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INITIAL VALUE PROBLEMS FOR SECOND ORDER IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS IN BANACH SPACES**

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Abstract

This paper investigates the maximal and minimal solutions of initial value problem for second order nonlinear impulsive integro-differential equation in a Banach space by establishing a comparison result and using the upper and lower solutions.

Keywords Impulsive integro-differential equation, Banach space, Initial value problem, Comparison result
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§1. Introduction

The theory of impulsive differential equations in Banach spaces has become an important area of investigation in recent years (see [1]). In paper [2], we have discussed the existence of solutions of boundary value problem for second order nonlinear impulsive differential equation in a Banach space by means of fixed point theory. Now, in this paper, we shall investigate the existence of extremal solutions of initial value problem (IVP) for second order nonlinear impulsive integro-differential equation in a Banach space by means of completely different method, that is, by establishing a comparison result and using the upper and lower solutions. Consider the IVP for impulsive integro-differential equation in Banach space E:

$$\begin{cases} x'' = f(t,x,Tx), & t \in J,\ t \neq t_k, \\ \triangle x|_{t=t_k} = L_k x'(t_k), & \\ \triangle x'|_{t=t_k} = L'_k x(t_k) & (k=1,2,\cdots,m), \\ x(0)=x_0, \quad x'(0)=x_1, & \end{cases} \tag{1.1}$$

where $f \in C(J \times E \times E, E)$, $J=[0,a]$ $(a>0)$, $0<t_1<\cdots<t_k<\cdots<t_m<a$, L_k, L'_k $(k=1,2,\cdots,m)$ are constants, $x_0,\ x_1 \in E$, and

$$(Tx)(t) = \int_0^t k(t,s)x(s)ds, \tag{1.2}$$

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$k \in C(D, R_+)$, $D = \{(t,s) \in J \times J:\ t \geq s\}$, R_+ is the set of all nonnegative numbers. $\triangle x|_{t=t_k}$ denotes the jump of $x(t)$ at $t = t_k$, i.e.

$$\triangle x|_{t=t_k} = x(t_k^+) - x(t_k^-),$$

where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, and $\triangle x'|_{t=t_k}$ has a similar meaning for $x'(t)$. Let $PC(J, E) = \{x:\ x$ is a map from J into E such that $x(t)$ is continuous at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$ exist, $k = 1, 2, \cdots, m\}$ and $PC^1(J, E) = \{x:\ x$ is a map from J into E such that $x(t)$ is continuously differentiable at $t \neq t_k$, left continuous at $t = t_k$, and $x(t_k^+)$, $x'(t_k^-)$, $x'(t_k^+)$ exist, $k = 1, 2, \cdots, m\}$. Evidently, $PC(J, E)$ is a Banach space with norm

$$\|x\|_{PC} = \sup_{t \in J} \|x(t)\|.$$

For $x \in PC^1(J, E)$, by virtue of the mean value theorem

$$x(t_k) - x(t_k - h) \in h\ \overline{\text{co}}\{x'(t):\ t_k - h < t < t_k\} \quad (h > 0),$$

it is easy to see that the left derivative $x'_-(t_k)$ exists and

$$x'_-(t_k) = \lim_{h \to 0^+} h^{-1}[x(t_k) - x(t_k - h)] = x'(t_k^-).$$

In (1.1) and in the following, $x'(t_k)$ is understood as $x'_-(t_k)$. It is clear that $PC^1(J, E)$ is a Banach space with norm

$$\|x\|_{PC^1} = \max\{\|x\|_{PC},\ \|x'\|_{PC}\}.$$

Let $J' = J \backslash \{t_1, \cdots, t_m\}$. A map $x \in PC^1(J, E) \cap C^2(J', E)$ is called a solution of IVP(1.1) if it satisfies (1.1).

§2. Comparison Result

Let E be partially ordered by a cone P of E, i.e. $x \leq y$ if and only if $y - x \in P$. P is said to be normal if there exists a positive constant N such that $\theta \leq x \leq y$ implies $||x|| \leq N||y||$, where θ denotes the zero element of E, and P is said to be regular if $x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$ implies $||x_n - x|| \to 0$ as $n \to \infty$ for some $x \in E$. It is well known that the regularity of P implies the normality of P. For details on cone theory, see [3].

In the following, let $J_0 = [0, t_1]$, $J_1 = (t_1, t_2], \cdots, J_{m-1} = (t_{m-1}, t_m]$, $J_m = (t_m, a]$, $r = \max\{t_{k+1} - t_k:\ k = 0, 1, \cdots, m\}$ (here $t_0 = 0$, $t_{m+1} = a$) and $k_0 = \max\{k(t,s):\ (t,s) \in D\}$.

