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NEW SYMPLECTIC MAPS: INTEGRABILITY AND LAX REPRESENTATION***

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Abstract

New family of integrable symplectic maps are reduced from the Toda hierarchy via constraint for a higher flow of the hierarchy in terms of square eigenfunctions. Their integrability and Lax representation are deduced systematically from the discrete zero-curvature representation of the Toda hierarchy. Also a discrete zero-curvature representation for the Toda hierarchy with sources is presented.

Keywords Integrable symplectic map, Discrete zero-curvature representation, Lax representation, Higher-order constraint

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§1. Introduction

In recent years some methods to obtain integrable symplectic maps (ISM) have been developed. An attempt to introduce a general procedure to construct ISMs from the stationary flows of discrete integrable systems (DIS) (nonlinear differential-difference equations) was made in [1]. Discrete versions of some classical integrable systems were investigated based on factorization of matrix polynomials[2]. In the last few years an approach has been developed to reduce finite-dimensional integrable Hamiltonian systems from soliton equations via the constraints relating potential and eigenfunctions (see, for example, [3–9]). Obviously, this approach can be applied to get ISMs from DISs. In this approach, we suppose that the hierarchy of DISs is associated with a discrete spectral problem and possesses hamiltonian structure. Then we consider the system consisting of N copies of the spectral problem and of (higher-order) constraint relating the variational derivatives of Hamiltonian functions and square eigenfunctions. This system is invariant under all flows in the hierarchy, so is expected to give rise to an ISM.

The main problem in this approach is how to construct integrals of motion and Lax representation for the symplectic maps, and to show their integrability. In [10], explicit constraint was considered and integrability of the map was shown by following the Moser's method. It seems to be difficult to apply the method in [10] to higher-order constraint. In this paper, we derive new ISMs from Toda hierarchy under higher-order constraints, present

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a general method for constructing integrals of motion and Lax representation, and showing integrability of these maps by using the zero-curvature representation of Toda hierarchy. The advantage of our method is to deduce properties of ISM directly from that of DIS. This method is also different from that in [9].

§2. Discrete Zero-Curvature Description of the Toda Hierarchy

We now briefly describe a discrete zero-curvature representation for the Toda hierarchy as presented in [11]. Consider the following discrete isospectral problem,

$$E\psi = U\psi, \qquad U = U(u,\lambda) = \begin{pmatrix} 0 & 1 \\ -v & \lambda - w \end{pmatrix}, \qquad \psi = (\psi_1, \psi_2)^t, \tag{2.1}$$

where $u = (w,v)^t, w = w(n,t)$ and $v = v(n,t)$ depend on integers $n \in \mathbf{Z}$ and $t \in \mathbf{R}$, λ is the spectral parameter, shift operator E and difference operator D are defined as

$$(Ef)(n) = f(n+1), \qquad (Df)(n) = (E-1)f(n), \qquad f^{(k)} = E^{(k)}f, \qquad n \in \mathbf{Z}. \tag{2.2}$$

We proceed first to solve the stationary discrete zero-curvature equation[11]

$$(E\Gamma)U - U\Gamma = 0. \tag{2.3}$$

The substitution of

$$\Gamma = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{i=0}^{\infty} \Gamma_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{-i} \tag{2.4}$$

into (2.3) gives

$$c_i = -vb_i^{(1)}, \qquad b_{i+1}^{(1)} = wb_i^{(1)} - (a_i^{(1)} + a_i), \tag{2.5a}$$

$$a_{i+1}^{(1)} - a_{i+1} = w(a_i^{(1)} - a_i) + vb_i - v^{(1)}b_i^{(2)}. \tag{2.5b}$$

The first coefficients are given as follows:

$$a_0 = \frac{1}{2}, \qquad b_0 = 0, \qquad b_1 = -1. \tag{2.6a}$$

$$a_1 = 0, \qquad a_2 = v, \qquad a_3 = v(w + w^{(-1)}), \tag{2.6b}$$

$$b_2 = -w^{(-1)}, \qquad b_3 = -(v^{(-1)} + v + w^{(-1)^2}), \cdots . \tag{2.6c}$$

Let us denote

$$V_m = (\Gamma\lambda^m)_+ + \triangle_m \equiv \sum_{i=0}^{m} \Gamma_i \lambda^{m-i} + \triangle_m, \qquad \triangle_m = \begin{pmatrix} b_{m+1} & 0 \\ 0 & 0 \end{pmatrix}, \tag{2.7}$$

in the auxiliary linear problem

$$\psi_{t_m} = V_m \psi, \qquad m = 1, 2, \cdots . \tag{2.8}$$

The compatibility condition of (2.1) and (2.8) gives rise to a discrete hierarchy of zero-curvature equations (assuming $\lambda_{t_m} = 0$)

$$U_{t_m} = (EV_m)U - UV_m, \qquad m = 1, 2, \cdots . \tag{2.9}$$

It describes the Toda hierarchy

$$w_{t_m} = -a_{m+1}^{(1)} + a_{m+1}, \qquad v_{t_m} = v(b_{m+1}^{(1)} - b_{m+1}), \tag{2.10}$$

which can be written as the following Hamiltonian equation[11]

$$u_{t_m} = \begin{pmatrix} w \\ v \end{pmatrix}_{t_m} = JK_{m+1} = J\frac{\delta H_{m+1}}{\delta u}, \qquad m = 1, 2, \cdots, \tag{2.11}$$

where $\frac{\delta}{\delta u} = (\frac{\delta}{\delta w}, \frac{\delta}{\delta v})^t$ stands for the discrete variational derivative defined as

$$\frac{\delta f}{\delta v} = \sum_{k \in \mathbf{Z}} E^{(-k)} \frac{\partial f}{\partial v^{(k)}}.$$

and

$$K_m = \frac{\delta H_m}{\delta u} = \begin{pmatrix} -b_m^{(1)} \\ \frac{a_m}{v} \end{pmatrix}, \quad J = \begin{pmatrix} 0 & (1-E)v \\ v(E^{(-1)} - 1) & 0 \end{pmatrix}, \quad H_m = -\frac{b_{m+1}}{m}. \tag{2.12}$$

