Chin. Ann. of Math.
18B: 4(1997),467-480.

ON Δ–GOOD MODULE CATEGORIES OF QUASI-HEREDITARY ALGEBRAS**

Deng Bangming* Xi Changchang*

Abstract

A useful reduction is presented to determine the finiteness of Δ–good module category $\mathcal{F}(\Delta)$ of a quasi–hereditary algebra. As an application of the reduction, the $\mathcal{F}(\Delta)$–finiteness of quasi–hereditary M–twisted double incidence algebras of posets is discussed. In particular, a complete classification of $\mathcal{F}(\Delta)$-finite M–twisted double incidence algebras is given in case the posets are linearly ordered.

Keywords Quasi–hereditary algebra, Δ–good module, $\mathcal{F}(\Delta)$–finiteness, M–twisted double incidence algebra
1991 MR Subject Classification 16G10, 16G60
Chinese Library Classification O153.3

§0. Introduction

Quasi–hereditary algebras are introduced by Cline, Parshall and Scott in [2] in order to study highest weight categories in representation theory of semisimple Lie algebras and algebraic groups. Many important algebras such as hereditary algebras, Schur algebras and algebras to blocks of the category $\mathcal{O}$ in [1] are typical examples of quasi–hereditary algebras. They can be defined recursively in terms of the existence of a particular idempotent ideal and appear quite common.

0.1 For each given quasi–hereditary algebra A, there is a partial order $(\Lambda, \leq)$ on the set of simple modules, and one studies the standard modules

$$\Delta = \{\Delta(\lambda) \mid \lambda \in \Lambda\}.$$

Of particular interest is the Δ-good module category $\mathcal{F}(\Delta)$ of all modules which have a Δ-filtration. As a notable example, considering the Schur algebra in [9], then the category $\mathcal{F}(\Delta)$ becomes just the category consisting of all modules which have a Weyl module filtration and is investigated by many authors. Recently, C. M. Ringel proved that $\mathcal{F}(\Delta)$ has almost split sequences[15].

One of the interesting questions on $\mathcal{F}(\Delta)$ is when it is finite (i.e., there are only finitely many pairwise non-isomorphic indecomposable modules in $\mathcal{F}(\Delta)$). If it is the case, the endomorphism ring of the direct sum of non–isomorphic indecomposable objects in $\mathcal{F}(\Delta)$

Manuscript received June 12, 1995. Revised May 3, 1996.
*Department of Mathematics, Beijing Normal University, Beijing 100875, China.
**Project supported by the Young Teachers Foundation of the State Education Commission and the National Natural Science Foundation of China.

is again quasi-hereditary as shown in [4]. The purpose of this paper is to provide a useful reduction to reduce the question from the given quasi-hereditary algebra to a smaller one by means of vectorspace categories. The advantage of this method is that one can use the well–developed theory of vectorspace categories to handle the question, there a lot of beautiful results such as Kleiner's criterion can serve as a tool.

In the first section we establish a reduction to decide whether $\mathcal{F}(\Delta)$ is finite and give some necessary conditions. Then we apply these results to the quasi-hereditary algebras defined in [7] and determine when $\mathcal{F}(\Delta)$ is finite. The last section, as an explanation of our method, offers a complete classification of $\mathcal{F}(\Delta)$-finite quasi-hereditary algebras $\mathcal{A}_{(X,M)}$ with X a linearly ordered set.

0.2 Now let us recall some definitions and fix notation. Let A be a finite dimensional algebra over an algebraically closed field k. We will consider (almost in all cases finitely generated left) A–modules, maps between A–modules will be written on the right side of the argument, thus the composition of maps $f : M_1 \longrightarrow M_2$ and $g : M_2 \longrightarrow M_3$ will be denoted by fg. The category of all finitely generated A–modules will be denoted by A–mod. Given a class Θ of A–modules, we denote by $\mathcal{F}(\Theta)$ the class of all A–modules which have a Θ–filtration, that is, a filtration

$$0 = M_t \subset M_{t-1} \subset \cdots \subset M_1 \subset M_0 = M$$

such that each factor M_{i-1}/M_i is isomorphic to one object in Θ for $1 \leq i \leq t$. The modules in $\mathcal{F}(\Theta)$ are called Θ-good modules, and the category $\mathcal{F}(\Theta)$ is called the Θ-good module category.

Let $\wedge$ be a finite poset in bijective correspondence with the isomorphism classes of simple A–modules. For each $\lambda \in \wedge$, let $E(\lambda)$ be a simple module in the isomorphism class corresponding to λ and $P(\lambda)$ (or $P_A(\lambda)$) a projective cover of $E(\lambda)$ and denote by $\Delta(\lambda)$ the maximal factor module of $P(\lambda)$ with composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Dually, let $Q(\lambda)$ (or $Q_A(\lambda)$) be an injective hull of $E(\lambda)$ and denote by $\nabla(\lambda)$ the maximal submodule of $Q(\lambda)$ with the composition factors of the form $E(\mu)$, $\mu \leq \lambda$. Let Δ (respectively, ∇) be the full subcategory consisting of all $\Delta(\lambda)$, $\lambda \in \wedge$ (respectively, all $\nabla(\lambda)$, $\lambda \in \wedge$). We call modules in Δ standard modules and ones in ∇ costandard modules.

The algebra A is said to be quasi–hereditary with respect to $(\wedge, \leq)$ if for each $\lambda \in \wedge$ we have

(i) $\mathrm{End}_A(\Delta(\lambda)) \cong k$;

(ii) $P(\lambda) \in \mathcal{F}(\Delta)$, and moreover, $P(\lambda)$ has a Δ–filtration with factors $\Delta(\mu)$ for $\mu \geq \lambda$ in which $\Delta(\lambda)$ occurs exactly once.

For a quasi–hereditary algebra A with respect to a poset $\wedge$ we call the elements in $\wedge$ weights and $\wedge$ the weight poset of A. By $(A, \wedge)$ we denote a quasi–hereditary algebra A with the weight poset $\wedge$.

