, y) defined by (3), then the divergence of (1) is

$$\begin{aligned}\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}&=-4n(n+b)\{x[am(n+b+2l)\\&-(n+b)(n+l)(b+2l)]+(1-ny)W_1\}x^2L_+^{\alpha-1}L_-^{\beta-1}.\end{aligned}\tag{5}$$

If the condition (2) holds, we have

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=-4n(n+b)W_1[\frac{(n+b)(n+l)}{a}x+1-ny]x^2L_+^{\alpha-1}L_-^{\beta-1}.\tag{6}$$

Similar to the above proof, when $m\le-\frac{a(b+2l)}{n+l}$, since the slope of the line $L'\triangleq\frac{(n+b)(n+l)}{a}x+1-ny=0$ is less than that of $L_+=0$, (6) does not change its sign in the domain under the line $L_+=0$ and $L_-=0$ (see Fig.2), and the conjecture in [1] is correct. Since $a(b+2l)/(n+l)>0$, the previous condition $m\le 0$ is better. Even though, making use of (6), under certain conditions, we can prove a concentrated distribution theorem of LCs of the system (1) as follows.

Theorem. *When $W_1\neq 0$, limit cycles can not co-exist both around O and any other focus of* (1).

For, otherwise, there would be three contact points on the line L'.

Ye Yanqian[2,3] have investigated the LC bifurcation of the system

$$\dot{x}=-y+\delta x+lx^2+ny^2\triangleq P(x,\,y),\quad \dot{y}=x(1+ax-y)\triangleq Q(x,\,y),\tag{7}$$

under the conditions: $-1<l<0$, $n+l-1>0$, $a\le 0$, and $-1\le l<0$, $n+l>0$, $n>1$, $a\le 0$, respectively. He proposes the following

Conjecture 2. *Under the conditions*

$$-l<na^2<(n-1)(n+l)^2,\qquad a^2-4(n-1)(1-l)>0,\tag{8}$$

the system (7) *can not simultaneously have limit cycles around O and S_1, respectively*[3].

We now will prove this conjecture. Use the Dulac fuction

$$B(x,\,y)=(1-y)^{2l-1},\tag{9}$$

$$\frac{\partial(BP)}{\partial x}+\frac{\partial(BQ)}{\partial y}=(1-y)^{2l-2}[\delta(1-y)-(2l-1)ax^2].\tag{10}$$

Since $y-1=0$ is a straight line having the same property as $1+by=0$ in Fig.1, LCs of (7) must locate in the domain $y>1$ or $y<1$. The x coordinate of the critical points $S_i(x_i,\,y_i)$ $(i=1,\,2,\quad x_1<x_2)$ on $1+ax-y=0$ satisfy

$$(na^2+l)x^2+(\delta+2na-a)x+n-1=0.\tag{11}$$

If the system (7) has four finite critical points, δ must satisfy

$$\delta>a(1-2n)+2\sqrt{(na^2+l)(n-1)}>0,\tag{12}$$

or

$$\delta<a(1-2n)-2\sqrt{(na^2+l)(n-1)}>0.\tag{13}$$

When δ satisfies (12), S_1 and S_2 both locate in the domain $y > 1$, making use of (10), we know that (7) has no LC around S_1. If δ satisfies (13), S_1 and S_2 both locate in the domain $y < 1$. (7) has no LC around O or S_1 as $\delta \le 0$, by (10). Therefore, when $\delta > a(1-2n) + 2\sqrt{(na^2+l)(n-1)}$ or $\delta \le 0$, the conjecture is correct.

As for the case $0 < \delta < a(1-2n) - 2\sqrt{(na^2+l)(n-1)}$, since the y coordinate of any critical point $\mathrm{P}(1, \mathrm{y}, 0)$ of (7) at infinity satisfies

$$f(y) \triangleq ny^3 + (1+l)y - a = 0, \tag{14}$$

and $f'(y) = 3ny^2 + 1 + l > 0$, (7) has a unique stable node $P(1,\, y_0,\, 0)(y_0 < 0, \quad \lambda_1 = -l - ny_0^2 < 0, \quad \lambda_2 = -1 - l - 3ny_0^2 < 0)$ at infinity. Let $\overline{L} \triangleq y - kx - \frac{1}{n} = 0$, and $k \triangleq k(\delta) = \frac{-\delta - \sqrt{\delta^2 + 4(1-\frac{1}{n})}}{2}$ be the slope of the tangent of the separatrice at $N(0,\, \frac{1}{n})$. Then

$$\left.\frac{d\overline{L}}{dt}\right|_{(7)} = \dot{y} - k\dot{x} = -[nk^3 + (1+l)k - a]x^2. \tag{15}$$