It is known[12] that equations (2.11) have the bi-hamiltonian formulation

$$GK_{i-1} = JK_i, \qquad i = 1, 2, \cdots, \tag{2.13}$$

$$G = \begin{pmatrix} vE^{(-1)} - v^{(1)}E & w(1-E)v \\ v(E^{(-1)} - 1)w & v(E^{(-1)} - E)v \end{pmatrix}.$$

Let us define V in terms of Γ by $\Gamma = VU$. Then it is deduced from (2.3) that

$$D\Gamma = [U, V], \qquad \Gamma^{(1)} = UV, \tag{2.14}$$

$$D(a^2 + bc) = \frac{1}{2} D(\mathrm{Tr}\Gamma^2) = \frac{1}{2}(\mathrm{Tr}(UV)^2 - \mathrm{Tr}(VU)^2) = 0, \tag{2.15}$$

where Tr means trace of a matrix. In the same way given by [13], we get from (2.8)

$$\Gamma_{t_m} = [V_m, \Gamma], \tag{2.16}$$

which yields

$$2\frac{d}{dt_m}(a^2 + bc) = \frac{d}{dt_m}\mathrm{Tr}\Gamma^2 = \frac{d}{dt_m}\mathrm{Tr}[V_m, \Gamma^2] = 0. \tag{2.17}$$

The adjoint equations of (2.1) and (2.8) read, respectively

$$E^{(-1)}\phi = \phi U, \qquad \phi = (\phi_1, \phi_2), \tag{2.18}$$

$$E^{(-1)}\phi_{t_m} = -(E^{(-1)}\phi)V_m. \tag{2.19}$$

It can be found by a direct calculation that

$$\frac{\delta \lambda}{\delta u} = \begin{pmatrix} \frac{\delta \lambda}{\delta w} \\ \frac{\delta \lambda}{\delta v} \end{pmatrix} = -\begin{pmatrix} \psi_2 \phi_2 \\ \psi_1 \phi_2 \end{pmatrix}, \qquad G\frac{\delta \lambda}{\delta u} = \lambda J \frac{\delta \lambda}{\delta u}. \tag{2.20}$$

§3. New Integrable Symplectic Maps

We consider for N distinct λ_j, j=1, … ,N, the following system of equations consisting of replicas of (2.1) and (2.18) as well as of the constraint for variational derivatives for conserved quantities H_{k_0} (for a fixed k_0) and eigenvalue λ_j

$$E\psi_{1j} = \psi_{2j}, \qquad E\psi_{2j} = -v\psi_{1j} + (\lambda_j - w)\psi_{2j}, \qquad j = 1, \cdots, N, \tag{3.1a}$$

$$E^{(-1)}\phi_{1j} = -v\phi_{2j}, \qquad E^{(-1)}\phi_{2j} = \phi_{1j} + (\lambda_j - w)\phi_{2j}, \qquad j = 1, \cdots, N, \tag{3.1b}$$

$$\frac{\delta H_{k_0}}{\delta u} - \sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u} = 0. \tag{3.1c}$$

We shall denote the inner product in $\mathbf{R}^N$ by $\langle \cdot\ ,\ \cdot \rangle$ and shall use the following notations

$$\Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^t, \qquad \Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^t, \qquad i = 1, 2, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N).$$

By substituting (2.20) into (3.1), we get

$$E\Psi_1 = \Psi_2, \qquad E\Psi_2 = -v\Psi_1 + (\Lambda - w)\Psi_2, \tag{3.2a}$$

$$E^{(-1)}\Phi_1 = -v\Phi_2, \qquad E^{(-1)}\Phi_2 = \Phi_1 + (\Lambda - w)\Phi_2, \tag{3.2b}$$

$$\frac{\delta H_{k_0}}{\delta w} = -\langle \Psi_2, \Phi_2 \rangle, \qquad \frac{\delta H_{k_0}}{\delta v} = -\langle \Psi_1, \Phi_2 \rangle, \tag{3.2c}$$

which are discrete Euler-Lagrange equations[1]:

$$\frac{\delta \mathcal{L}}{\delta \Phi_i} = 0, \qquad \frac{\delta \mathcal{L}}{\delta \Psi_i} = 0, \qquad i = 1, 2, \qquad \frac{\delta \mathcal{L}}{\delta w} = 0, \qquad \frac{\delta \mathcal{L}}{\delta v} = 0, \tag{3.3a}$$

$$\begin{aligned} \mathcal{L} = {} & \langle \Psi_1^{(1)}, \Phi_1 \rangle + \langle \Psi_2^{(1)}, \Phi_2 \rangle - \langle \Psi_2, \Phi_1 \rangle + v\langle \Psi_1, \Phi_2 \rangle \\ & - \langle \Lambda\Psi_2, \Phi_2 \rangle + w\langle \Psi_2, \Phi_2 \rangle + H_{k_0}. \end{aligned} \tag{3.3b}$$