If a quasi–hereditary algebra has a duality δ which fixes simple modules, we call it a BGG–algebra (see [2, 11]).

For a quasi-hereditary algebra A, if the Δ-module category $\mathcal{F}(\Delta)$ of A is finite we say that the algebra A is $\mathcal{F}(\Delta)$–finite.

Definition 0.1.[17] *Let $\mathcal{K}$ be a Krull–Schmidt k–category and* $|\cdot| : \mathcal{K} \longrightarrow k$–mod *an* *additive functor. The pair* $(\mathcal{K}, |\cdot|)$ *is called a vectorspace category. We denote by* $\check{\mathcal{U}}(\mathcal{K}, |\cdot|)$, *called subspace category of* $(\mathcal{K}, |\cdot|)$, *the category of all triples*

$$V = (V_0, V_\omega, \gamma_V : V_\omega \to |V_0|),$$

where $V_\omega \in k - mod$, $V_0 \in \mathcal{K}$ *and* γ_V *is a k–linear map. A morphism from* $V \to V'$ *by definition is a pair* (f_0, f_ω), *where*

$$f_0 : V_0 \to V_0' \quad and \quad f_\omega : V_\omega \to V_\omega'$$

are morphisms in $\mathcal{K}$ *and in* k-mod *respectively, such that* $f_\omega \gamma_{V'} = \gamma_V |f_0|$.

For a module $M \in A$–mod, we denote by $\mathrm{add}(M)$ the full additive subcategory of A–mod consisting of all finite direct sums of direct summands of M. An additive k-category $\mathcal{K}$ is called finite provided there are only finitely many pairwise non-isomorphic indecomposable objects in $\mathcal{K}$.

§1. Criteria for the Finiteness of $\mathcal{F}(\Delta)$

In the representation theory one of the main questions is when an algebra is representation finite, i.e. there are only finitely many isoclasses (=isomorphism classes) of indecomposable modules. But by [16] the most BGG–algebras are representation infinite. Thus one considers, however, another interesting concept of finite type for a quasi–hereditary algebra (with the given ordering of simple modules), namely, that of $\mathcal{F}(\Delta)$–finite type, where $\mathcal{F}(\Delta)$ is the Δ–good module category. In this section we give a reduction to determine the finiteness of $\mathcal{F}(\Delta)$ by applying the well–developed theory of vectorspace categories. We shall use the results in this section to characterize $\mathcal{F}(\Delta)$–finite quasi–hereditary algebras $\mathcal{A}_{(X,M)}$ associated with the labelling matrix (X, M) in the last two sections. Compared with the method in [6], this reduction is more general.

The main idea is to establish a functor between the category $\mathcal{F}(\Delta_A)$ for a quasi–hereditary algebra A and a vectorspace category, and thus reduce the finiteness of $\mathcal{F}(\Delta_A)$ to that of a vectorspace category.

1.1 Let A be a quasi–hereditary algebra with a weight poset $\wedge$. Suppose that $\omega \in \wedge$ is a maximal element. Thus the standard module $\Delta_A(\omega)$ corresponding to ω is the indecomposable projective module $P(\omega) = Ae_\omega$. Let us denote by A_0 the factor algebra of A by the heredity ideal $Ae_\omega A$. Then A_0 is automatically a quasi-hereditary algebra with the standard modules $\Delta_A(\lambda)$, $\lambda \in \wedge \backslash \{\omega\}$.

Lemma 1.1. *Let A be a quasi–hereditary algebra and* ω *a maximal element in* $\wedge$. *Then*

(1) $\mathrm{End}(P(\omega)) \cong k$;

(2) *For each module* M *in* $\mathcal{F}(\Delta_A)$, *there is a unique submodule* M' *of* M *such that* $M' \in \mathrm{add}(P(\omega))$ *and*

$$M/M' \in \mathcal{F}(\{\Delta_A(\lambda) | \lambda \neq \omega\}) = \mathcal{F}(\Delta_{A_0}).$$

The proof of the second statement is referred to [4].

Theorem 1.1. *There is a functor* $\eta : \mathcal{F}(\Delta_A)^{\mathrm{op}} \to \check{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-, P(\omega)))$ *such that*

(1) η *is dense and full;*

(2) *If* M, N *are modules in* $\mathcal{F}(\Delta_A)$ *such that* $\eta(M) \cong \eta(N)$, *then* $M \cong N$;

(3) *For each module* $0 \neq M \in \mathcal{F}(\Delta_A)$*, there holds* $\eta(M) \neq 0$.

Proof. The construction of the functor η is based on Lemma 1.1. A very similar construction was already used in [17]. To define the functor η, we take an arbitrary module $M \in \mathcal{F}(\Delta_A)$, we have by Lemma 1.1 an exact sequence

$$0 \longrightarrow P(\omega)^m \xrightarrow{\alpha_M} M \xrightarrow{\pi_M} M_0 \longrightarrow 0,$$

where α_M denotes the canonical inclusion and where π_M is the canonical surjection from M onto the factor module $M_0 \in \mathcal{F}(\Delta_{A_0})$. Applying $\mathrm{Hom}_A(-, P(\omega))$ to this sequence, we get the following long exact sequence

$$\begin{aligned} &0 \longrightarrow \mathrm{Hom}_A(M_0, P(\omega)) \longrightarrow \mathrm{Hom}_A(M, P(\omega)) \longrightarrow \mathrm{Hom}_A(P(\omega)^m, P(\omega)) \\ &\xrightarrow{\delta_M} \mathrm{Ext}^1_A(M_0, P(\omega)) \longrightarrow \mathrm{Ext}^1_A(M, P(\omega)) \longrightarrow 0\,. \end{aligned}$$

