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ASYMPTOTIC STABILITY FOR A CLASS OF NONAUTONOMOUS NEUTRAL DIFFERENTIAL EQUATIONS**

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Abstract

Consider the scalar nonlinear delay differential equation

$$\frac{d}{dt}[x(t)-f(t,x(t-\tau))]+g(t,x(t-\delta))=0, \quad t\geq t_0,$$

where $\tau,\delta\in(0,\infty), f,g\in C([t_0,\infty)\times \boldsymbol{R},\boldsymbol{R})$ and $xg(t,x)\geq 0$ for $t\geq t_0, x\in \boldsymbol{R}$. The author obtains sufficient conditions for the zero solution of this equation to be uniformly stable as well as asymptotically stable.

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§1. Introduction

Consider the following neutral delay differential equation

$$\frac{d}{dt}[x(t)-f(t,x(t-\tau))]+g(t,x(t-\delta))=0, \quad t\geq t_0, \tag{1.1}$$

where $\tau,\delta\in(0,\infty), f,g\in C([t_0,\infty)\times \boldsymbol{R},\boldsymbol{R})$ and $xg(t,x)\geq 0$, for $t\geq t_0, x\in \boldsymbol{R}$.

When $f(t,x)\equiv 0$ and $g(t,x)=Q(t)x$, Equation (1.1) reduces to the nonneutral equation

$$\frac{dx(t)}{dt}+g(t,x(t-\delta))=0, \tag{1.2}$$

whose stability of the zero solution has been extensively investigated in the literature (see for example [1, 2, 4–12]). The best result known to us is obtained in [10] (see also [8]), which says that if there is $H>0$ such that $0\leq xg(t,x)\leq Q(t)x^2$ for $t\geq t_0, |x|<H$ and

$$\int_t^{t+\delta} Q(s)ds\leq\frac{3}{2}, \quad t\geq t_0, \tag{1.3}$$

then the zero solution of Equation (1.2) is uniformly stable. In addition, if

$$\sup_{t\geq t_0}\int_t^{t+\delta} Q(s)ds<\frac{3}{2} \text{ and } \inf_{t\geq t_0}\int_t^{t+\delta} Q(s)ds>0, \tag{1.4}$$

then the zero solution of Equation (1.2) is asymptotically stable.

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It was also pointed in [10] that the upper bound $\frac{3}{2}$ in (1.3) is the best possible for Equation (1.2). For several other related results on the stability of Equation (1.2), we may refer to [2, 4, 5, 6, 7] for the linear case $g(t,x) = Q(t)x$. Recently, conditions (1.3) and (1.4) have been developed to equations with unbounded delays, one can refer to [1, 10]. In [6], the asymptotic behavior of all solutions of the linear equation

$$\frac{d}{dt}[x(t) - px(t-\tau)] + Q(t)x(t-\delta) = 0, \quad t \geq t_0 \tag{1.5}$$

was studied when $Q(t)$ is eventually positive. In this case, it was shown that if

$$\int_{t_0}^{\infty} Q(s)\,ds = \infty \tag{1.6}$$

and

$$2|p| + \limsup_{t\to\infty} \int_{t-\delta}^{t} Q(s)\,ds \ < \ 1, \tag{1.7}$$

then every solution of Equation (1.5) tends to zero as $t \to \infty$.

When $\delta = 0, g(t,x) = g(x)$ and $f(t,x) = px$, the stability of Equation (1.1) has been investigated in [3]. It was proved in [3, p.300] that if $|p| < \frac{1}{2}$ and $xg(x) > 0$ for $x \neq 0, |g(x)| \to \infty$ as $|x| \to \infty$, then the zero solution of Equation (1.1) is uniformly asymptotically stable and every solution tends to zero as $t \to \infty$. Our purpose in this paper is to develop conditions (1.3) and (1.4) to Equation (1.1) of neutral type. The main results in this paper are the following two theorems, which are brand-new to the present time.