Lemma 2.1 (a) *If* $x \in PC(J, E) \cap C^1(J', E)$, *then*

$$x(t) = x(0) + \int_0^t x'(s)ds + \sum_{0 < t_k < t} [x(t_k^+) - x(t_k)], \quad t \in J. \tag{2.1}$$

(b) *If* $x \in PC^1(J, E) \cap C^2(J', E)$, *then*

$$x'(t) = x'(0) + \int_0^t x''(s)ds + \sum_{0 < t_k < t} [x'(t_k^+) - x'(t_k)], \quad t \in J, \tag{2.2}$$

and

$$x(t) = x(0) + tx'(0) + \int_0^t (t-s)x''(s)ds + \sum_{0 < t_k < t} \{[x(t_k^+) - x(t_k)] + (t - t_k)[x'(t_k^+) - x'(t_k)]\}, \quad t \in J. \tag{2.3}$$

Proof. (a) Let $x \in PC(J,E) \cap C^1(J',E)$ and $t_k < t \le t_{k+1}$. Then

$$x(t_1) - x(0) = \int_0^{t_1} x'(s)ds, \quad x(t_2) - x(t_1^+) = \int_{t_1}^{t_2} x'(s)ds,$$
$$\cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots$$
$$x(t_k) - x(t_{k-1}^+) = \int_{t_{k-1}}^{t_k} x'(s)ds, \quad x(t) - x(t_k^+) = \int_{t_k}^{t} x'(s)ds.$$

Adding together, we get

$$x(t) - x(0) - \sum_{i=1}^{k}[x(t_i^+) - x(t_i)] = \int_0^t x'(s)ds,$$

i.e. (2.1) holds.

(b) Let $x \in PC^1(J,E) \cap C^2(J',E)$. Replacing $x(t)$ by $x'(t)$ in (2.1), we get (2.2). Finally, substituting (2.2) into (2.1), we can obtain (2.3).

Lemma 2.2. (Comparison result) *Assume that* $p \in PC^1(J,E) \cap C^2(J',E)$ *satisfies*

$$\begin{cases} p'' \le -Mp - NTp, & t \in J,\ t \ne t_k, \\ \triangle p|_{t=t_k} \le L_k p'(t_k), & \\ \triangle p'|_{t=t_k} \le L_k' p(t_k) & (k = 1, 2, \cdots, m), \\ p(0) \le \theta, \quad p'(0) \le \theta, & \end{cases} \tag{2.4}$$

where $M \ge 0,\ N \ge 0,\ L_k \ge 0,\ L_k' \le 0\ (k = 1, 2, \cdots, m)$ *are constants and*

$$\Big(\sum_{k=1}^{m} L_k + (m+1)r\Big)\ \Big(-\sum_{k=1}^{m} L_k' + a(M + ak_0N)\Big) \le 1. \tag{2.5}$$

Then $p(t) \le \theta$ *for* $t \in J$.

Proof. For any $g \in P^*$ (P^* denotes the dual cone of P), let $u(t) = g(p(t))$. Then $u \in PC^1(J,R) \cap C^2(J',R)$, where R is the set of all real numbers, and

$$u'(t) = g(p'(t)), \quad g((Tp)(t)) = (Tu)(t).$$

By (2.4), we have

$$\begin{cases} u'' \le -Mu - NTu, & t \in J,\ t \ne t_k, \\ \triangle u|_{t=t_k} \le L_k u'(t_k), & \\ \triangle u'|_{t=t_k} \le L_k' u(t_k) & (k = 1, 2, \cdots, m), \\ u(0) \le 0, \quad u'(0) \le 0. & \end{cases} \tag{2.6}$$

We now prove

$$u(t) \le 0, \quad t \in J. \tag{2.7}$$

Suppose that (2.7) is not true. Then, there is a $0 < t^* \le a$ such that $u(t^*) > 0$. Let $t^* \in J_j$ and $\inf\{u(t) : 0 \le t \le t^*\} = -b$. We have $b \ge 0$.

In case of $b = 0$: $u(t) \ge 0$ for $0 \le t \le t^*$, so (2.6) implies that $u''(t) \le 0$ for $0 \le t \le t^*,\ t \ne t_k$, and

$$\triangle u'|_{t=t_k} \le L_k' u(t_k) \le 0 \quad \text{for} \quad t_k \le t^*.$$

Hence, $u'(t)$ is nonincreasing in $[0, t^*]$, and therefore $u'(t) \le u'(0) \le 0$ for $0 \le t \le t^*$ and

$$\triangle u|_{t=t_k} \le L_k u'(t_k) \le 0, \quad \text{for} \quad t_k \le t^*.$$

Consequently, $u(t)$ is nonincreasing in $[0,t^*]$, so $u(t)\le u(0)\le 0$ for $0\le t\le t^*$, which contradicts $u(t^*)>0$.