As argued in [4-8], the system (3.2) is invariant with respect to the action of all flows of the Toda hierarchy. So (3.2) is expected to give an integrable symplectic map (ISM). Following the procedure in [1], we can introduce canonical coordinates (q, p) for (3.2):

$$q = (q_1, \cdots, q_{N_1})^t, \qquad p = (p_1, \cdots, p_{N_1})^t, \tag{3.4a}$$

and define Poisson bracket for any pair of functions f, g and any (q, p) as follows:

$$\{f, g\}_{q,p} = \sum_{j=1}^{N_1} \Big(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \Big), \tag{3.4b}$$

such that (3.2) can be cast in canonical form of a symplectic map:

$$Eq_i = f_i(q(n), p(n)), \qquad Ep_i = g_i(q(n), p(n)), \qquad i = 1, \cdots, N_1, \tag{3.5}$$

where f_i, g_i satisfy

$$\{f_i, f_j\} = \{g_i, g_j\} = 0, \qquad \{f_i, g_j\} = \delta_{i,j}. \tag{3.6}$$

Now we present the first two symplectic maps obtained from (3.2) as examples.

(1) For $k_0 = 2$, (3.2c) reads

$$K_2 = \begin{pmatrix} -b_2^{(1)} \\ \frac{a_2}{v} \end{pmatrix} = \begin{pmatrix} w \\ 1 \end{pmatrix} = \begin{pmatrix} -\langle \Psi_2, \Phi_2 \rangle \\ -\langle \Psi_1, \Phi_2 \rangle \end{pmatrix}, \tag{3.7a}$$

which together with (3.2a,b) leads to

$$v = \langle \Psi_1, \widetilde{\Phi}_1 \rangle, \qquad w = -\langle \Psi_2, \Phi_2 \rangle = \frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle} \langle \Psi_2, \widetilde{\Phi}_1 \rangle. \tag{3.7b}$$

Throughout this paper, we denote $\widetilde{\Phi}_i = \Phi_i^{(-1)}, i = 1, 2$. By substitution of (3.7b), (3.2a,b) can be rewritten as

$$E\Psi_1 = \Psi_2, \qquad E\Psi_2 = -\langle \Psi_1, \widetilde{\Phi}_1 \rangle \Psi_1 + \Big(\Lambda - \frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle} \langle \Psi_2, \widetilde{\Phi}_1 \rangle \Big) \Psi_2, \tag{3.8a}$$

$$E\widetilde{\Phi}_1 = \frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle} \Big(\Lambda - \frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle} \langle \Psi_2, \widetilde{\Phi}_1 \rangle \Big) \widetilde{\Phi}_1 + \widetilde{\Phi}_2, \quad E\widetilde{\Phi}_2 = -\frac{1}{\langle \Psi_1, \widetilde{\Phi}_1 \rangle} \widetilde{\Phi}_1. \tag{3.8b}$$

For (3.8) the canonical coordinates (q, p) are defined as follows:

$$q = (q_1, \cdots, q_{2N})^t \equiv (\psi_{11}, \cdots, \psi_{1N}, \psi_{21}, \cdots, \psi_{2N})^t, \qquad N_1 = 2N,$$
$$p = (p_1, \cdots, p_{2N})^t \equiv (\tilde{\phi}_{11}, \cdots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \cdots, \tilde{\phi}_{2N})^t. \tag{3.9}$$

It is easy to verify that (3.6) for (3.8) holds, so (3.8) defines a symplectic map. From (3.8b), we have $\langle\Psi_1, \Phi_2\rangle = \langle\Psi_1, E\widetilde{\Phi}_2\rangle = -1$, so $\langle\Psi_1, \Phi_2\rangle = -1$ in (3.7a) is not really a constraint.

(2) For $k_0 = 3$, it is found from (3.2c) that

$$K_3 = \begin{pmatrix} -b_3^{(1)} \\ \frac{a_3}{v} \end{pmatrix} = \begin{pmatrix} v + v^{(1)} + w^2 \\ w + w^{(-1)} \end{pmatrix} = \begin{pmatrix} -\langle\Psi_2, \Phi_2\rangle \\ -\langle\Psi_1, \Phi_2\rangle \end{pmatrix}. \tag{3.10}$$

Then the system (3.2) with (3.2c) given by (3.10) can be rewritten in the canonical form:

$$E\Psi_1 = \Psi_2, \quad E\Psi_2 = -v\Psi_1 + \Big(\Lambda + \widetilde{w} - \frac{1}{v}\langle\Psi_1, \widetilde{\Phi}_1\rangle\Big)\Psi_2, \tag{3.11a}$$

$$Ev = -v - \widetilde{w}^2 + \frac{2}{v}\widetilde{w}\langle\Psi_1, \widetilde{\Phi}_1\rangle - \frac{1}{v^2}\langle\Psi_1, \widetilde{\Phi}_1\rangle^2 + \frac{1}{v}\langle\Psi_2, \widetilde{\Phi}_1\rangle, \tag{3.11b}$$

$$E\widetilde{\Phi}_1 = \frac{1}{v}\Big(\Lambda + \widetilde{w} - \frac{1}{v}\langle\Psi_1, \widetilde{\Phi}_1\rangle\Big)\widetilde{\Phi}_1 + \widetilde{\Phi}_2, \quad E\widetilde{\Phi}_2 = -\frac{1}{v}\widetilde{\Phi}_1, \tag{3.11c}$$

$$E\widetilde{w} = -\widetilde{w} + \frac{1}{v}\langle\Psi_1, \widetilde{\Phi}_1\rangle, \tag{3.11d}$$

where $\widetilde{w} = w^{(-1)}$. For (3.11) the canonical coordinates (q, p) are defined as follows:

$$q = (q_1, \cdots, q_{2N+1})^t \equiv (\psi_{11}, \cdots, \psi_{1N}, \psi_{21}, \cdots, \psi_{2N}, v)^t, \qquad N_1 = 2N + 1,$$
$$p = (p_1, \cdots, p_{2N+1})^t \equiv (\tilde{\phi}_{11}, \cdots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \cdots, \tilde{\phi}_{2N}, \widetilde{w})^t. \tag{3.12}$$