Theorem 1.1. *Assume that there are $p \in [0,1)$ and $H > 0$ such that*

$$|f(t,x)| \leq p|x| \ \text{ and } \ xg(t,x) \leq Q(t)x^2, \ \ t \geq t_0, |x| < H \tag{1.8}$$

and that

$$2p(2-p) + \int_{t}^{t+\delta} Q(s)\,ds \leq \frac{3}{2}, \quad t \geq t_0. \tag{1.9}$$

Then the zero solution of Equation (1.1) *is uniformly stable.*

Theorem 1.2. *Assume that* (1.6) *and* (1.8) *hold and that*

$$\lambda = 2p(2-p) + \sup_{t\geq t_0} \int_{t}^{t+\delta} Q(s)\,ds \ < \ \frac{3}{2}. \tag{1.10}$$

Then the zero solution of Equation (1.1) *is asymptotically stable.*

We should note that (1.9) becomes (1.3), and (1.10) becomes the first condition (1.4) when $p = 0$, while in this case the second condition of (1.4) is much improved by (1.6). In the meantime, by Theorem 1.2 we see that if (1.6) holds and

$$2|p|(2-|p|) + \limsup_{t\to\infty} \int_{t}^{t+\delta} Q(s)\,ds \ < \ \frac{3}{2}, \tag{1.11}$$

then every solution of Equation (1.5) tends to zero as $t \to \infty$.

Clearly, (1.11) is an improvement on (1.7).

Finally, we shall apply the above theorems to the following neutral delay logistic equation

$$\dot{x}(t) = rx(t)\Big[1 - \frac{x(t-\tau) + \rho\dot{x}(t-\tau)}{K}\Big], \tag{1.12}$$

where $r, K, \tau \in (0,\infty)$ and $\rho \in \mathbb{R}$. This equation was first introduced and investigated in [12] (see also [13]).

§2. Proofs of Main Results

In this section, one will prove Theorems 1.1 and 1.2 listed in the above section. The method we will use is not the usual Liapunov functional (or function) method.

Proof of Theorem 1.1. Let $\rho = \max\{\tau, \delta\}, m = \min\{\tau, \delta\}$. Then both ρ and m are positive constants. Now choose a positive integer l such that $lm \geq 2\delta$. For any $\varepsilon \in (0, H)$, define $\eta = (1-p)\varepsilon/(1+p)(2p+\frac{5}{2})^l$. We will prove that for any $\bar{t} \geq t_0, \phi \in C([\bar{t}-\rho, \bar{t}], (-\eta, \eta))$ implies

$$|x(t;\bar{t},\phi)| < \varepsilon, \quad t \geq \bar{t}, \tag{2.1}$$

where $x(t;\bar{t},\phi)$ denotes the solution of Equation (1.1) satisfying the initial condition $x(s;\bar{t},\phi) = \phi(s)$ for $s \in [\bar{t}-\rho, \bar{t}]$. For the convenience, in the sequel we denote $x(t) = x(t;\bar{t},\phi)$ and always set

$$z(t) \;=\; x(t) \;-\; f(t, x(t-\tau)). \tag{2.2}$$

Denote $\rho_1 = (2p+\frac{5}{2})\eta, \rho_i = (2p+\frac{5}{2})\rho_{i-1},\ i = 2, \cdots, l$. Then $\rho_i = (2p+\frac{5}{2})^i\eta,\ \ i = 1, \cdots, l$. Clearly, $\eta < \rho_1 < \rho_2 < \cdots < \rho_l < \varepsilon$. We will first prove that

$$|x(t)| < \rho_i, \quad t \in [\bar{t} + (i-1)m, \bar{t} + im], \ \ i = 1, 2, \cdots, l. \tag{2.3}$$