In case of $b>0$: there exists a J_i $(i\le j)$ such that $u(t_*)=-b$ for some $t_*\in J_i$ or $u(t_i^+)=-b$. From (2.6), we have

$$u''(t)\le Mb+Nak_0b,\quad 0\le t\le t^*,\ t\ne t_k,$$

$$\triangle u'|_{t=t_k}\le L_k'u(t_k)\le -bL_k',\quad t_k\le t^*,\quad \text{and}\quad u'(0)\le 0,$$

so, by (2.2),

$$u'(t)\le\int_0^t(Mb+Nak_0b)ds-\sum_{0<t_k<t}bL_k'\le bM_0,\quad 0\le t\le t^*, \tag{2.8}$$

where

$$M_0=aM+a^2k_0N-\sum_{k=1}^m L_k'. \tag{2.9}$$

Now, mean value theorem implies

$$u(t^*)-u(t_j^+)=u'(s_j)(t^*-t_j)\quad (t_j<s_j<t^*),$$
$$u(t_j)-u(t_{j-1}^+)=u'(s_{j-1})(t_j-t_{j-1})\quad (t_{j-1}<s_{j-1}<t_j),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u(t_{i+2})-u(t_{i+1}^+)=u'(s_{i+1})(t_{i+2}-t_{i+1})\quad (t_{i+1}<s_{i+1}<t_{i+2}),$$

$$\begin{cases} u(t_{i+1})-u(t_*)=u'(s_i)(t_{i+1}-t_*) & (t_*<s_i<t_{i+1}),\ \text{if}\ u(t_*)=-b,\\ u(t_{i+1})-u(t_i^+)=u'(s_i^*)(t_{i+1}-t_i) & (t_i<s_i^*<t_{i+1}),\ \text{if}\ u(t_i^+)=-b,\end{cases}$$

and, by (2.6) and (2.8),

$$u(t_k^+)-u(t_k)=\triangle u|_{t=t_k}\le L_ku'(t_k)\le bM_0L_k,\quad t_k\le t^*,$$

hence

$$u(t^*)-u(t_j)-bM_0L_j\le u'(s_j)(t^*-t_j),$$
$$u(t_j)-u(t_{j-1})-bM_0L_{j-1}\le u'(s_{j-1})(t_j-t_{j-1}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{cases} u(t_{i+1})+b=u'(s_i)(t_{i+1}-t_*), & \text{if}\ u(t_*)=-b,\\ u(t_{i+1})+b=u'(s_i^*)(t_{i+1}-t_i), & \text{if}\ u(t_i^+)=-b.\end{cases}$$

Adding together and using (2.8), we obtain

$$u(t^*)+b-bM_0\sum_{k=i+1}^{j}L_k\le (j-i+1)bM_0r,$$

and so

$$0<u(t^*)\le -b+bM_0\sum_{k=1}^m L_k+(m+1)bM_0r,$$

which contradicts (2.5).

Hence (2.7) holds. Since $g\in P^*$ is arbitrary, (2.7) implies that $p(t)\le\theta$ for $t\in J$. The proof is complete.

Lemma 2.3. *Let $z \in PC(J,E)$. Then, $x \in PC^1(J,E) \cap C^2(J',E)$ is a solution of the linear* IVP

$$\begin{cases} x'' = -Mx - NTx + z(t), & t \in J,\ t \neq t_k, \\ \triangle x|_{t=t_k} = L_k x'(t_k), & \\ \triangle x'|_{t=t_k} = L'_k x(t_k) & (k = 1, 2, \cdots, m), \\ x(0) = x_0, \quad x'(0) = x_1, & \end{cases} \tag{2.10}$$

if and only if $x \in PC^1(J,E)$ is a solution of the following linear impulsive integral equation

$$x(t) = x_0 + tx_1 + \int_0^t (t-s)(-Mx(s) - N(Tx)(s) + z(s))ds + \sum_{0<t_k<t} [L_k x'(t_k) + (t-t_k)L'_k x(t_k)]. \tag{2.11}$$

Proof. If $x \in PC^1(J,E) \cap C^2(J',E)$ is a solution of IVP(2.10), then, substituting (2.10) into (2.3), we get (2.11).

