Chin. Ann. of Math.
18B: 4(1997),435-438.

A SURGERY REPRESENTATION OF 3-DIMENSIONAL HOMOLOGY SPHERES**

Zhou Qing*

Abstract

It is proved that for any 3-dimensional homology sphere M, there is a sequence of homology spheres $M_0, M_1, \cdots, M_s$ such that $M_0 = S^3$, $M_s = M$ and each M_i is a result of ± 1 surgery of M_{i-1} along a knot.

Keywords 3-dimensional homology sphere, Surgery, Knot
1991 MR Subject Classification 57R65, 57M25, 57N10
Chinese Library Classification O189.32, O189.33

Let M be a 3-dimensional homology sphere, $K \subset M$ be a knot. ± 1 surgery of M along K yields another homology sphere M_K. A long exact sequence of instanton homology groups related to this surgery has been studied in [1–3, 7]. This provides a way to calculate the instanton homology.

It is known that every oriented 3-dimensional manifold M can be represented as the result of an integral surgery on a link in S^3 (see [5]), or we can say that there is a sequence of 3-manifolds $M_0, M_1, \ldots, M_s$, such that $M_0 = S^3$, $M_s = M$ and each M_i is the result of a surgery of M_{i-1} along a knot $K_{i-1} \subset M_{i-1}$. To use the idea of long exact sequence calculating the instanton homology for all homology spheres, we need a special surgery representation for homology spheres. Namely, for every homology sphere M, the 3-manifolds M_i in the above sequence must be all homology spheres. In this note we will give an affirmative answer to the existence of this special surgery representation for homology spheres.

Theorem 1. *For every* 3*-dimensional homology sphere* M*, we can find a sequence of homology spheres* $M_0, M_1, \ldots, M_s$ *and knots* $K_i \subset M_i$ *for all* $0 \le i < s$*, such that* $M_0 = S^3$*,* $M_s = M$ *and each* M_i *is a result of* ± 1 *surgery of* M_{i-1} *along* K_{i-1}*.*

We start with several well-known results on surgery of 3-dimensional manifolds.

Let M be the result of a surgery of S^3 along a link $L \subset S^3$, where $L = \{L_1, L_2, \cdots, L_n\}$ has n components and the surgery coefficient on the component L_i is c_i. The link together with these coefficients (they are also called framings) is called a framed link. Put an orientation on each component L_i of the link L, then let $c_{ii} = c_i$ and for $i \ne j$, c_{ij} be the linking number $\mathrm{Lk}(L_i, L_j)$ of L_i and L_j. The matrix $C = (c_{ij})$ is called the linking matrix of the framed link. It is easy to see that the first homology group $\mathbf{H}_1(M, \mathbf{Z})$ of M is $\mathbf{Z}^n$ modulo

Manuscript received August 23, 1995.
*Institute of Mathematics, East China Normal University, Shanghai 200062, China.
**Project supported by the National Natural Science Foundation of China, SEDC and the Hok Ying Tong Eduction Foundation.

the lattice generated by the rows of the linking matrix. So the manifold M is a homology sphere iff the linking matrix is unimodular.

The linking matrix C of a framed link L depends on the choice of the orientation on L and how to order the components of L. If we reverse the orientation on the component L_i, c_{ik} and c_{ki} are changed by a sign for all $k \neq i$, and the rest of entries remain the same, i.e.,

$$C \Rightarrow (I_{i-1} \oplus (-1) \oplus I_{n-i}) C (I_{i-1} \oplus (-1) \oplus I_{n-i}).$$

If we interchange the orders of two components L_i and L_j, the change of the linking matrix obeys the following rule

$$C \Rightarrow \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} C \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

The same manifold M may be represented by surgeries of S^3 along two or more different framed links. The connection between these two framed links is the so-called the Kirby calculus. Kirby introduced two kinds of moves: I) add or remove an isolated copy of an unknotted circle with framing ± 1; II) slide one 2-handle over another. Then he proved the following theorem[4].

Theorem 2. *The results of surgeries of S^3 along two framed links L and L' are the same if and only if L can be changed to L' by a sequence of moves of type* I *and type* II.

The proof of the "if part" of the theorem is easy, and in fact we only need that part.