In fact, for $t \in [\bar{t}, \bar{t} + m]$, we have by (1.1)

$$\begin{aligned}
|x(t)| &= \Big|f(t, x(t-\tau)) + x(\bar{t}) - f(\bar{t}, x(\bar{t}-\tau)) - \int_{\bar{t}}^{t} g(s, x(s-\delta))ds\Big| \\
&\leq p|x(t-\tau)| + |x(\bar{t})| + p|x(t-\tau)| + \int_{\bar{t}}^{t} Q(s)|x(s-\delta)|ds \\
&\leq \eta\Big[2p + 1 + \int_{\bar{t}}^{t} Q(s)ds\Big] \\
&< \eta[2p + 1 + \frac{3}{2}] = \rho_1,
\end{aligned}$$

which shows that (2.3) holds when $i = 1$, and so

$$|x(t)| < \rho_1, \quad \text{for } \ t \in [\bar{t}-\rho, \bar{t}+m],$$

which follows similarly by repeating the above arguments

$$|x(t)| < [2p + \frac{5}{2}]\rho_1 = \rho_2, \ \ \text{for } \ t \in [\bar{t}+m, \bar{t}+2m].$$

Thus, by the induction we may prove that (2.3) holds. Next, we return to the proof of (2.1). By way of contradiction, we assume that (2.1) is not true. Then by (2.3) there must be some $T > \bar{t} + lm$ such that $|x(T)| = \varepsilon$ and $|x(t)| < \varepsilon$ for $\bar{t} \leq t < T$. Without loss of generality, we may suppose $x(T) = \varepsilon$. Thus, we have

$$z(T) = x(T) - f(T, x(T-\tau)) \geq (1-p)\varepsilon > 0. \tag{2.4}$$

Again since

$$z(\bar{t}+lm) = x(\bar{t}+lm) - f(\bar{t}+lm, x(\bar{t}+lm-\tau)) < \rho_l(1+p) = (1-p)\varepsilon \leq z(T),$$

it follows by (2.4) that there exists $\xi \in (\bar{t}+lm, T]$ such that $z(\xi) = \max\limits_{\bar{t}+lm \leq t \leq T} z(t)$ and $z(\xi) > z(t)$ for $\bar{t}+lm \leq t < \xi$. By (1.9), we see that $2p < 1$, and so

$$x(\xi) = z(\xi) + f(\xi, x(\xi-\tau)) \geq z(T) - p\varepsilon \geq (1-2p)\varepsilon > 0.$$

Next we prove $x(\xi-\delta)\le 0$. Otherwise, $x(\xi-\delta)>0$. Thus, there is a left neighbor of $\xi-\delta$ which is denoted by $(\xi-\delta-h,\xi-\delta)$ for some $h>0$, such that $x(t)>0$ for $t\in(\xi-\delta-h,\xi-\delta)$, which shows that $x(t-\delta)>0$ for $t\in(\xi-h,\xi)$, and therefore by (1.1) we know that $z(t)$ is not increasing on $(\xi-h,\xi)$. This contradicts the definition of ξ and so $x(\xi-\delta)\le 0$. Hence there exists $T_0\in[\xi-\delta,\xi)$ such that $x(T_0)=0$. From (1.1) and (1.8), we have

$$\dot z(t)\le Q(t)\varepsilon,\qquad \bar t\le t\le T. \tag{2.5}$$

Since $t\in[T_0,\xi]$ implies $t-\delta\le T_0$, we have by integrating (2.5) from $t-\delta$ to T_0,

$$z(T_0)-z(t-\delta)\le\varepsilon\int_{t-\delta}^{T_0}Q(s)\,ds.$$

That is,

$$\begin{aligned}-x(t-\delta)&\le -f(t-\delta,x(t-\delta-\tau))+f(T_0,x(T_0-\tau))+\varepsilon\int_{t-\delta}^{T_0}Q(s)ds\\&\le\varepsilon\Big[2p+\int_{t-\delta}^{T_0}Q(s)ds\Big],\quad T_0\le t\le\xi\end{aligned}$$

and so

$$-g(t,x(t-\delta))\le\varepsilon Q(t)\Big[2p+\int_{t-\delta}^{T_0}Q(s)ds\Big],\quad T_0\le t\le\xi.$$