Conversely, if $x \in PC^1(J,E)$ is a solution of Equation (2.11), then direct differentiation of (2.11) gives

$$x'(t) = x_1 + \int_0^t (-Mx(s) - N(Tx)(s) + z(s))ds + \sum_{0<t_k<t} L'_k x(t_k), \quad t \in J,\ t \neq t_k,$$

and

$$x''(t) = -Mx(t) - N(Tx)(t) + z(t), \qquad t \in J,\ t \neq t_k.$$

Hence $x \in PC^1(J,E) \cap C^2(J',E)$ and $x(t)$ satisfies (2.10).

Lemma 2.4. *Let $z \in PC(J,E)$ and $M \geq 0,\ N \geq 0,\ L_k \geq 0,\ L'_k \leq 0$ $(k = 1, 2, \cdots, m)$. If*

$$b_1 = \frac{a^2}{2}(M + ak_0N) + \sum_{k=1}^m [L_k - (a - t_k)L'_k] < 1, \tag{2.12}$$

$$b_2 = a(M + ak_0N) - \sum_{k=1}^m L'_k < 1, \tag{2.13}$$

then Equation (2.11) *has a unique solution in $PC^1(J,E)$.*

Proof. Define operator F by

$$(Fx)(t) = x_0 + tx_1 + \int_0^t (t-s)(-Mx(s) - N(Tx)(s) + z(s))ds + \sum_{0<t_k<t} [L_k x'(t_k) + (t-t_k)L'_k x(t_k)]. \tag{2.14}$$

Then

$$(Fx)'(t) = x_1 + \int_0^t (-Mx(s) - N(Tx)(s) + z(s))ds + \sum_{0<t_k<t} L'_k x(t_k), \tag{2.15}$$

and F is an operator from $PC^1(J,E)$ into $PC^1(J,E)$. For $x, y \in PC^1(J,E)$, we have by (2.14),

$$||(Fx)(t) - (Fy)(t)|| \leq (M||x-y||_{PC} + ak_0N||x-y||_{PC}) \int_0^t (t-s)ds + \sum_{0<t_k<t} [L_k||x'-y'||_{PC} - (t-t_k)L'_k||x-y||_{PC}],$$

so

$$
\begin{aligned}
||Fx-Fy||_{PC} &\le \frac{a^2}{2}(M+ak_0N)||x-y||_{PC}+\Big(\sum_{k=1}^{m}L_k\Big)||x'-y'||_{PC}\\
&\quad-\Big(\sum_{k=1}^{m}(a-t_k)L_k'\Big)||x-y||_{PC}\\
&\le b_1||x-y||_{PC^1},
\end{aligned}
$$

where b_1 is defined by (2.12). Similarly, (2.15) implies

$$||(Fx)'-(Fy)'||_{PC}\le b_2||x-y||_{PC^1},$$

where b_2 is defined by (2.13). Hence

$$||Fx-Fy||_{PC^1}\le b^*||x-y||_{PC^1},\qquad x,y\in PC^1(J,E),\tag{2.16}$$

where

$$b^*=\max\{b_1,b_2\}<1.\tag{2.17}$$

Consequently, the Banach fixed point theorem implies that F has a unique fixed point in $PC^1(J,E)$, and the lemma is proved.

§3. Main Theorem

Let us list some conditions.

(H_1) There exist $u_0,\ v_0\in PC^1(J,E)\cap C^2(J',E)$ with $u_0(t)\le v_0(t)$ $(t\in J)$ such that

$$
\begin{cases}
u_0''\le f(t,u_0,Tu_0), & t\in J,\ t\ne t_k,\\
\triangle u_0|_{t=t_k}\le L_k u_0'(t_k), & \\
\triangle u_0'|_{t=t_k}\le L_k' u_0(t_k) & (k=1,2,\cdots,m),\\
u_0(0)\le x_0,\quad u_0'(0)\le x_1 &
\end{cases}
$$

and

$$
\begin{cases}
v_0''\ge f(t,v_0,Tv_0), & t\in J,\ t\ne t_k,\\
\triangle v_0|_{t=t_k}\ge L_k v_0'(t_k), & \\
\triangle v_0'|_{t=t_k}\ge L_k' v_0(t_k) & (k=1,2,\cdots,m),\\
v_0(0)\ge x_0,\quad v_0'(0)\ge x_1, &
\end{cases}
$$

where constants $L_k\ge 0,\ L_k'\le 0$ $(k=1,2,\cdots,m)$, i.e. u_0 and v_0 are lower and upper solution of IVP(1.1) respectively.