It is easy to verify that (3.6) for (3.11) holds, so (3.11) defines a symplectic map.

We now use (3.11) ($k_0 = 3$) as an example to illustrate how the integrability of the symplectic map can be deduced from that of the Toda hierarchy (2.11).

Lemma 3.1. *Under* (3.11), *let us define*

$$\tilde{a}_i = a_i, \qquad \tilde{b}_i = b_i, \qquad \tilde{c}_i = c_i, \qquad i = 0, 1, 2, \tag{3.13a}$$

$$\tilde{a}_i = \frac{1}{2}(\langle\Lambda^{i-3}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{i-3}\Psi_2, \widetilde{\Phi}_2\rangle), \qquad i = 3, 4, \cdots, \tag{3.13b}$$

$$\tilde{b}_i = \langle\Lambda^{i-3}\Psi_1, \widetilde{\Phi}_2\rangle, \qquad \tilde{c}_i = \langle\Lambda^{i-3}\Psi_2, \widetilde{\Phi}_1\rangle, \qquad i = 3, 4, \cdots, \tag{3.13c}$$

$$\widetilde{\Gamma} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & -\tilde{a} \end{pmatrix} = \sum_{i=0}^{\infty} \widetilde{\Gamma}_i \lambda^{-i} = \sum_{i=0}^{\infty} \begin{pmatrix} \tilde{a}_i & \tilde{b}_i \\ \tilde{c}_i & -\tilde{a}_i \end{pmatrix} \lambda^{-i}.$$

Then under (3.11) $\widetilde{\Gamma}$ *satisfies* (2.3), *and*

$$D(\tilde{a}^2 + \tilde{b}\tilde{c}) = 0. \tag{3.14}$$

Proof. Notice that the kernal of J is $\{K_0, K_1\}$, i.e., $J(\alpha K_0 + \beta K_1) = 0$. By using (2.13) and (2.20), we find from (3.10)

$$K_4 = J^{-1}GK_3 = J^{-1}G\begin{pmatrix} -\langle\Psi_2, \Phi_2\rangle \\ -\langle\Psi_1, \Phi_2\rangle \end{pmatrix} = \begin{pmatrix} -\langle\Lambda\Psi_2, \Phi_2\rangle \\ -\langle\Lambda\Psi_1, \Phi_2\rangle \end{pmatrix} + h_3K_1 + \alpha_4K_0,$$

$$K_5 = J^{-1}GK_4 = \begin{pmatrix} -\langle\Lambda^2\Psi_2, \Phi_2\rangle \\ -\langle\Lambda^2\Psi_1, \Phi_2\rangle \end{pmatrix} + h_3K_2 + (\alpha_4 + h_4)K_1 + \alpha_5K_0,$$

and in general, we have

$$K_k = \sum_{i=0}^{k-3}(\alpha_i + h_i)\begin{pmatrix} -\langle \Lambda^{k-i-3}\Psi_2, \Phi_2\rangle \\ -\langle \Lambda^{k-i-3}\Psi_1, \Phi_2\rangle \end{pmatrix} + (\alpha_{k-2} + h_{k-2})K_2 + (\alpha_{k-1} + h_{k-1})K_1 + \alpha_k K_0, \qquad k = 3, 4, \cdots,$$

where h_i, α_i are some undetermined constants and $\alpha_0 = 1, \alpha_1 = \alpha_2 = \alpha_3 = 0, h_0 = h_1 = h_2 = 0$. This formula together with (2.12) and (3.11) yields

$$b_k = \sum_{i=0}^{k-3}(\alpha_i + h_i)\langle \Lambda^{k-i-3}\Psi_1, \widetilde{\Phi}_2\rangle + (\alpha_{k-2} + h_{k-2})b_2 + (\alpha_{k-1} + h_{k-1})b_1 + \alpha_k b_0, \tag{3.15a}$$

$$a_k = \sum_{i=0}^{k-3}(\alpha_i + h_i)\langle \Lambda^{k-i-3}\Psi_1, \widetilde{\Phi}_1\rangle + (\alpha_{k-2} + h_{k-2})a_2 + (\alpha_{k-1} + h_{k-1})a_1 + \alpha_k a_0. \tag{3.15b}$$

Since a_k, b_k have to satisfy (2.5), by inserting (3.15) for k and $k+1$ into (2.5), and by using (3.11), we find

$$h_k = \sum_{i=0}^{k-3}(\alpha_i + h_i)[\langle \Lambda^{k-i-3}\Psi_1, \widetilde{\Phi}_1\rangle + \langle \Lambda^{k-i-3}\Psi_2, \widetilde{\Phi}_2\rangle].$$

Notice that $a_0 = \frac{1}{2}$, the above formular and (3.15b) give rise to

$$a_k = \frac{1}{2}\sum_{i=0}^{k-3}(\alpha_i + h_i)(\langle \Lambda^{k-i-3}\Psi_1, \widetilde{\Phi}_1\rangle - \langle \Lambda^{k-i-3}\Psi_2, \widetilde{\Phi}_2\rangle) + (\alpha_{k-2} + h_{k-2})a_2 + (\alpha_{k-1} + h_{k-1})a_1 + (\alpha_k + h_k)a_0. \tag{3.16}$$

Equations (3.15a), (3.16), (2.5a) and (3.11) lead to the definition (3.13). The above procedure guarantees that (3.13) under (3.11) satisfies (2.5). Then (2.15) yields (3.14). This completes the proof.