Substituting this into (1.1), we obtain

$$\dot z(t)\le\varepsilon Q(t)\Big[2p+\int_{t-\delta}^{T_0}Q(s)ds\Big],\quad T_0\le t\le\xi. \tag{2.6}$$

Since $\xi-T_0\le\delta$, we have by (1.9)

$$2p(2-p)+\int_{T_0}^{\xi}Q(s)ds\le\frac32. \tag{2.7}$$

The proof will be complete if we can conclude that

$$(z(T)\le)z(\xi)<(1-p)\varepsilon, \tag{2.8}$$

which is due to the contradiction to (2.4). There are two possible cases.

Case 1. $2p+\int_{T_0}^{\xi}Q(s)ds\le 1$.

In this case, we have by integrating (2.6) from T_0 to ξ,

$$\begin{aligned}z(\xi)&\le z(T_0)+\varepsilon\int_{T_0}^{\xi}Q(t)\Big[2p+\int_{t-\delta}^{T_0}Q(s)ds\Big]dt\\&=-f(T_0,x(T_0-\tau))+\varepsilon\int_{T_0}^{\xi}Q(t)\Big[2p+\int_{t-\delta}^{t}Q(s)ds-\int_{T_0}^{t}Q(s)ds\Big]dt\\&\le p\varepsilon+\varepsilon\int_{T_0}^{\xi}Q(t)\Big[\frac32-2p(1-p)-\int_{T_0}^{t}Q(s)ds\Big]dt\\&=\varepsilon\Big[p+(\frac32-2p(1-p))\int_{T_0}^{\xi}Q(t)dt-\frac12\Big(\int_{T_0}^{\xi}Q(t)dt\Big)^2\Big].\end{aligned}$$

Noting that the function $p+(\frac32-2p(1-p))x-\frac12x^2$ is increasing on $x\in(0,1-2p)$, we have

$$z(\xi)\le\varepsilon[p+(\frac32-2p(1-p))(1-2p)-\frac12(1-2p)^2]<(1-p)\varepsilon.$$

Case 2. $2p+\int_{T_0}^{\xi}Q(s)ds>1$.

Since $2p < 1$, there is $T_1 \in (T_0, \xi)$ such that $2p + \int_{T_1}^{\xi} Q(s)ds = 1$. Integrating first (2.5) from T_0 to T_1 and then (2.6) from T_1 to ξ, we have

$$\begin{aligned}
z(\xi) &\le z(T_0) + \varepsilon \int_{T_0}^{T_1} Q(t)dt + \varepsilon \int_{T_1}^{\xi} Q(t)\Big[2p + \int_{t-\delta}^{T_0} Q(s)ds\Big]dt \\
&< p\varepsilon + \varepsilon \int_{T_1}^{\xi} Q(t)dt \int_{T_0}^{T_1} Q(s)ds 2p\varepsilon \int_{T_0}^{T_1} Q(s)ds + \varepsilon \int_{T_1}^{\xi} Q(t)\Big[2p + \int_{t-\delta}^{T_0} Q(s)ds\Big]dt \\
&= \varepsilon\Big[p + 2p(1-2p) + 2p \int_{T_0}^{T_1} Q(s)ds + \int_{T_1}^{\xi} Q(t) \int_{t-\delta}^{T_1} Q(s)dsdt\Big] \\
&\le \varepsilon\Big[(3-4p)p + 2p\Big(-1 + 2p + \int_{T_0}^{\xi} Q(s)ds\Big) \\
&\quad + \int_{T_1}^{\xi} Q(t)\Big(\frac{3}{2} - 2p(2-p) - \int_{T_1}^{t} Q(s)ds\Big)dt\Big] \\
&\le \varepsilon\Big[p + 2p\Big(\frac{3}{2} - 2p(2-p)\Big) + \Big(\frac{3}{2} - 2p(2-p)\Big)(1-2p) - \frac{1}{2}(1-2p)^2\Big] \\
&= (1-p)\varepsilon.
\end{aligned}$$

And therefore, the proof is complete.