(H_2) There exist constants $M\ge 0$ and $N\ge 0$ such that

$$f(t,x,y)-f(t,\overline{x},\overline{y})\ge -M(x-\overline{x})-N(y-\overline{y})$$

whenever $t\in J,\ u_0(t)\le\overline{x}\le x\le v_0(t)$ and $(Tu_0)(t)\le\overline{y}\le y\le(Tv_0)(t)$.

In the following, let

$$[u_0,v_0]=\{x\in PC(J,E):\ u_0(t)\le x(t)\le v_0(t)\ \text{for}\ t\in J\}.$$

Theorem 3.1. *Let cone P be regular and f be uniformly continuous on $J\times B_r\times B_r$ for any $r>0$, where $B_r=\{x\in E:\ ||x||\le r\}$. Suppose that conditions (H_1) and (H_2) are satisfied and inequlities (2.5), (2.12) and (2.13) hold. Then there exist monotone sequences $\{u_n\},\ \{v_n\}\subset PC^1(J,E)\cap C^2(J',E)$ which converge in $PC^1(J,E)$ to the minimal and maximal solutions $\overline{x},\ x^*\in PC^1(J,E)\cap C^2(J',E)$ of IVP(1.1) in $[u_0,v_0]$ respectively.*

Proof. For any $w \in [u_0, v_0]$, consider the linear IVP(2.10) with

$$z(t) = f(t, w(t), (Tw)(t)) + Mw(t) + N(Tw)(t). \tag{3.1}$$

By Lemma 2.3 and Lemma 2.4, IVP(2.10) has a unique solution $x \in PC^1(J, E) \cap C^2(J', E)$ which is the unique solution of Equation (2.11) in $PC^1(J, E)$. Let $x = Aw$. Then A is an operator from $[u_0, v_0]$ into $PC^1(J, E) \cap C^2(J', E)$. We now show that (a) $u_0 \le Au_0$, $Av_0 \le v_0$ and (b) A is nondecreasing in $[u_0, v_0]$. To prove (a), we set $u_1 = Au_0$ and $p = u_0 - u_1$. From (2.10) and (3.1), we have

$$\begin{cases} u_1'' = -Mu_1 - NTu_1 + f(t, u_0, Tu_0) + Mu_0 + NTu_0, \quad t \in J,\ t \ne t_k, \\ \triangle u_1|_{t=t_k} = L_k u_1'(t_k), \\ \triangle u_1'|_{t=t_k} = L_k' u_1(t_k) \quad (k = 1, 2, \cdots, m), \\ u_1(0) = x_0, \quad u_1'(0) = x_1, \end{cases}$$

so, by (H$_1$),

$$\begin{cases} p'' = u_0'' - u_1'' \le -Mp - NTp, \quad t \in J,\ t \ne t_k, \\ \triangle p|_{t=t_k} = \triangle u_0|_{t=t_k} - \triangle u_1|_{t=t_k} \le L_k p'(t_k), \\ \triangle p'|_{t=t_k} = \triangle u_0'|_{t=t_k} - \triangle u_1'|_{t=t_k} \le L_k' p(t_k), \\ p(0) = u_0(0) - u_1(0) \le \theta, \quad p'(0) = u_0'(0) - u_1'(0) \le \theta, \end{cases}$$

which implies by virtue of Lemma 2.2 that $p(t) \le \theta$ for $t \in J$, i.e. $u_0 \le Au_0$. Similarly, we can show that $Av_0 \le v_0$. To prove (b), let $w_1,\ w_2 \in [u_0, v_0]$ such that $w_1 \le w_2$ and let $p = \overline{w}_1 - \overline{w}_2$, where $\overline{w}_1 = Aw_1$ and $\overline{w}_2 = Aw_2$. Then, from (2.10), (3.1) and (H$_2$), we have

$$\begin{cases} p'' = -Mp - NTp - [f(t, w_2, Tw_2) - f(t, w_1, Tw_1) + M(w_2 - w_1) + N(Tw_2 - Tw_1)] \\ \qquad \le -Mp - NTp, \quad t \in J,\ t \ne t_k, \\ \triangle p|_{t=t_k} = L_k p'(t_k), \\ \triangle p'|_{t=t_k} = L_k' p(t_k) \quad (k = 1, 2, \cdots, m), \\ p(0) = \theta, \quad p'(0) = \theta. \end{cases}$$

So, Lemma 2.2 implies that $p(t) \le \theta$ for $t \in J$, i.e. $Aw_1 \le Aw_2$, and (b) is proved.