Since $\frac{dk}{d\delta} < 0$ as $\delta > 0$, $k(0) = -\sqrt{1-\frac{1}{n}} < 0$, we have $k(\delta) < 0$ for all $\delta > 0$, $f'_\delta = [3nk^2 + 1 + l]k_\delta < 0$. Note that

$$f(k(0)) = -[(n+l)\sqrt{1-\frac{1}{n}} + a]. \tag{16}$$

Since $na^2 < (n-1)(n+l)^2$, we have $(n+l)\sqrt{1-\frac{1}{n}} + a > 0$, i.e., $f(k(0)) < 0$, hence,

$$f(k(\delta)) = nk^3 + (1+l)k - a < 0 \quad \text{as} \quad \delta > 0.$$

Furthermore, $\overline{L} = 0$ is a straight line without contact, and $k < y_0 < 0$ for all $\delta > 0$. When $\delta = 0$, O is a stable weak focus, S_1 is a stable strong critical point, and the system (7) has no limit cycle (see [4, Theorem 15.1]). In this case we have Fig.3. From the Fig.4 in [3], for $0 < \delta \ll 1$, when $\mathrm{div}(7) = 0$ locates on the left of S_1, since O is an unstable focus, S_1 is a stable critical point, we can obtain the phase- portrait Fig.4. When S_1 locates on the right of $\mathrm{div}(7) = 0$, S_1 is always a strong critical point, LC can not be generated from S_1. Since the line $\mathrm{div}(7) = 0$ moves rightward with the increase of δ, when $\mathrm{div}(7) = 0$ locates on the left of S_1, $\mathrm{div}|_{S_2} < 0$. Since S_1 is a stable critical point, two separatrices passing through S_2 can not form a separatrix loop around S_1. Therefore, when $\mathrm{div}(7) = 0$ locates on the left of S_1, (7) has no limit cycle around S_1 (here we have not considered the semi-stable LC which suddenly appears).

Fig.3 Fig.4 Fig.5

When div(7) = 0 passes through S_1 (see Fig.5), δ satisfies

$$\delta = \delta_1 \triangleq (1-2l)\frac{a(1-2n)-\sqrt{a^2-4(1-l)(n-1)}}{2(na^2+1-l)}, \tag{17}$$

and when the intersection point of $\overline{L}$ and div(7) = 0 locates on the x-axis, δ must satisfy

$$\delta = \delta_2 \triangleq \frac{(1-2l)}{\sqrt{n(n-2l)}}. \tag{18}$$

Now, we want to prove $\delta_2 < \delta_1$, that is, when δ increases from 0, div(7) = 0 moves rightward, it first passes through M as $\delta = \delta_2$, and then passes through S_1 as $\delta = \delta_1$. Since LC around O locates on the left of B, and M locates on the right of B, they have disappeared as $\delta = \delta_2$. So, if $\delta_2 < \delta_1$, then (7) has no LC around O as $\delta = \delta_1$.

Propersition 1. *For* $-1 \le l \le -\frac{1}{4}$, $\delta_2 < \delta_1$ *if and only if*

$$n^2(n-1)^2a^4 + n(n-1)^2[2-n-2n(n-1)(n-2l)]a^2 + [n(n-2l)(n-1)^2 + (n-1)(1-l)]^2 > 0. \tag{19}$$

Proof. The formula (19) is equivalent to

$$n^2(n-1)^2a^4 - (1-2n+2n^2)[n(n-2l)(n-1)^2 + (n-1)(1-l)]a^2 + [n(n-2l)(n-1)^2 + (n-1)(1-l)]^2 > -(1-l)(n-1)(1-2n)^2a^2,$$

that is

$$a^2(1-2n)^2[a^2-4(1-l)(n-1)] < [2n(n-2l)(n-1)^2+2(n-1)(1-l)-a^2(1-2n+2n^2)]^2. \tag{20}$$

From the condition (8), we have $4n(1-l) < (n+l)^2 < n(n+l)$, i.e., $n > 4-5l$. So, when $-1 \le l \le -\frac{1}{4}$,

$$\begin{aligned} &2n(n-2l)(n-1)^2 + 2(n-1)(1-l) - a^2(1-2n+2n^2) \\ &> 2n(n-2l)(n-1)^2 + 2(n-1)(1-l) - (1-\frac{1}{n})(n+l)^2(1-2n+2n^2) \\ &= 2n(n-1)(-4nl-n+2l-l^2) + 2(n-1)(1-l) + (1-\frac{1}{n})(2n-1)(n+l)^2 > 0. \end{aligned}$$

Hence, (20) is equivalent to

$$\frac{1}{\sqrt{n(n-2l)}} < \frac{2(n-1)}{a(1-2n)+\sqrt{a^2-4(1-l)(n-1)}},$$

i.e., $\delta_2 < \delta_1$. This completes the proof of Propersition 1.