We define

$$\eta(M) = (\mathrm{Hom}_A(P(\omega)^m, P(\omega)), M_0, \delta_M) \in \breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-, P(\omega)))$$

since $\mathrm{Hom}_A(P(\omega)^m, P(\omega)) \cong k^m$ by Lemma 1.1. For each $f \in \mathrm{Hom}_A(N, M)$ we define $\eta(f) = (f_\omega, f_0)$ by the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & P(\omega)^m \arrow[r, "\alpha_M"] & M \arrow[r, "\pi_M"] & M_0 \arrow[r] & 0 \\ 0 \arrow[r] & P(\omega)^n \arrow[r, "\alpha_N"] \arrow[u, "f'_\omega"] & N \arrow[r, "\pi_N"] \arrow[u, "f"] & N_0 \arrow[r] \arrow[u, "f_0"] & 0 \end{tikzcd}$$

(Here the existence of f'_ω follows from the fact $\mathrm{Hom}_A(P(\omega), M_0) = 0$ and hence f'_ω is the restriction of f onto the submodule $P(\omega)^n$). Put $f_\omega = \mathrm{Hom}_A(f'_\omega, P(\omega))$. Then $\eta(f)$ is a morphism from $\eta(M)$ to $\eta(N)$ since we have the desired commutative diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{Hom}_A(P(\omega)^m, P(\omega)) \arrow[r, "\delta_M"] \arrow[d, "\mathrm{Hom}_A(f'_\omega, P(\omega))"] & \mathrm{Ext}^1_A(M_0, P(\omega)) \arrow[r] \arrow[d, "\mathrm{Ext}^1_A(f_0, P(\omega))"] & \cdots \\ \cdots \arrow[r] & \mathrm{Hom}_A(P(\omega)^n, P(\omega)) \arrow[r, "\delta_N"] & \mathrm{Ext}^1_A(N_0, P(\omega)) \arrow[r] & \cdots \end{tikzcd}$$

Clearly, η is a well–defined functor.

(1) η is dense. In fact, given an arbitrary object in $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}}, \mathrm{Ext}^1_A(-, P(\omega)))$, say

$$(M_\omega, M_0, \phi : M_\omega \longrightarrow \mathrm{Ext}^1_A(M_0, P(\omega))),$$

we can write M_ω in the form $\overset{m}{\underset{i=1}{\oplus}} k$ with $m = \dim_k M_\omega$, and

$$\phi = (\phi_1, \cdots, \phi_m)^t \text{ with } \ \phi_i : k \longrightarrow \mathrm{Ext}^1_A(M_0, P(\omega)).$$

In this way, we obtain a map

$$\widetilde{\phi} : k \longrightarrow \mathrm{Ext}^1_A\big(M_0, \overset{m}{\underset{i=1}{\oplus}} P(\omega)\big) \text{ with } \widetilde{\phi} = (\phi_1, \cdots, \phi_m),$$

and the image of $1 \in k$ under $\widetilde{\phi}$ gives an element in $\mathrm{Ext}^1_A\big(M_0, \overset{m}{\underset{i=1}{\oplus}} P(\omega)\big)$; thus an exect sequence of the form

$$0 \longrightarrow \overset{m}{\underset{i=1}{\oplus}} P(\omega) \longrightarrow M \longrightarrow M_0 \longrightarrow 0$$

exists and it follows that $\eta(M)$ is isomorphic to (M_ω, M_0, ϕ). This shows that η is dense.

The functor η is full. Indeed, for given objects M, N in $\mathcal{F}(\Delta_A)$, suppose $\overline{f} = (f_\omega,\, f_0) : \eta(M) \longrightarrow \eta(N)$ is a morphism in $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\rm op},\, {\rm Ext}^1_A(-, P(\omega)))$, where

$$f_\omega : {\rm Hom}_A(P(\omega)^m,\, P(\omega)) \to {\rm Hom}_A(P(\omega)^n,\, P(\omega)),$$

and $f_0 : N_0 \longrightarrow M_0$ is a homomorphism, that is, we have the following commutative diagram

$$\begin{array}{ccc} {\rm Hom}_A(P(\omega)^m,\, P(\omega)) & \xrightarrow{\delta_M} & {\rm Ext}^1_A(M_0,\, P(\omega)) \\ \Big\downarrow{\scriptstyle f_\omega} & & \Big\downarrow{\scriptstyle {\rm Ext}^1(f_0,\, P(\omega))} \\ {\rm Hom}_A(P(\omega)^n,\, P(\omega)) & \xrightarrow{\delta_N} & {\rm Ext}^1_A(N_0,\, P(\omega)) \end{array}$$

We may write $f_\omega = {\rm Hom}(f'_\omega,\, P(\omega))$ with $f'_\omega : P(\omega)^n \longrightarrow P(\omega)^m$. Then the diagram above induces the following commutative diagram

$$\begin{array}{ccc} {\rm Hom}_A(P(\omega)^m,\, P(\omega)^m) & \xrightarrow{\delta_M^m} & {\rm Ext}^1_A(M_0,\, P(\omega)^m) \\ \Big\downarrow{\scriptstyle {\rm Hom}(f_{\omega'},P(\omega)^m)} & & \Big\downarrow{\scriptstyle {\rm Ext}^1(f_0,\, P(\omega)^m)} \\ {\rm Hom}_A(P(\omega)^n,\, P(\omega)^m) & \xrightarrow{\delta_N^m} & {\rm Ext}^1_A(N_0,\, P(\omega)^m) \end{array}$$

Thus the images of the identity map of $P(\omega)^m$ under

$${\rm Hom}(f'_\omega, P(\omega)^m){\delta_N}^m \quad \text{and} \quad {\delta_M}^m{\rm Ext}^1(f_0,\, P(\omega)^m)$$

coincide, so we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P(\omega)^m & \xrightarrow{\alpha_M} & M & \xrightarrow{\pi_M} & M_0 & \longrightarrow & 0 \\ & & \Big\| & & \Big\uparrow{\scriptstyle f_2} & & \Big\uparrow{\scriptstyle f_0} & & \\ 0 & \longrightarrow & P(\omega)^m & \xrightarrow{\gamma} & X & \xrightarrow{\varepsilon} & N_0 & \longrightarrow & 0 \\ & & \Big\uparrow{\scriptstyle f'_\omega} & & \Big\uparrow{\scriptstyle f_1} & & \Big\| & & \\ 0 & \longrightarrow & P(\omega)^n & \xrightarrow{\alpha_N} & N & \xrightarrow{\pi_N} & N_0 & \longrightarrow & 0 \end{array}$$

Set $f = f_1 f_2$. It is clear that $\eta(f) = \overline{f}$. Hence the functor η is full.