Let $u_n = Au_{n-1}$ and $v_n = Av_{n-1}$ $(n = 1, 2, 3, \cdots)$. By (a) and (b) just proved, we have

$$u_0(t) \le u_1(t) \le \cdots \le u_n(t) \le \cdots \le v_n(t) \le \cdots \le v_1(t) \le v_0(t), \quad t \in J. \tag{3.2}$$

On account of the definition of u_n and (2.11), we have

$$\begin{aligned} u_n(t) = x_0 + tx_1 + \int_0^t (t - s)(-Mu_n(s) - N(Tu_n)(s) + z_{n-1}(s))ds \\ + \sum_{0<t_k<t} [L_k u_n'(t_k) + (t - t_k) L_k' u_n(t_k)], \end{aligned} \tag{3.3}$$

where

$$z_{n-1}(t) = f(t, u_{n-1}(t), (Tu_{n-1})(t)) + Mu_{n-1}(t) + N(Tu_{n-1})(t), \tag{3.4}$$

so

$$u_n'(t) = x_1 + \int_0^t (-Mu_n(s) - N(Tu_n)(s) + z_{n-1}(s))ds + \sum_{0<t_k<t} L_k' u_n(t_k). \tag{3.5}$$

Similar to the proof of (2.16), by using (3.3) and (3.5) instead of (2.14) and (2.15), we can get

$$||u_{n+i} - u_n||_{PC^1} \le b^*||u_{n+i} - u_n||_{PC^1} + a^*||z_{n+i-1} - z_{n-1}||_{PC},$$

where b^* is defined by (2.17), (2.12), (2.13) and $a^* = \max\{\frac{a^2}{2}, a\}$. Hence

$$||u_{n+i} - u_n||_{PC^1} \le \frac{a^*}{1-b^*}||z_{n+i-1} - z_{n-1}||_{PC} \quad (n, i = 1, 2, 3, \cdots). \tag{3.6}$$

Since the regularity of P implies the normality of P, we see from (3.2) that $V = \{u_n : n = 0, 1, 2, \cdots\}$ is a bounded set in $PC(J, E)$. It is easy to show that the uniform continuity of f on $J \times B_r \times B_r$ implies the boundedness of f on $J \times B_r \times B_r$, so, by (3.4), there is a constant $c > 0$ such that

$$||z_{n-1}||_{PC} \le c \quad (n = 1, 2, 3, \cdots),$$

and therefore, from (3.3) we know that functions $\{u_n(t)\}$ $(n = 1, 2, 3, \cdots)$ are equicontinuous on each J_k $(k = 0, 1, \cdots, m)$. On the other hand, (3.2) and the regularity of P imply that $\alpha(V(t)) = 0$ $(t \in J)$, where $V(t) = \{u_n(t) : n = 0, 1, 2, \cdots\}$ and α denotes the Kuratowski measure of noncompactness in E. Hence V is relatively compact in $PC(J, E)$, and so, there is a subsequence of $\{u_n\}$, which converges uniformly in $t \in J$ to some $\overline{x} \in PC(J, E)$. Since $\{u_n\}$ is nondecreasing and P is normal, the entire sequence $\{u_n\}$ converges uniformly in $t \in J$ to $\overline{x}$, i.e.

$$||u_n - \overline{x}||_{PC} \to 0 \quad (n \to \infty). \tag{3.7}$$

From (3.4) and (3.7), we find

$$||z_{n-1} - \overline{z}||_{PC} \to 0 \quad (n \to \infty), \tag{3.8}$$

where

$$\overline{z}(t) = f(t, \overline{x}(t), (T\overline{x})(t)) + M\overline{x}(t) + N(T\overline{x})(t). \tag{3.9}$$

Now, (3.6) and (3.8) imply that the sequence $\{u_n\}$ is convergent in $PC^1(J, E)$, and hence, by (3.7), $\overline{x} \in PC^1(J, E)$ and

$$||u_n - \overline{x}||_{PC^1} \to 0 \quad (n \to \infty). \tag{3.10}$$

Observing (3.10), (3.8) and taking limits in (3.3), we obtain

$$\begin{aligned}\overline{x}(t) = x_0 + tx_1 + &\int_0^t (t-s)(-M\overline{x}(s) - N(T\overline{x})(s) + \overline{z}(s))ds \\ &+ \sum_{0<t_k<t} [L_k\overline{x}'(t_k) + (t - t_k)L_k'\overline{x}(t_k)],\end{aligned}$$

which implies by virtue of Lemma 2.3 that $\overline{x} \in PC^1(J, E) \cap C^2(J', E)$ and $\bar{x}$ is a solution of IVP(1.1).