Set

$$\tilde{a}^2 + \tilde{b}\tilde{c} = \sum_{i=0}^{\infty} F_k \lambda^{-k}, \qquad F_k = \sum_{i=0}^{k}(\tilde{a}_i\tilde{a}_{k-i} + \tilde{b}_i\tilde{c}_{k-i}). \tag{3.17}$$

Then substituting (3.13) into (3.17), we obtain

$$F_0 = \frac{1}{4}, \qquad F_1 = F_2 = 0, \qquad F_3 = -\frac{1}{2}(\langle \Psi_1, \widetilde{\Phi}_1\rangle + \langle \Psi_2, \widetilde{\Phi}_2\rangle), \tag{3.18a}$$

$$F_4 = \frac{1}{2}(\langle \Lambda\Psi_1, \widetilde{\Phi}_1\rangle - \langle \Lambda\Psi_2, \widetilde{\Phi}_2\rangle) - \langle \Psi_2, \widetilde{\Phi}_1\rangle - \widetilde{w}\langle \Psi_1, \widetilde{\Phi}_1\rangle + v\langle \Psi_1, \widetilde{\Phi}_2\rangle + v^2 + v\widetilde{w}^2, \tag{3.18b}$$

$$\begin{aligned} F_k = {} & \frac{1}{2}(\langle \Lambda^{k-3}\Psi_1, \widetilde{\Phi}_1\rangle - \langle \Lambda^{k-3}\Psi_2, \widetilde{\Phi}_2\rangle) - \langle \Lambda^{k-4}\Psi_2, \widetilde{\Phi}_1\rangle + v\langle \Lambda^{k-4}\Psi_1, \widetilde{\Phi}_2\rangle \\ & + v\langle \Lambda^{k-5}\Psi_1, \widetilde{\Phi}_1\rangle - v\langle \Lambda^{k-5}\Psi_2, \widetilde{\Phi}_2\rangle - \widetilde{w}\langle \Lambda^{k-5}\Psi_2, \widetilde{\Phi}_1\rangle - v\widetilde{w}\langle \Lambda^{k-5}\Psi_1, \widetilde{\Phi}_2\rangle \\ & + \langle \Psi_1, \widetilde{\Phi}_1\rangle\langle \Lambda^{k-5}\Psi_1, \widetilde{\Phi}_2\rangle + \sum_{i=0}^{k-6}\Big[\langle \Lambda^{i}\Psi_1, \widetilde{\Phi}_2\rangle\langle \Lambda^{k-i-6}\Psi_2, \widetilde{\Phi}_1\rangle \\ & + \frac{1}{4}\langle \Lambda^{i}\Psi_1, \widetilde{\Phi}_1\rangle\langle \Lambda^{k-i-6}\Psi_1, \widetilde{\Phi}_1\rangle + \frac{1}{4}\langle \Lambda^{i}\Psi_2, \widetilde{\Phi}_2\rangle\langle \Lambda^{k-i-6}\Psi_2, \widetilde{\Phi}_2\rangle \\ & - \frac{1}{2}\langle \Lambda^{i}\Psi_1, \widetilde{\Phi}_1\rangle\langle \Lambda^{k-i-6}\Psi_2, \widetilde{\Phi}_2\rangle\Big], \qquad k = 5, 6, \cdots. \end{aligned} \tag{3.18c}$$

Equation (3.14) implies that F_k are integrals of motion for the symplectic map (3.11). In order to prove involutivity of F_k, we consider the equations following from (2.8), (2.19) and (2.10)

$$\Psi_{1,t_m} = \sum_{k=0}^{m}(a_k\Lambda^{m-k}\Psi_1 + b_k\Lambda^{m-k}\Psi_2) + b_{m+1}\Psi_1, \tag{3.19a}$$

$$\Psi_{2,t_m} = \sum_{k=0}^{m}(c_k\Lambda^{m-k}\Psi_1 - a_k\Lambda^{m-k}\Psi_2), \tag{3.19b}$$

$$v_{t_m} = v(b_{m+1}^{(1)} - b_{m+1}), \tag{3.19c}$$

$$\widetilde{\Phi}_{1,t_m} = -\sum_{k=0}^{m}(a_k\Lambda^{m-k}\widetilde{\Phi}_1 + c_k\Lambda^{m-k}\widetilde{\Phi}_2) - b_{m+1}\widetilde{\Phi}_1, \tag{3.19d}$$

$$\widetilde{\Phi}_{2,t_m} = -\sum_{k=0}^{m}(b_k\Lambda^{m-k}\widetilde{\Phi}_1 - a_k\Lambda^{m-k}\widetilde{\Phi}_2), \tag{3.19e}$$