(2) This statement follows directly from (1) and the fact that a morphism f is an isomorphism if so is $\eta(f)$.

(3) This statement is trivial.

This finishes the proof.

Indeed, in the above theorem the functor η induces a bijection between the isoclasses of indecomposable objects in $\mathcal{F}(\Delta_A)$ and those in the corresponding vectorspace category $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\rm op}, {\rm Ext}^1_A(-, P(\omega)))$. Thus the theorem gives a method to see whether $\mathcal{F}(\Delta_A)$ is finite.

The theorem can be formulated more generally for a subcategory of A–mod with certain properties. However, for our purpose it is more convenient to work with $\mathcal{F}(\Delta_A)$ only. Now let us see some corollaries of the theorem.

Corollary 1.1. *Let A be a quasi–hereditary algebra with standard modules $\Delta(1), \cdots, \Delta(n)$. If A is $\mathcal{F}(\Delta)$–finite, there holds $\dim_k {\rm Ext}^1_A(\Delta(i),\, \Delta(j)) \leq 1$ for all i and j.*

Proof. We may assume that $i < j$. In this case we can assume further that $\Delta(j) = \Delta(n)$

is the projective module corresponding to the maximal weight n. Then the condition

$$\dim \mathrm{Ext}_A^1(\Delta(i),\, \Delta(j)) \geq 2$$

implies that $\mathrm{Ext}_A^1(\Delta(i),\, \Delta(j))$ as a right module over $\mathrm{End}\Delta(i) \cong k$ is not uniserial. Thus the vectorspace category $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}},\, \mathrm{Ext}_A^1(-,\, \Delta(j))$ is infinite by [10, Proposition 4.7]. Therefore, $\mathcal{F}(\Delta_A)$ is infinite, a contradiction.

Corollary 1.2. *If A is an $\mathcal{F}(\Delta_A)$–finite quasi–hereditary algebra, then for each indecomposable module M in $\mathcal{F}(\Delta_A)$ with Δ–composition factors of the form $\Delta(j)$ for $j < i$, there holds*

$$\dim_k \mathrm{Ext}_A^1(M,\, \Delta(i)) \leq 3.$$

This is also a direct consequence of a result in the theory of finite vectorspace categories (see [10, Lemma 4.8]).

The following three lemmas describe the behaviour of relative irreducible maps under the functor η. Since their proofs are straightforward, we omit them.

Lemma 1.2. *Suppose $f : M \to N$ is a relative irreducible map in $\mathcal{F}(\Delta_A)$ between indecomposable modules M and N. Then $\eta(f) = 0$ if and only if $M \in \mathcal{F}(\Delta_{A_0})$ and $N = P(\omega)$.*

Lemma 1.3. *Let $f : M \longrightarrow N$ be a relative irreducible map in $\mathcal{F}(\Delta_A)$ between indecomposable modules M and N such that $\eta(f) \neq 0$. Then $\eta(f) : \eta(N) \longrightarrow \eta(M)$ is irreducible in $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{A_0})^{\mathrm{op}},\, \mathrm{Ext}_A^1(-, P(\omega)))$.*

Lemma 1.4. *If $f : M \longrightarrow N$ is a morphism between two indecomposable modules in $\mathcal{F}(\Delta_A)$ such that $\eta(f)$ is irreducible, then f is irreducible in $\mathcal{F}(\Delta_A)$.*

1.2 To end this section, let us point out another necessary condition which will be useful.

Assume that A is a quasi–hereditary algebra with a poset $(\wedge, \leq)$. Take a set of orthogonal primitive idempotents $\{e_\lambda \mid \lambda \in \wedge\}$ such that $Ae_\lambda \cong P_A(\lambda)$, where $P_A(\lambda)$ is the projective A–module corresponding to the weight λ. Let Γ be a subset of $\wedge$. We denote by e_Γ the idempotent $e_\Gamma = \sum\limits_{\lambda \in \Gamma} e_\lambda$ and set $A_\Gamma = e_\Gamma A e_\Gamma$.

Recall that a subset Γ of $\wedge$ is called an ideal in $\wedge$ if $\mu \in \Gamma$ and $\lambda \leq \mu$ imply $\lambda \in \Gamma$. Dually, a subset Γ of $\wedge$ is called a coideal in $\wedge$ if $\wedge \backslash \Gamma$ is an ideal in $\wedge$.

In what follows we always assume that A is a quasi–hereditary with a poset $(\wedge, \leq)$ and that Γ is a coideal in $\wedge$. For simplicity, we write $e = e_\Gamma$.

Lemma 1.5. *Under the above assumptions, we have*

(1) *The algebra A_Γ is a quasi–hereditary with standard modules $\{e\Delta_A(\gamma) \mid \gamma \in \Gamma\}$ and with the poset $(\Gamma, \leq)$ induced by the order relation of $\wedge$.*

(2) *The costandard modules of A_Γ are the module $e\nabla_A(\gamma)$ for $\gamma \in \Gamma$.*

(3) *If $\lambda \notin \Gamma$, then $e\Delta_A(\lambda) = 0$ and $e\nabla_A(\lambda) = 0$.*

For the proof of this lemma one may see [8].