In the same way, we can show that $\|v_n - x^*\|_{PC^1} \to 0$ $(n \to \infty)$ for some $x^* \in PC^1(J, E) \cap C^2(J', E)$ and x^* is a solution of IVP(1.1).

Finally, let $x \in PC^1(J, E) \cap C^2(J', E)$ be any solution of IVP(1.1) satisfying $u_0(t) \le x(t) \le v_0(t)$ for $t \in J$. Assume that $u_{n-1}(t) \le x(t) \le v_{n-1}(t)$ for $t \in J$, and set $p(t) =$

$u_n(t)-x(t)$. Then, by (H_2),

$$\begin{cases} p''=-Mp-NTp-[f(t,x,Tx)-f(t,u_{n-1},Tu_{n-1}) \\ \qquad\qquad +M(x-u_{n-1})+N(Tx-Tu_{n-1})] \\ \quad \leq -Mp-NTp, \qquad t\in J,\ t\neq t_k, \\ \triangle p|_{t=t_k}=L_kp'(t_k), \\ \triangle p'|_{t=t_k}=L_k'p(t_k) \quad (k=1,2,\cdots,m), \\ p(0)=\theta, \quad p'(0)=\theta, \end{cases}$$

which implies by virtue of Lemma 2.2 that $p(t)\leq\theta$ for $t\in J$, i.e. $u_n(t)\leq x(t)$ for $t\in J$. In the same way, we can show that $x(t)\leq v_n(t)$ for $t\in J$. Hence, by induction, $u_n(t)\leq x(t)\leq v_n(t)$ for $t\in J$ $(n=0,1,2,\cdots)$, which implies that $\overline{x}(t)\leq x(t)\leq x^*(t)$ for $t\in J$. The proof is complete.

Remark 3.1. The condition that P is regular will be satisfied if E is weakly complete (reflexive, in particular) and P is normal (see [4, Theorem 2]).

§4. An Example

Example 4.1. Consider the IVP of infinite system for nonlinear scalar second order integro-differential equations

$$\begin{cases} x_n''=\dfrac{1}{30}\Big(\dfrac{1}{4n^2}-x_n+x_{2n}\Big)+\dfrac{t}{60n^2}\left(\displaystyle\int_0^t e^{-ts}x_{n+1}(s)ds\right) \\ \qquad\quad -\dfrac{1}{50(n+1)^2}\Big(\displaystyle\int_0^t e^{-ts}x_n(s)ds\Big)^2, \qquad 0\leq t\leq 2,\ t\neq 1; \\ \triangle x_n|_{t=1}=\dfrac{1}{2}x_n'(1), \\ \triangle x_n'|_{t=1}=-\dfrac{1}{6}x_n(1), \\ x_n(0)=\dfrac{1}{n^2}, \quad x_n'(0)=0 \qquad (n=1,2,3,\cdots). \end{cases} \tag{4.1}$$

Conclusion. IVP(4.1) *admits minimal and maximal solutions which are continuously differentiable on* $[0,1)\cup(1,2]$ *and satisfy*

$$0\leq x_n(t)\leq\begin{cases}\frac{1}{n^2}, & 0\leq t\leq 1;\\ \frac{t+1}{n^2}, & 1<t\leq 2.\end{cases}$$

Proof. Let $E=l^1=\{x=(x_1,\cdots,x_n,\cdots): \sum\limits_{n=1}^{\infty}|x_n|<\infty\}$ with norm $||x||=\sum\limits_{n=1}^{\infty}|x_n|$ and $P=\{x=(x_1,\cdots,x_n,\cdots)\in l^1:\ x_n\geq 0,\ n=1,2,3,\cdots\}$. Then P is a normal cone in E. Since l^1 is weakly complete, from Remark 3.1 we know that P is regular. System (4.1) can be regarded as an IVP of form (1.1), where $a=2,\ k(t,s)=e^{-ts},\ x=(x_1,\cdots,x_n,\cdots),\ y=(y_1,\cdots,y_n,\cdots),\ f=(f_1,\cdots,f_n,\cdots)$, in which

$$f_n(t,x,y)=\frac{1}{30}\Big(\frac{1}{4n^2}-x_n+x_{2n}\Big)+\frac{t}{60n^2}y_{n+1}-\frac{1}{50(n+1)^2}y_n^2,$$

and $m=1,\ t_1=1,\ L_1=\frac{1}{2},\ L_1'=-\frac{1}{6},\ x_0=(1,\cdots,\frac{1}{n^2},\cdots),\ x_1=(0,\cdots,0,\cdots)$. Evidently, $f\in C(J\times E\times E,E)$ $(J=[0,2])$. Let $u_0(t)=(0,\cdots,0,\cdots)$ $(t\in J)$ and

$$v_0(t)=\begin{cases}\Big(1,\cdots,\frac{1}{n^2},\cdots\Big), & 0\leq t\leq 1;\\ \Big(t+1,\cdots,\frac{t+1}{n^2},\cdots\Big), & 1<t\leq 2.\end{cases}$$