Proof of Theorem 1.2. In view of Theorem 1.1, the zero solution of Equation (1.1) is uniformly stable. Therefore, for any $\bar{t} \ge t_0$, there exists an $\eta > 0$ such that $\phi \in C([\bar{t} - \rho, \bar{t}], (-\eta, \eta))$ implies

$$|x(t)| = |x(t; \bar{t}, \phi)| < \frac{1}{2}H, \quad t \ge \bar{t}.$$

It suffices to prove that

$$\lim_{t\to\infty} x(t) = 0.$$

Let $z(t)$ be defined by (2.2). Now we consider the following two possible cases.

Case 1. $x(t)$ itself is nonoscillatory. We may assume that $x(t)$ is eventually positive. The case when $x(t)$ is eventually negative is similar and will be omitted. Choose $T \ge t_0 + \rho$ such that $x(t-\rho) > 0$ for $t \ge T$. Then

$$\dot{z}(t) = -g(t, x(t-\delta)) \le 0, \quad \text{for} \ \ t \ge T,$$

which means that $z(t)$ is nonincreasing on $t \in [T, \infty)$ and so the limit

$$\beta = \lim_{t\to\infty} z(t)$$

exists and is finite. Set

$$x_1 = \limsup_{t\to\infty} x(t) \ \text{ and } \ x_2 = \liminf_{t\to\infty} x(t).$$

Then it is easy to see by (1.6) that $x_2 = 0$. Using (1.8), we have

$$x(t) \ge z(t) - px(t-\tau), \quad \text{for} \ \ t \ge T.$$

It follows that

$$0 = \liminf_{t\to\infty} x(t) \ge \beta - p \limsup_{t\to\infty} x(t-\tau) = \beta - px_1$$

and so $\beta \le px_1$. Similarly, we have

$$x_1 = \limsup_{t\to\infty} x(t) \le \beta + p \limsup_{t\to\infty} x(t-\tau) = \beta + px_1,$$

which yields

$$x_1 \le \frac{\beta}{1-p} \le \frac{px_1}{1-p}.$$

By virtue of (1.10), we have $p < \frac{1}{2}$ and so $\frac{p}{1-p} < 1$. Therefore, we have $x_1 = 0$. This proves Case 1.

Case 2. $x(t)$ itself is oscillatory in the sense that it has arbitrarily large zeros. For this case, set

$$\mu = \limsup_{t\to\infty} |x(t)|.$$

Then $0 \le \mu \le \frac{H}{2}$. The proof will be finished when we prove $\mu = 0$. Suppose $\mu > 0$. Then for any $\epsilon \in (0, (1-2|p|)\mu)$, there is $t_1 \ge t_0 + \rho$ such that $|x(t)| < \mu + \epsilon,\ \ t \ge t_1 - \rho$. Let $z(t)$ be defined by (1.11). Then

$$|z(t)| \ge |x(t)| - |f(t, x(t-\tau))| \ge |x(t)| - p(\mu+\epsilon),\ \ t \ge t_1,$$

which yields

$$M = \limsup_{t\to\infty} |z(t)| \ge (1-p)\mu \tag{2.9}$$

since $\epsilon > 0$ is arbitrary. Note that $\dot{z}(t)$ is oscillatory, there is an increasing sequence $\{u_n\}$ such that $u_n \ge t_1 + \tau + 2\delta, u_n \to \infty, |z(u_n)| \to M$ as $n \to \infty, |z(u_n)| > (1-p)(\mu-\epsilon)$ and u_n is left local maximum of $|z(t)|$. We may assume $z(u_n) > 0$. The case when $z(u_n) < 0$ is similar and the proof will be omitted. Thus

$$x(u_n) = z(u_n) + f(u_n, x(u_n - \tau)) > (1-p)(\mu-\epsilon) - p(\mu+\epsilon) = (1-2p)\mu - \epsilon > 0.$$