Theorem 1.2. *Suppose A is a quasi–hereditary algebra with a poset $(\wedge, \leq)$ and with standard modules $\Delta_A(\lambda)$ for $\lambda \in \wedge$. Let Γ be a coideal in $\wedge$. Then the exact functor*

$$Ae \otimes_{eAe} - : \mathcal{F}(\Delta_{eAe}) \longrightarrow \mathcal{F}(\{\Delta_A(\gamma) \mid \gamma \in \Gamma\})$$

is an equivalence.

The proof follows from [5, Theorem 2] since $Ae \in \mathcal{F}(\{\Delta_A(\gamma) \mid \gamma \in \Gamma\})$. This result suggests that one may use subalgebra to see whether $\mathcal{F}(\Delta_A)$ is finite.

§2. The Finiteness of $\mathcal{F}(\Delta)$ for Algebras of the Form $\mathcal{A}_{(X,M)}$

In this section we use the results in the previous section to study the finiteness of $\mathcal{F}(\Delta)$ for quasi–hereditary algebras of the form $\mathcal{A}_{(X,M)}$ associated with a matrix labelling poset (X, M) and present some necessary conditions on (X, M) such that $\mathcal{F}(\Delta)$ is finite.

2.1 We recall some definitions from [7] (see also [3]). Let X be a finite poset. For $x, y \in X$ we write $x \lessdot y$ (or $y \gtrdot x$) to signify that $x < y$ and that there is no $z \in X$ satisfying $x < z < y$. For $x, y \in X$ with $x \le y$ the closed subinterval $[x, y]$ is defined to be the full convex subposet of X formed by all $z \in X$ with $x \le z \le y$. A maximal chain of length n from x to y is a sequence

$$x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y.$$

The minimum (resp. maximum) of the lengths of all maximal chains from x to y is called the minimal (resp. maximal) length of $[x, y]$.

We consider each closed subinterval $[x, y]$ of X with minimal length 2. Suppose $u_1, \cdots, u_n$ are elements in $[x, y]$ such that

$$x \lessdot u_i \lessdot y, \quad 1 \le i \le n,$$

are all maximal chains from x to y of length 2, i.e. the Hasse diagram of $[x, y]$ has a subdiagram of the following form

$$\begin{array}{ccccc} & & u_1 & & \\ & \diagup & \vdots & \diagdown & \\ x & — & u_i & — & y \\ & \diagdown & \vdots & \diagup & \\ & & u_n & & \end{array}$$

and we call it for simplicity a mesh diagram of x and y. With such a mesh we associate a matrix $M_n(x,y) \in k^{n\times n}$, say

$$M_n(x,y) = (a^{(x,y)}_{u_iu_j})_{u_iu_j}.$$

Then we say that X is labelled by matrices, denoted by (X, M), where M is the set of all the labelling matrices, and call M a matrix labelling on X.

We first define an associative k–algebra $\mathcal{A}'_X$ with a k–basis consisting of all symbols $x_n \cdots x_1x_0$, where $n \ge 0$ and $x_i \in X$, $0 \le i \le n$, satisfy either $x_{i-1} \lessdot x_i$ or $x_{i-1} \gtrdot x_i$. The multiplication is defined on the basis elements by setting $(y_m \cdots y_1y_0)(x_n \cdots x_1x_0)$ equal to $y_m \cdots y_1x_n \cdots x_1x_0$ if $y_0 = x_n$ and 0 otherwise, and then extended to $\mathcal{A}'_X$ by linearity. Obviously, there is a k–algebra anti-involution ε of $\mathcal{A}'_X$ defined on the basis vectors by

$$\varepsilon : x_n \cdots x_1x_0 \longmapsto x_0x_1 \cdots x_n.$$

Further, for $x \lessdot u$, $x \lessdot v$ in X, we define

$$r_{uxv} = uxv - \sum_y a^{(x,y)}_{uv} uyv,$$

where the sum runs over all y satisfying $u \lessdot y$ and $v \lessdot y$. Note that we allow in the above definition that $u = v$ and that if there is no y satisfying $u \lessdot y$ and $v \lessdot y$ then the summation is zero.

Finally, by $\mathcal{A}_{(X,M)}$ we denote the quotient algebra of $\mathcal{A}'_X$ by the ideal $I(X, M)$ generated by elements of the following two types

$$r_{uxv}, \qquad x,\, u,\, v \in X \;\text{ with }\; x \lessdot u \;\text{ and }\; x \lessdot v, \tag{2.1.1}$$

$$u_n \cdots u_1 u_0 - v_m \cdots v_1 v_0,\;\; u_0 u_1 \cdots u_n - v_0 v_1 \cdots v_m, \quad \text{for }\; u < v \;\text{ in }\; X, \tag{2.1.2}$$

where $u = u_n \lessdot \cdots \lessdot u_1 \lessdot u_0 = v$ and $u = v_m \lessdot \cdots \lessdot v_1 \lessdot v_0 = v$ are maximal chains from u to v.

It is known that $\mathcal{A}_{(X,M)}$ is quasi–hereditary if (X, M) satisfies the conditions in [3] or if the Hasse diagram of X is a tree (see [7]). Moreover, if $\mathcal{A}_{(X,M)}$ is quasi–hereditary, it admits a strong Δ–subalgebra which is just the incidence algebra $\mathcal{I}(X)$ of X generated by all elements $x + I(X, M)$ and $yz + I(X, M)$ for $x, y, z \in X$ with $y \lessdot z$ and an exact Borel subalgebra B generated by all elements $x + I(X, M)$ and $yz + I(X, M)$ for $x, y, z \in X$ with $y \gtrdot z$ (see [7, 3.7]). In this case the standard $\mathcal{A}_{(X,M)}$–modules are just the indecomposable projective $\mathcal{I}(X)$–modules and the costandard $\mathcal{A}_{(X,M)}$–modules are just the indecomposable injective B–modules.