$$\widetilde{w}_{t_m} = a_{m+1}^{(-1)} - a_{m+1}. \tag{3.19f}$$

By using (3.13) and (3.18), it is easy to verify by a straightforword calculation that equation (3.19) with a_k, b_k, c_k replaced by $\tilde{a}_k, \tilde{b}_k, \tilde{c}_k$ becomes a finite-dimensional Hamiltonian system (FDHS), i.e.,

$$Psi_{i,t_m} = \frac{\partial F_{m+3}}{\partial \widetilde{\Phi}_i}, \qquad \widetilde{\Phi}_{i,t_m} = -\frac{\partial F_{m+3}}{\partial \Psi_i}, \qquad i = 1, 2, \tag{3.20a}$$

$$v_{t_m} = \frac{\partial F_{m+3}}{\partial \widetilde{w}}, \qquad \widetilde{w}_{t_m} = -\frac{\partial F_{m+3}}{\partial v}. \tag{3.20b}$$

According to (2.17), one gets

$$\frac{d}{dt_m}(\tilde{a}^2 + \tilde{b}\tilde{c}) = 0, \qquad \frac{d}{dt_m}F_k = 0, \qquad k, m = 0, 1, \cdots, \tag{3.21}$$

which implies that the F_k are also integrals of motion for FDHS (3.20). The Poisson bracket for (3.20) are the same as (3.4b). So immediately from (3.20) and (3.21) we have

$$\{F_k, F_{m+3}\} = -\frac{d}{dt_m}F_k = 0, \qquad k, m = 0, 1, \cdots, \tag{3.22}$$

which means that integrals of motion F_k are in involution with respect to (3.4b).

Notice that we assume all λ_j to be distinct to have the Vandermonde determinant of $\lambda_1, \cdots, \lambda_N$ different from zero. For a specific N, it can be verified that

$$\frac{\partial(F_3, F_4, \cdots, F_{2N+3})}{\partial(\tilde{\phi}_{11}, \cdots, \tilde{\phi}_{1N}, \tilde{\phi}_{21}, \cdots, \tilde{\phi}_{2N}, \widetilde{w})} \neq 0, \tag{3.23}$$

so grad$F_k, k = 3, \cdots, 2N+3$, are linear independent. Thus we have

Proposition 3.1. *The* F_k *given by* (3.18) *are functionally independent integrals of motion in involution for* (3.11), *and the symplectic map* (3.11) *is completely integrable in the Liouville sense*[14].

Similarly, we obtain the integrals of motion for (3.8) as follows:

$$F_0 = \frac{1}{4}, \qquad F_1 = 0, \qquad F_2 = -\frac{1}{2}(\langle\Psi_1, \widetilde{\Phi}_1\rangle + \langle\Psi_2, \widetilde{\Phi}_2\rangle), \tag{3.24a}$$

$$F_3 = \frac{1}{2}(\langle\Lambda\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda\Psi_2, \widetilde{\Phi}_2\rangle) - \langle\Psi_2, \widetilde{\Phi}_1\rangle + \langle\Psi_1, \widetilde{\Phi}_1\rangle\langle\Psi_1, \widetilde{\Phi}_2\rangle, \tag{3.24b}$$

$$\begin{aligned}F_k = &\frac{1}{2}(\langle\Lambda^{k-2}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{k-2}\Psi_2, \widetilde{\Phi}_2\rangle) - \langle\Lambda^{k-3}\Psi_2, \widetilde{\Phi}_1\rangle \\ &+ \langle\Psi_1, \widetilde{\Phi}_1\rangle\langle\Lambda^{k-3}\Psi_1, \widetilde{\Phi}_2\rangle + \sum_{i=0}^{k-4}\Big[\langle\Lambda^i\Psi_1, \widetilde{\Phi}_2\rangle\langle\Lambda^{k-i-4}\Psi_2, \widetilde{\Phi}_1\rangle \\ &+ \frac{1}{4}\langle\Lambda^i\Psi_1, \widetilde{\Phi}_1\rangle\langle\Lambda^{k-i-4}\Psi_1, \widetilde{\Phi}_1\rangle + \frac{1}{4}\langle\Lambda^i\Psi_2, \widetilde{\Phi}_2\rangle\langle\Lambda^{k-i-4}\Psi_2, \widetilde{\Phi}_2\rangle \\ &- \frac{1}{2}\langle\Lambda^i\Psi_1, \widetilde{\Phi}_1\rangle\langle\Lambda^{k-i-4}\Psi_2, \widetilde{\Phi}_2\rangle\Big], \quad k = 4, 5, \cdots, \end{aligned} \tag{3.24c}$$

and conclude that the map (3.8) is an ISM.

Finally, for the system (3.2), we define

$$\tilde{a}_i = a_i, \qquad \tilde{b}_i = b_i, \qquad \tilde{c}_i = c_i, \qquad i = 0, \cdots, k_0 - 1, \tag{3.25a}$$

$$\tilde{a}_i = \frac{1}{2}(\langle\Lambda^{i-k_0}\Psi_1, \widetilde{\Phi}_1\rangle - \langle\Lambda^{i-k_0}\Psi_2, \widetilde{\Phi}_2\rangle), \qquad i = k_0, k_0 + 1, \cdots, \tag{3.25b}$$

$$\tilde{b}_i = \langle\Lambda^{i-k_0}\Psi_1, \widetilde{\Phi}_2\rangle, \qquad \tilde{c}_i = \langle\Lambda^{i-k_0}\Psi_2, \widetilde{\Phi}_1\rangle, \qquad i = k_0, k_0 + 1, \cdots. \tag{3.25c}$$

Then under (3.2) $\tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ satisfy (2.5), so integrals of motion F_k for (3.2) can be calculated from (3.17) and (3.25). Integrals of motion for (3.5) can be obtained by expressing the F_k in terms of (q, p). Similarly, we consider the time evolution equations for q and p which can be constructed out from (3.19), and show that the F_k are in involution. So (3.5) is an integrable symplectic map.