It is also easy to see that $x(u_n - \delta) \le 0$ in view of the way of the definition of u_n. Therefore, there exists $\xi_n \in [u_n - \delta, u_n)$ such that $x(\xi_n) = 0$. Set

$$\theta = \max\Big\{(1-2p)\Big(\lambda - \frac{1}{2}\Big) + 4p^2(1-p), \frac{1}{2}(1-2p)^2 + p, \lambda - \frac{1}{2} - p\Big\}.$$

We will prove

$$z(u_n) \le \theta(\mu+\epsilon). \tag{2.10}$$

From (1.1) and the fact $|x(t)| < \mu + \epsilon,\ t \ge t_1 - \rho$, we have

$$\dot{z}(t) \le Q(t)(\mu+\epsilon),\ \ t \ge t_1. \tag{2.11}$$

Since $t \in [\xi_n, u_n]$ implies $t - \delta \in [t_1, \xi_n]$, integrate (2.11) from $t-\delta$ to ξ_n to get

$$z(\xi_n) - z(t-\delta) \le (\mu+\epsilon)\int_{t-\delta}^{\xi_n} Q(s)ds.$$

Furthermore, we have

$$-x(t-\delta) \le \Big[2p + \int_{t-\delta}^{\xi_n} Q(s)ds\Big](\mu+\epsilon), \quad t \in [\xi_n, u_n].$$

This yields by (1.1) and (1.8),

$$\dot{z}(t) \le Q(t)\Big[2p + \int_{t-\delta}^{\xi_n} Q(s)ds\Big](\mu+\epsilon), \quad t \in [\xi_n, u_n]. \tag{2.12}$$

There are two possibilities as follows.

Subcase 1. $2p + \int_{\xi_n}^{u_n} Q(s)ds \le 1$.

Then integrating (2.12) from ξ_n to u_n, we get

$$\begin{aligned}
z(u_n) &\le z(\xi_n) + (\mu+\epsilon)\int_{\xi_n}^{u_n} Q(t)\Big[2p + \int_{t-\delta}^{\xi_n} Q(s)ds\Big]dt \\
&= -f(\xi_n, x(\xi_n-\tau)) + (\mu+\epsilon)\int_{\xi_n}^{u_n} Q(t)\Big[2p + \int_{t-\delta}^{t} Q(s)ds - \int_{\xi_n}^{t} Q(s)ds\Big]dt \\
&\le (\mu+\epsilon)\Big[p + \int_{\xi_n}^{u_n} Q(t)\Big(\lambda - 2p(1-p) - \int_{\xi_n}^{t} Q(s)ds\Big)dt\Big] \\
&= (\mu+\epsilon)\Big[p + (\lambda - 2p(1-p))\int_{\xi_n}^{u_n} Q(t)dt - \frac{1}{2}\Big(\int_{\xi_n}^{u_n} Q(t)dt\Big)^2\Big] \\
&\le (\mu+\epsilon)\Big[p + \max\{\lambda - 2p(1-p), 1-2p\}\int_{\xi_n}^{u_n} Q(t)dt - \frac{1}{2}\Big(\int_{\xi_n}^{u_n} Q(t)dt\Big)^2\Big].
\end{aligned}$$

Noting that the function in $w:\ \ p + \max\{\lambda - 2p(1-p), 1-2p\}w - \frac{1}{2}w^2$ is increasing on $w \in [0, 1-2p]$, we have

$$z(u_n) \le (\mu+\epsilon)\Big[p + \max\{\lambda - 2p(1-p), 1-2p\}(1-2p) - \frac{1}{2}(1-2p)^2\Big] \le \theta(\mu+\epsilon).$$

Subcase 2. $2p + \int_{\xi_n}^{u_n} Q(s)ds > 1$.