Obviously, (19) is equivalent to

$$a^2 < [n-2+2n(n-1)(n-2l)-\sqrt{D}]/2n \triangleq R, \tag{21}$$

or

$$a^2 > [n-2+2n(n-1)(n-2l)+\sqrt{D}]/2n, \tag{22}$$

where $D = -(4-n-2l)(n-2l)(1-2n)^2 > 0$. By straight calculation, we have the following

Propersition 2. *For* $-1 \le l \le -\frac{1}{4}$,

$$(n-1)(n+l)^2 < [n-2+2n(n-1)(n-2l)-\sqrt{D}]/2n. \tag{23}$$

We know from (8) and (21) that for $-1 \le l \le -\frac{1}{4}$, (19) is correct, that is, $\delta_2 < \delta_1$, hence, when div(7) = 0 passing through S_1, the LC around O has disappeared. Therefore, for $-1 \le l \le -\frac{1}{4}$, $\delta \ge \delta_1$, (7) can not simultaneously have LCs around O and S_1, respectively.

When $-\frac{1}{4} < l < 0$, if the intersection point of div(7) $= 0$ and $\overline{L}$ locates on $P(x, y) = 0$, $\delta(= \delta_3)$ satisfies

$$\delta_3^2 = \frac{E - \sqrt{E^2 + 4(n-1)(1-l)(1-2l)^2}}{2n(l-1)}, \tag{24}$$

where $E = n + n^2 + 5l - l^2 - 4nl - 2$. We do not know if $\delta_3 < \delta_1$ generally, but for $l = -0.20$, -0.10 and -0.05, we have the following numerical results respectively:

n	6	10	20	30	50	100
a	-5.294	-9.297	-19.298	-29.299	-49.299	-99.299
δ_1	0.2340	0.13754	0.06915	0.04625	0.02784	0.01396
δ_3	0.19261	0.12377	0.06557	0.04466	0.02725	0.01359

n	6	10	20	30	50	100
a	-5.392	-9.391	-19.396	-29.397	-49.398	-99.399
δ_1	0.19364	0.11651	0.05895	0.03951	0.02382	0.01195
δ_3	0.16887	0.10779	0.05669	0.03849	0.02345	0.01168

n	6	10	20	30	50	100
a	-5.431	-9.439	-19.444	-29.446	-49.447	-99.449
δ_1	0.18225	0.10616	0.06044	0.03616	0.02181	0.01095
δ_3	0.15662	0.09956	0.05215	0.03536	0.02153	0.01089

where n, a and l satisfy the condition (8) and $0 < a + (n+l)\sqrt{1 - \frac{1}{n}} < 0.001$. Since

$$\delta'_{1a} = -\frac{2(n-1)(1-2l)}{[a(1-2n) + \sqrt{a^2 - 4(1-l)(n-1)}]^2}\Big[1 - 2n + \frac{a}{\sqrt{a^2 - 4(1-l)(n-1)}}\Big] > 0, \tag{25}$$

$4n(n-1)(1-l) > -l$ as $4 - n - 5l < 0$, hence we know from the selection of a that for all a satisfying $4(n-1)(1-l) < a^2 < (1 - \frac{1}{n})(n+l)^2$, $\delta_3 < \delta_1$ as $l = -0.20$, -0.10 and -0.05. Since the LC around O locates on the left of B, when div(7) $= 0$ passes through S_1, (7) has no LC around O. By the above numerical results, we can affirm that for $-\frac{1}{4} < l < 0$, when div(7) $= 0$ passes through S_1, (7) has no LC around O. Furthermore, (7) can not simultaneously have LCs around O and S_1, respectively.

Remark 2. The problem whether or not semi-stable LC can appear suddenly around S_1 as δ increases from 0 to $a(1-2n) - 2\sqrt{(na^2 + l)(n-1)}$ will be considered in a forthcoming paper.

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