§4. The Lax Representation and Toda Hierarchy with Source

By following the method in [6], we will show here how the Lax representation for (3.2) can be deduced from the stationary zero-curvature equation (2.3). By using (3.25), we obtain

$$\begin{aligned}\lambda^{k_0-1}\sum_{i=k_0}^{\infty}\tilde{b}_i\lambda^{-i} &= \lambda^{-1}\sum_{i=k_0}^{\infty}\langle\Lambda^{i-k_0}\Psi_1, \widetilde{\Phi}_2\rangle\lambda^{-i+k_0} \\ &= \lambda^{-1}\sum_{i=0}^{\infty}\sum_{j=1}^{N}\frac{\lambda_j^i}{\lambda^i}\psi_{1j}\tilde{\phi}_{2j} = \sum_{j=1}^{N}\frac{\psi_{1j}\tilde{\phi}_{2j}}{\lambda - \lambda_j},\end{aligned} \tag{4.1a}$$

$$\lambda^{k_0-1}\sum_{i=k_0}^{\infty}\tilde{a}_i\lambda^{-i} = \frac{1}{2}\sum_{j=1}^{N}\frac{\psi_{1j}\tilde{\phi}_{1j} - \psi_{2j}\tilde{\phi}_{2j}}{\lambda - \lambda_j}, \quad \lambda^{k_0-1}\sum_{i=k_0}^{\infty}\tilde{c}_i\lambda^{-i} = \sum_{j=1}^{N}\frac{\psi_{2j}\tilde{\phi}_{1j}}{\lambda - \lambda_j}. \tag{4.1b}$$

According to (2.7), set

$$M_{k_0} \equiv \lambda^{k_0-1}\widetilde{\Gamma} = (\lambda^{k_0-1}\widetilde{\Gamma})_+ + N_0 = \overline{V}_{k_0-1} + N_0, \tag{4.2a}$$

where

$$\overline{V}_{k_0-1} = \sum_{i=0}^{k_0-1}\begin{pmatrix} a_i & b_i \\ c_i & -a_i\end{pmatrix}\lambda^{k_0-1-i} = V_{k_0-1} - \triangle_{k_0-1}, \tag{4.2b}$$

$$N_0 = \sum_{j=1}^{N}\frac{1}{\lambda - \lambda_j}\begin{pmatrix}\frac{1}{2}(\psi_{1j}\tilde{\phi}_{1j} - \psi_{2j}\tilde{\phi}_{2j}) & \psi_{1j}\tilde{\phi}_{2j} \\ \psi_{2j}\tilde{\phi}_{1j} & -\frac{1}{2}(\psi_{1j}\tilde{\phi}_{1j} - \psi_{2j}\tilde{\phi}_{2j})\end{pmatrix}. \tag{4.2c}$$

Since $\widetilde{\Gamma}$ satisfies (2.3), M_{k_0} satisfies (2.3), too. So we have

Proposition 4.1. *By substituting the expression M_{k_0} for $\lambda^{k_0-1}\widetilde{\Gamma}$, the stationary zero-curvature equation* (2.3) *reduces to the Lax representation for* (3.2)*:*

$$(EM_{k_0})U - UM_{k_0} = 0, \tag{4.3}$$

with the linear problem equations given by

$$E\psi = U(u,\lambda)\psi, \qquad M_{k_0}\psi = \mu\psi. \tag{4.4}$$

Proof. Comparing (2.3) with (2.9) and (2.11) (taking $m = k_0 - 1$), one finds

$$(EV_{k_0-1})U - UV_{k_0-1} = \begin{pmatrix} 0 & 0 \\ -v(E^{(-1)}-1)\frac{\delta H_{k_0}}{\delta w} & (E-1)v\frac{\delta H_{k_0}}{\delta v} \end{pmatrix}. \tag{4.5a}$$

Using (2.7), one finds

$$-(E\triangle_{k_0-1})U + U\triangle_{k_0-1} = \begin{pmatrix} 0 & -b^{(1)}_{k_0} \\ -vb_{k_0} & 0 \end{pmatrix}. \tag{4.5b}$$

It is easy to calculate the matrix elements of $((EN_0)U - UN_0) \equiv Q_0 = (Q_{ij})$. For instance, we get

$$Q_{12} = \sum_{j=1}^{N} \frac{1}{\lambda-\lambda_j}\Big[\frac{1}{2}(\psi^{(1)}_{1j}\phi_{1j} - \psi^{(1)}_{2j}\phi_{2j}) + (\lambda - w)\psi_{2j}\phi_{2j} + \frac{1}{2}\psi_{1j}\tilde{\phi}_{1j} - \frac{1}{2}\psi_{2j}\tilde{\phi}_{2j}\Big]$$

$$= \langle\Psi_2,\Phi_2\rangle + \frac{1}{2}\sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}[\psi^{(1)}_{1j}\phi_{1j} - \psi^{(1)}_{2j}\phi_{2j} + 2(\lambda_j - w)\psi_{2j}\phi_{2j} + \psi_{1j}\tilde{\phi}_{1j} - \psi_{2j}\tilde{\phi}_{2j}].$$

Then it is easy to see that the coefficients at $\frac{1}{\lambda-\lambda_j}$ in (4.3) which are just given by that in Q_0 are satisfied by (3.2a,b) and the remaining terms in Q_0 together with (4.5a,b) give rise to (3.2c). This completes the proof.