Since $2p < 1$, there is $T_n \in (\xi_n, u_n)$ such that

$$2p + \int_{T_n}^{u_n} Q(s)ds = 1.$$

Integrating first (2.11) from ξ_n to T_n, and then (2.12) from T_n to u_n, we have

$$\begin{aligned}
z(u_n) &\le z(\xi_n) + (\mu+\epsilon)\int_{\xi_n}^{T_n} Q(s)ds + (\mu+\epsilon)\int_{T_n}^{u_n} Q(t)\Big[2p + \int_{t-\delta}^{\xi_n} Q(s)ds\Big]dt \\
&\le (\mu+\epsilon)\Big[p + 2p\int_{\xi_n}^{T_n} Q(s)ds + \int_{T_n}^{u_n} Q(t)dt\int_{\xi_n}^{T_n} Q(s)ds \\
&\quad + \int_{T_n}^{u_n} Q(t)\Big(2p + \int_{t-\delta}^{\xi_n} Q(s)ds\Big)dt\Big] \\
&= (\mu+\epsilon)\Big[(3-4p)p + 2p\int_{\xi_n}^{T_n} Q(s)ds + \int_{T_n}^{u_n} Q(t)\int_{t-\delta}^{T_n} Q(s)dsdt\Big] \\
&\le (\mu+\epsilon)\Big[(3-4p)p + 2p\Big(\int_{\xi_n}^{u_n} Q(s)ds - 1 + 2p\Big) \\
&\quad + \int_{T_n}^{u_n} Q(t)\Big(\lambda - 2p(2-p) - \int_{T_n}^{t} Q(s)ds\Big)dt\Big] \\
&\le (\mu+\epsilon)[(3-4p)p + 2p(\lambda - 1 - 2p(1-p)) \\
&\quad + (\lambda - 2p(2-p))\int_{T_n}^{u_n} Q(t)dt - \frac{1}{2}\Big(\int_{T_n}^{u_n} Q(t)dt\Big)^2] \\
&= (\mu+\epsilon)[(3-4p)p + 2p(\lambda - 1 - 2p(1-p)) + (\lambda - 2p(2-p))(1-2p) - \frac{1}{2}(1-2p)^2] \\
&= (\mu+\epsilon)[(1-2p)(\lambda - \frac{1}{2}) + 2p(\lambda - 1)] \\
&= (\mu+\epsilon)(\lambda - \frac{1}{2} - p) \le \theta(\mu+\epsilon).
\end{aligned}$$

This has proved (2.10). Letting $n \to \infty$ in (2.10) and noting the arbitrariness of ϵ, we have

$M \leq \theta\mu$, which combining with (2.9) yields $\theta \geq 1-p$, that is,

$$\max\Big\{(1-2p)\Big(\lambda-\frac{1}{2}\Big)+4p^2(1-p),\ \frac{1}{2}(1-2p)^2+p,\ \lambda-\frac{1}{2}-p\Big\} \geq 1-p,$$

which is impossible since $\lambda < \frac{3}{2}$ and $2p < 1$. Therefore, $\mu = 0$, and so the proof is complete.

Finally, let us study the asymptotic stability of the positive equilibrium K of Equation (1.12). For a given initial function $\phi \in C^1([-\tau, 0], [0, \infty))$ with $\phi(0) > 0$, we may easily prove that Equation (1.12) has a unique solution $x(t)$ defined on $[-\tau, \infty)$ which satisfies $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$. By the change of variable $y(t) = \ln \frac{x(t)}{K}$, Equation (1.12) becomes

$$\frac{d}{dt}[y(t) + r\rho(e^{y(t-\tau)} - 1)] + r(e^{y(t-\tau)} - 1) = 0. \tag{2.13}$$

By virtue of Theorem 1.2, one can prove that if

$$2r|\rho|(2 - r|\rho|) + r\tau < \frac{3}{2},$$

then the zero solution of Equation (2.13) is asymptotically stable, i.e., the positive equilibruim K of Equation (1.12) is asymptotically stable.

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