2.2 The following lemma will be used in our proofs.

Lemma 2.1. *Let A be a quasi–hereditary algebra and B a strong exact Borel subalgebra of A. Then for each natural number l, each B–module M, and each A–module N, there is an isomorphism*

$$\mathrm{Ext}^l_A(A \otimes_B M,\, N) \cong \mathrm{Ext}^l_B(M,\, N|_B),$$

where $N|_B$ denotes the restriction of A–module N to B.

From now on, we assume that (X, M) is a matrix labelling poset such that $\mathcal{A}_{(X,M)}$ is quasi–hereditary, and we simply write $\mathcal{A}$ for $\mathcal{A}_{(X,M)}$ if there is no confusion arising.

Lemma 2.2. *If $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite, then the Hasse diagram of X is a tree and the algebra $\mathcal{A}$ is a BGG–algebra.*

Proof. Suppose that X is not a tree, then there are elements $a < b$ in X such that there exist $x, y \in X$ satisfying $a \lessdot x < b$ and $a \lessdot y < b$. Now let us compute the dimension of $\mathrm{Ext}^1_{\mathcal{A}}(\Delta(a), \Delta(b))$. By Lemma 2.1, there holds

$$\dim_k \mathrm{Ext}^1_{\mathcal{A}}(\Delta(a),\, \Delta(b)) = \dim_k \mathrm{Ext}^1_B(E(a),\, \Delta(b)|_B),$$

where $E(a)$ denotes the simple module corresponding to the weight $a \in X$ and where B is the strong exact Borel subalgebra of $\mathcal{A}$. Note that as a B–module

$$\Delta(b)|_B = \mathop{\oplus}_{z \le b} E(z).$$

Therefore, one has

$$\begin{aligned} &\dim_k \mathrm{Ext}^1_{\mathcal{A}}(\Delta(a),\, \Delta(b)) \\ = &\mathop{\oplus}_{z \le b} \dim_k \mathrm{Ext}^1_B(E(a),\, E(z)) \\ \ge &\dim_k \mathrm{Ext}^1_B(E(a),\, E(x)) + \dim_k \mathrm{Ext}^1_B(E(a),\, E(y)) = 2. \end{aligned}$$

By Corollary 1.3, the algebra $\mathcal{A}$ is then $\mathcal{F}(\Delta)$–infinite. This contradiction implies that the Hasse diagram of X must be a tree.

Proposition 2.1. *If $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite, then the Hasse diagram of X is of Dynkin type.*

Proof. Since the strong exact Borel subalgebra B of $\mathcal{A}$ is also a factor algebra of $\mathcal{A}$ (that is, $B \cong \mathcal{A}/(xy|x \lessdot y)$), the functor $\mathcal{A}\otimes_B - : B$–mod$\longrightarrow A$–mod preserves indecomposability and isomorphism classes. Hence, if $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite, then the algebra B is hereditary by 2.3 and of finite type. Hence the Hasse diagram of X is of Dynkin type.

Proposition 2.2. *Suppose that X is a bipartite poset, i.e. there do not exist x, y, z in X with $x < y < z$. Then $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite if and only if the Hasse diagram of X is a Dynkin diagram.*

Proof. Since X is a bipartite poset, the algebra $\mathcal{A}$ is just the algebra $\mathcal{A}(\mathcal{I}(X))$ defined in [19] with $\mathcal{I}(X)$ a hereditary algebra of radical square zero. According to [6, Theorem 4.1] and Lemma 2.4, the algebra $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite if and only if the Hasse diagram of X is a Dynkin diagram.

Remark 2.1. If the poset X satisfies the condition 3.1(a) in [3], then $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite if and only if X is a bipartite poset with a Hasse diagram of Dynkin type.

Let T be the characteristic module of the quasi–hereditary algebra $\mathcal{A}$ (for the definition of T see [15]). We set

$$\mathcal{H}(T) = \{Y \in \mathcal{A} - \text{mod} \mid \text{Hom}_{\mathcal{A}}(T, Y) = 0\}.$$

Then we may describe $\mathcal{F}(\Delta)$ by means of $\mathcal{H}(T)$.

Proposition 2.3. *If the Hasse diagram of X is a tree, then $\mathcal{H}(T)$ is equivalent to the category $\mathcal{F}(\Delta)/\langle T\rangle$, where $\langle T\rangle$ denotes the ideal of $\mathcal{F}(\Delta)$ generated by morphisms which factor through the objects in add(T).*

Proof. Since the Hasse diagram of X is a tree, the incidence algebra $\mathcal{I}(X)$ is a hereditary algebra. This implies that each standard module of $\mathcal{A}$ has projective dimension at most 1. Since $\mathcal{A}$ is a BGG–algebra, each costandard module has injective dimension at most 1. Hence, by [15, Theorem 3] the proposition follows.

2.3 Recall that a subset Y of X is called a coideal in X if $y \in Y$ and $y < z$ imply $z \in Y$. For a coideal Y in X, each matrix labelling M of X gives rise to a matrix labelling of Y in the obvious way, and we denote it by N. Finally, we set $e = e_Y = \sum\limits_{y\in Y}(y + I(X, M))$ in $\mathcal{A}_{(X,\,M)}$.

Theorem 2.1. *Let (X, M) be a matrix labelling poset such that $\mathcal{A}_{(X,M)}$ is quasi–hereditary and Y a coideal in X with the induced matrix labelling N. Then $\mathcal{F}(\Delta_{\mathcal{A}_{(Y,\,N)}})$ is finite if so is $\mathcal{F}(\Delta_{\mathcal{A}_{(X,M)}})$.*

The theorem follows from Theorem 1.2 and the following lemma.