In order to get the Lax representation for the symplectic map (3.5), we need to express M_{k_0} and U in terms of p and q.

For instance, for $k_0 = 3$, the Lax representation for (3.11) is given by (4.3) ($k_0 = 3$) with $M_3 = \overline{V}_2 + N_0$ where

$$\overline{V}_2 = \begin{pmatrix} \frac{1}{2}\lambda^2 + v & -\lambda - \widetilde{w} \\ v\lambda - v\widetilde{w} + \langle\Psi_1,\widetilde{\Phi}_1\rangle & -\frac{1}{2}\lambda^2 - v \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & 1 \\ -v & \lambda + \widetilde{w} - \frac{1}{v}\langle\Psi_1,\widetilde{\Phi}_1\rangle \end{pmatrix}.$$

For $k_0 = 2$, the Lax representation for (3.8) is given by (4.3) ($k_0 = 2$) with $M_2 = \overline{V}_1 + N_0$ where

$$\overline{V}_1 = \begin{pmatrix} \frac{1}{2}\lambda & -1 \\ \langle\Psi_1,\widetilde{\Phi}_1\rangle & -\frac{1}{2}\lambda \end{pmatrix}, \quad U = \begin{pmatrix} 0 & 1 \\ -\langle\Psi_1,\widetilde{\Phi}_1\rangle & \lambda - \frac{1}{\langle\Psi_1,\widetilde{\Phi}_1\rangle}\langle\Psi_2,\widetilde{\Phi}_1\rangle \end{pmatrix}.$$

Now we consider the Toda hierarchy with sources defined by

$$E\Psi_1 = \Psi_2, \qquad E\Psi_2 = -v\Psi_1 + (\Lambda - w)\Psi_2, \tag{4.6a}$$

$$E^{(-1)}\Phi_1 = -v\Phi_2, \qquad E^{(-1)}\Phi_2 = \Phi_1 + (\Lambda - w)\Phi_2, \tag{4.6b}$$

$$\begin{pmatrix} w \\ v \end{pmatrix}_{t_m} = J\frac{\delta H_{m+1}}{\delta u} + J\begin{pmatrix} \langle\Psi_2,\Phi_2\rangle \\ \langle\Psi_1,\Phi_2\rangle \end{pmatrix}. \tag{4.6c}$$

As a consequence of Proposition 4.1, noting (4.5a) for $k_0 = m+1$, we have immediately

Proposition 4.2. *The Toda hierarchy with sources* (4.6) *admits the following discrete zero-curvature representation*

$$U_{t_m} = (EM_{m+1})U - UM_{m+1}, \tag{4.7}$$

with the linear problem equations given by

$$E\psi = U(u,\lambda)\psi, \qquad \psi_{t_m} = M_{m+1}\psi, \tag{4.8}$$

where

$$M_{m+1} = \sum_{i=0}^{m} \begin{pmatrix} a_i & b_i \\ c_i & -a_i \end{pmatrix} \lambda^{m-i} + N_0.$$

References

[1] Bruschi, M., Ragnisco, O., Santini, P. M. & Tu Guizhang, Integrable symplectic maps, *Physica D,* **49**(1991), 273–294.

[2] Moser, J. & Veselov, A. P., Discrete version of some classical integrable systems and factorization of matrix polynomials, *Commun. Math. Phys.,* **139**(1991), 217–243.

[3] Cao Cewen, A cubic system which generates Bargmann potential and N-gap potential, *Chinese Quarterly J. of Math.,* **3**(1988), 90–96.

[4] Zeng Yunbo & Li Yishen, The constraints of potential and the finite-dimensional integrable systems, *J. Math. Phys.,* **30**(1989), 1679–1689.

[5] Zeng Yunbo, An approach to the deduction of the finite-dimensional integrability from the infinite-dimensional integrability, *Phys. Lett. A,* **160**(1991), 541–547.

[6] Zeng Yunbo & Li Yishen, The deduction of the Lax representation for constrained flows from the adjoint representation, *J. Phys. A: Math. Gen.,* **26**(1993), L273–L278.

[7] Zeng Yunbo, New factorizations of the Kaup-Newell hierarchy, *Physica D,* **73**(1994), 171–188.

[8] Zeng Yunbo & Li Yishen, On a general approach to finite-dimensional integrable Hamiltonian systems from infinite-dimensional ones, *Advance in Mathematics,* **24:**2(1995), 111–130.

[9] Rauch-Wojciechowski, S., New restricted flows of the KdV hierarchy and their bi-hamiltonian structure, *Phys. Lett. A,* **160**(1991), 241–246.

[10] Ragnisco, O., A discrete Neumann system, *Phys. Lett. A,* **167**(1992), 165–171.

[11] Tu Guizhang, A trace identity and its application to the theory of discrete integrable systems, *J. Phys. A,* **23**(1990), 3903–3922.

[12] Kupershmidt, B. A., Discrete Lax equations and differential-difference calculus, *Aste'rique,* **123** (1985).

[13] Newell, A.C., Soliton in mathematics and physics, *SIAM,* Philadelphia, 1985.

[14] Arnold, V. I., Mathematical methods of classical mechanics, Springer, New York, 1978.