We have $u_0 \in C^2(J,E)$, $v_0 \in PC^1(J,E) \cap C^2(J',E)$, $u_0(t) < v_0(t)$ $(t \in J)$ and

$$u_0(0) = (0,\cdots,0,\cdots) < \Big(1,\cdots,\frac{1}{n^2},\cdots\Big) = x_0, \quad u_0'(0) = (0,\cdots,0,\cdots) = x_1,$$

$$v_0(0) = \Big(1,\cdots,\frac{1}{n^2},\cdots\Big) = x_0, \quad v_0'(0) = (0,\cdots,0,\cdots) = x_1,$$

$$\triangle u_0|_{t=1} = (0,\cdots,0,\cdots) = \frac{1}{2}u_0'(1), \quad \triangle u_0'|_{t=1} = (0,\cdots,0,\cdots) = -\frac{1}{6}u_0(1),$$

$$\triangle v_0|_{t=1} = (1,\cdots,\frac{1}{n^2},\cdots) > (0,\cdots,0,\cdots) = \frac{1}{2}v_0'(1),$$

$$\triangle v_0'|_{t=1} = (1,\cdots,\frac{1}{n^2},\cdots) > -\frac{1}{6}(1,\cdots,\frac{1}{n^2},\cdots) = -\frac{1}{6}v_0(1),$$

$$u_0''(t) = (0,\cdots,0,\cdots)\ (t \in J), \quad v_0''(t) = (0,\cdots,0,\cdots)\ (t \in J,\ t \neq 1),$$

$$f_n(t,u_0(t),(Tu_0)(t)) = \frac{1}{120n^2} > 0 \ \ (t \in J),$$

$$0 \le t \le 1 \Longrightarrow f_n(t,v_0(t),(Tv_0)(t)) < \frac{1}{30}\Big(\frac{1}{4n^2} - \frac{1}{n^2} + \frac{1}{4n^2}\Big) + \frac{t}{60n^2}\Big(\frac{1}{(n+1)^2}\int_0^t e^{-ts}ds\Big)$$

$$\le -\frac{1}{60n^2} + \frac{1}{60n^2(n+1)^2} < 0,$$

$$1 < t \le 2 \Longrightarrow f_n(t,v_0(t),(Tv_0)(t)) < \frac{1}{30}\Big(\frac{1}{4n^2} - \frac{t+1}{n^2} + \frac{t+1}{4n^2}\Big)$$

$$+ \frac{t}{60n^2}\Big\{\frac{1}{(n+1)^2}\Big(\int_0^1 e^{-ts}ds + \int_1^t e^{-ts}(s+1)ds\Big)\Big\}$$

$$\le \frac{1}{30}\Big(\frac{1}{4n^2} - \frac{3(t+1)}{4n^2}\Big) + \frac{t}{60n^2}\Big\{\frac{1}{(n+1)^2}\left(1 + \frac{1}{e}\int_1^t (s+1)ds\right)\Big\}$$

$$< -\frac{1}{60n^2} + \frac{1}{60n^2} = 0.$$

Hence, u_0 and v_0 satisfy (H_1). On the other hand, for $t \in J$, $u_0(t) \le \overline{x} \le x \le v_0(t)$ and $(Tu_0)(t) \le \overline{y} \le y \le (Tv_0)(t)$, we have $0 \le \overline{x}_n \le x_n \le \frac{3}{n^2}$, $0 \le \overline{y}_n \le y \le \frac{2}{n^2}$ $(n = 1,2,3,\cdots)$, so

$$f_n(t,x,y) - f_n(t,\overline{x},\overline{y}) \ge -\frac{1}{30}(x_n - \overline{x}_n) - \frac{1}{50(n+1)^2}(y_n^2 - \overline{y}_n^2)$$

$$\ge -\frac{1}{30}(x_n - \overline{x}_n) - \frac{1}{50}(y_n - \overline{y}_n), \quad (n = 1,2,3,\cdots),$$

consequently, (H_2) is satisfied for $M = \frac{1}{30}$ and $N = \frac{1}{50}$. It is clear that $k_0 = 1$ and $r = 1$, and it is easy to verify that inequalities (2.5), (2.12) and (2.13) hold. Hence, our conclusion follows from Theorem 3.1.

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