Lemma 2.3. *With the same assumptions as in* 2.3, *we have an isomorphism*

$$\mathcal{A}_{(Y,N)} \cong e\mathcal{A}_{(X,M)}e.$$

Proof. There is a natural embedding

$$\begin{aligned} i: \quad & \mathcal{A}'_Y \longrightarrow \mathcal{A}'_X \\ y_0y_1\cdots y_m & \longmapsto y_0y_1\cdots y_m. \end{aligned}$$

Clearly, the morphism i induces an injective algebra homomorphism $\tilde{i} : \mathcal{A}'_Y \to \tilde{e}\mathcal{A}'_X\tilde{e}$, where $\tilde{e} = \sum\limits_{y\in Y} y$ is an idempotent in $\mathcal{A}'_X$.

Consider the following diagram

$$\begin{array}{ccc}
\mathcal{A}'_Y & \xrightarrow{\tilde{i}} & \tilde{e}\mathcal{A}'_X\tilde{e} \\
\big\downarrow{\scriptstyle \pi_Y} & & \big\downarrow{\scriptstyle \tilde{\pi}} \\
\mathcal{A}'_Y/I(Y,N) & & \tilde{e}\mathcal{A}'_X\tilde{e}/\tilde{e}I(X,M)\tilde{e} \\
\Big\| & & \big\downarrow{\scriptstyle \eta} \\
\mathcal{A}_{(Y,N)} & \xrightarrow{\Psi} & e\mathcal{A}_{(X,M)}e
\end{array}$$

where $\tilde{\pi}$ is the canonical projection and η the canonical isomorphism. It is easy to see that $\mathrm{Ker}(\tilde{i}\tilde{\pi}\eta) = I(Y,N)$, thus $\tilde{i}\tilde{\pi}\eta$ factors through the canonical projection

$$\pi_Y : \mathcal{A}'_Y \to \mathcal{A}'_Y/I(Y,N) = \mathcal{A}_{(Y,N)},$$

i.e., there is an injective algebra homomorphism $\Psi : \mathcal{A}_{(Y,N)} \to e\mathcal{A}_{(X,M)}e$ such that $\tilde{i}\tilde{\pi}\eta = \pi_Y\Psi$.

It remains to show that Ψ is surjective. Given an arbitrary element in $e\mathcal{A}_{(X,M)}e$ of the form $x_0x_1\cdots x_n + I(X,M)$, where x_0 and x_n lie in Y, one can easily use induction on n to get that $x_0x_1\cdots x_n + I(X,M)$ lies in $\mathrm{Im}(\Psi)$. Since each element in $e\mathcal{A}_{(X,M)}e$ is a k–linear combination of elements of the form $x_0x_1\cdots x_n + I(X,M)$ with $x_0, x_n \in Y$, the homomorphism Ψ is surjective. This finishes the proof.

Since the converse of Proposition 2.1 is not true, it would be interesting to give a complete list of matrix labelling posets (X, M) with the Hasse diagrams of X a Dynkin diagram such that $\mathcal{F}(\Delta_{\mathcal{A}_{(X,M)}})$ is finite. In the next section we shall deal with this question in a special case in detail.

§3. Examples

In this section we will classify all the matrix labelling posets (X, M) with X linearly ordered sets such that $\mathcal{A}_{(X,M)}$ is $\mathcal{F}(\Delta)$–finite. And meanwhile this gives also an explanation of the methods in section 1.

3.1 Let $X = \{1 < 2 < \cdots < n\}$ be a linearly ordered set. If $n \geq 3$, then each matrix labelling of X is given by $n-2$ elements

$$M(1,3) = a_1, \cdots, M(n-2,n) = a_{n-2} \text{ in } k.$$

For simplicity, we denote by the sequence $(a_1, a_2, \cdots a_{n-2})$ the matrix labelling M.

Further, if we attach to X the new matrix labelling $N = (b_1, b_2, \cdots, b_{n-2})$ such that $b_i = 1$ if $a_i \neq 0$ and that $b_i = 0$ if $a_i = 0$ for all $1 \leq i \leq n-2$, then the associated algebras $\mathcal{A}_{(X,N)}$ and $\mathcal{A}_{(X,M)}$ are isomorphic. Thus in the following we will assume that each labelling $M = (a_1, a_2, \cdots, a_{n-2})$ of X is such that $a_i = 0$ or 1 for all $1 \leq i \leq n-2$ and we simply write $\mathcal{A}$ for $\mathcal{A}_{(X,M)}$.

Proposition 3.1. *Let $X = \{1 < 2 < \cdots < n\}$ be a linearly ordered set with the matrix labelling $M = (a_1, a_2, \cdots, a_{n-2})$. Then $\mathcal{A} = \mathcal{A}_{(X,M)}$ is $\mathcal{F}(\Delta)$–finite if and only if one of the following conditions is satisfied:*

(i) $n \le 3$;

(ii) $n = 4$ *and* $(a_1, a_2) \neq (0, 0)$;

(iii) $n = 5$ *and* $(a_1, a_2, a_3) = (1, 1, 1)$.

Proof. I) We first prove that the algebras $\mathcal{A}$ given in the proposition are $\mathcal{F}(\Delta)$–finite. This will be done by examining the finiteness of $\mathcal{F}(\Delta_{\mathcal{A}})$ case by case.

1) In case $n \le 2$, the algebras $\mathcal{A}$ are trivialwise $\mathcal{F}(\Delta)$–finite.

2) In case $n = 3$ and $a_1 = 0$, the algebra $\mathcal{A}$ is then given by the quiver

$$1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3$$

with relations $\alpha\alpha' = \beta\beta' = 0$. From [6, 3.4], we know that $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite. For the use of the later computation we list the Auslander–Reiten quiver of $\mathcal{F}(\Delta_{\mathcal{A}})$ in Fig. 1.