We will see how the linking matrix changes under the Kirby moves. It is clear that the changes on the linking matrix corresponding to type I moves are

$$C \Leftrightarrow C \oplus (\pm 1).$$

Let a type II move be sliding L_i over L_j. Then c_{ii} changes to $c_{ii} \pm 2c_{ij} + c_{jj}$, for $k \neq i$, c_{ik}, c_{ki} change to $c_{ik} \pm c_{jk}, c_{ki} \pm c_{kj}$ respectively and the rest of entries remain the same, i.e.,

$$C \Rightarrow \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & \pm 1 & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix} C \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \vdots & \ddots & & & \\ & & \pm 1 & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}.$$

Proof of Theorem 1. If a homology sphere M is the result of a surgery of S^3 on a link L, then the linking matrix is an $n \times n$ integral symmetric unimodular matrix. If the linking matrix C is equal to $C' \oplus (\pm 1)$ (i.e., the linking numbers of L_n and L_i are zero for all $1 \leq i < n$), then C' is also an $(n-1) \times (n-1)$ integral symmetric unimodular matrix. Let $L' = \{L_1, \cdots, L_{n-1}\}$ and M' be the result of the surgery of S^3 along the link L'. Then M' is a homology sphere and doing surgery on M' along the knot L_n yields M. So to prove the theorem, it suffices to find a surgery representation with diagonal linking matrix.

To finish the proof, we need to know more about the integral symmetric unimodular matrices. Let S_n be the set of all $n \times n$ integral symmetric unimodular matrices. Two matrices $A, A' \in S_n$ are said to be equivalent, if there is an $n \times n$ matrix $B \in GL(n, \mathbf{Z})$, such that $A = B^T A' B$. The structure theorem for the integral symmetric unimodular matrices says that:

Theorem 3. *Let $A, A' \in S_n$ with the same signature ($\sigma(A) = \sigma(A')$). Then*

$$A \oplus (+1) \simeq A' \oplus (+1) \quad \text{or} \quad A \oplus (-1) \simeq A' \oplus (-1).$$

In particular, for every $A \in S_n$, either $A \oplus (+1)$ or $A \oplus (-1)$ is diagonalizable.

For a proof of the theorem, we refer readers to Serre's book (see [6, Theorem V.4, p.53]).

If we add an isolated copy of an unknotted circle with framing ± 1 to the framing link, the linking matrix C changes to $C \oplus (\pm 1)$. Now, using Theorem 3, to complete the proof of Theorem 1, we only need to check the following fact that

$$P(i) = I_{i-1} \oplus (-1) \oplus I_{n-i},$$

$$Q(i,j) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 0 & \cdots & 1 & & \\ & & \vdots & \ddots & \vdots & & \\ & & 1 & \cdots & 0 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix},$$

and

$$R_{\pm}(i,j) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & \vdots & \ddots & & & \\ & & \pm 1 & \cdots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

generate the whole group $GL(n, \mathbf{Z})$. If this is true, we can find a new framed link with diagonal linking matrix.

Lemma 1. *$P(i), Q(i,j)$ and $R(i,j)$ form a set of generators for the group $GL(n, \mathbf{Z})$.*

Proof. It is clear that

$$P(i)^{-1} = P(i), Q(i,j)^{-1} = Q(i,j)$$

and

$$R_{\pm}(i,j)^{-1} = R_{\mp}(i,j).$$

Let B be any matrix in $GL(n, \mathbf{Z})$. Since $\det B = \pm 1$, the maximal common divisor of the entries in the first row is 1. On the other hand,

$$BR_{\pm}(1,2) = \begin{pmatrix} b_{11} & b_{12} & \cdots \\ b_{21} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix} R_{\pm}(1,2) = \begin{pmatrix} b_{11} \pm b_{12} & b_{12} & \cdots \\ b_{21} \pm b_{22} & b_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

so we can do Euclidean algorithm on the first row to make only one entry on the first row nonzero. Then multiplying $Q(1,j)$ or $P(1)$, or both if necessary, on the right, we have

$$BT_1 = \begin{pmatrix} 1 & 0 \\ * & B_1 \end{pmatrix},$$

where T_1 is a product of $P(i)$'s, $Q(i,j)$'s and $R_\pm(i,j)$'s. Further, multiplying enough $R_\pm(1,j)$'s on the left, we can get

$$T_2BT_1 = \begin{pmatrix} 1 & 0 \\ 0 & B_1 \end{pmatrix},$$

where again T_2 is a product of $P(i)$'s, $Q(i,j)$'s and $R_\pm(i,j)$'s. Doing it inductively, we can get T_3 and T_4, both of which are products of $P(i)$'s, $Q(i,j)$'s and $R_\pm(i,j)$'s, such that $T_4BT_3 = I_n$, i.e., $B = T_4^{-1}T_3^{-1}$ is a product of $P(i)$'s, $Q(i,j)$'s and $R_\pm(i,j)$'s.

We complete the proof of the lemma and the proof of Theorem 1.

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