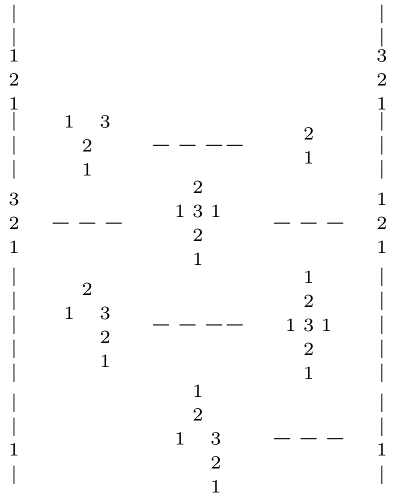

Fig.1

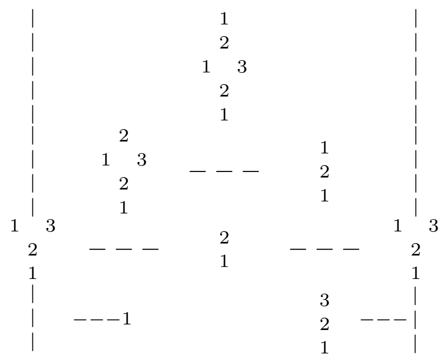

Fig.2

(In the figures above the indecomposable modules are displayed by their Loewy factors and the dotted vertical lines should be identified.)

3) In case $(X, M) = (\{1 < 2 < 3\}, (1))$, $(\{1 < 2 < 3 < 4\}, (1, 1))$ or $(\{1 < 2 < 3 < 4 < 5\}, (1, 1, 1))$, the algebra $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite according to [5]. Moreover, if $(X, M) = (\{1 < 2 < 3\}, (1))$, the Auslander–Reiten quiver of $\mathcal{F}(\Delta_{\mathcal{A}})$ is displayed in Fig.2.

4) In case $n = 4$ and $(a_1, a_2) = (1, 0)$, the algebra $\mathcal{A}$ is given by the quiver

$$1 \underset{\alpha'}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\beta'}{\overset{\beta}{\rightleftarrows}} 3 \underset{\gamma'}{\overset{\gamma}{\rightleftarrows}} 4$$

with relations

$$\alpha\alpha' = \beta'\beta \ \text{ and } \ \beta\beta' = \gamma\gamma' = 0.$$

According to Theorem 1.2, we consider the vectorspace category

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\rm op}, \operatorname{Ext}^1_{\mathcal{A}}(-, P_{\mathcal{A}}(4))),$$

where $\mathcal{A}_0$ is the algebra associated with the matrix labelling poset $(\{1 < 2 < 3\}, (1))$ (see 3)). A computation shows that $\dim_k \operatorname{Ext}^1_{\mathcal{A}}(M, P_{\mathcal{A}}(4)) \le 1$ for each indecomposable object in $\mathcal{F}(\Delta_{\mathcal{A}_0})$. Thus the study of the category $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^o p, \operatorname{Ext}^1_{\mathcal{A}}(-, P_{\mathcal{A}}(4)))$ is reduced to the study of representations of the poset S with the following Hasse diagram (see [10, 4.1]):

$$a — b — c — d, \quad e — f — g$$

and there is a natural bijection between the isoclasses of indecomposable objects in

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}},\ \mathrm{Ext}^1_{\mathcal{A}}(-,\ P_{\mathcal{A}}(4)))$$

and indecomposable representatons of S. By Kleiner's criterion (see [12]), the poset S is of finite type, hence the category $\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}},\ \mathrm{Ext}^1_{\mathcal{A}}(-,\ P_{\mathcal{A}}(4)))$ is finite. By Theorem 1.2, the algebra $\mathcal{A}$ is then $\mathcal{F}(\Delta)$–finite. In fact, $\mathcal{F}(\Delta_{\mathcal{A}})$ has 27 isoclasses of indecomposables.

5) In case $n = 4$ and $(a_1, a_2) = (0, 1)$, as in case 4) we consider the vectorspace category

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}}, \mathrm{Ext}^1_{\mathcal{A}}(-,\ P_{\mathcal{A}}(4))),$$

where $\mathcal{A}_0$ is the algebra associated with the matrix labelling poset $(\{1 < 2 < 3\}, (0))$ (see 2)). According to [10, 4.10, 4.11], there is a bijection between the isoclasses of indecomposable objects in

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}}, \mathrm{Ext}^1_{\mathcal{A}}(-,\ P_{\mathcal{A}}(4)))$$

and those of indecomposable representations of the following biinvolutive poset S^{**} whose Hasse diagram is of the following form:

$$\begin{tikzcd}
& b \arrow[rr, dash] & & d \arrow[rr, dash] & & f \arrow[rr, dash] & & h \arrow[rr, dash] & & v \\
a \arrow[ur, dash] \arrow[rr, dash] & & c \arrow[ur, dash] \arrow[rr, dash] & & e \arrow[ur, dash] \arrow[rr, dash] & & g \arrow[ur, dash] \arrow[rr, dash] & & u \arrow[ur, dash] & \\
& & & & & & & x \arrow[ur, dash] & &
\end{tikzcd}$$

whose involution on S is given by

$$a^* = u, \quad u^* = a, \quad c^* = v, \quad v^* = c, \text{ and } s^* = s$$

for all remaining points s in S, and whose involution on $S^2 = \{(s, t) | s \le t\}$ by

$$(a, b)^* = (u, v), \quad (u, v)^* = (a, c), \quad \text{and} \quad (s, t)^* = (s, t)$$

for all remaining pairs $(s, t) \in S^2$. According to [13, 1.2], one can associate with S^{**} a poset $C(S^{**})$ which is of finite type by Kleiner's criterion. It follows from the main theorem in [14] that S^{**} is of finite type. Thus the vectorspace category

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0}),\ \mathrm{Ext}^1_{\mathcal{A}}(-,\ P_{\mathcal{A}}(4)))$$

is finite, that is, $\mathcal{A}$ is $\mathcal{F}(\Delta)$–finite.

II) To prove the necessity, by Theorem 2.1 it suffices to prove that the algebras $\mathcal{A} = \mathcal{A}_{(X,M)}$ are $\mathcal{F}(\Delta)$–infinite for the matrix labelling posets $(X,\ M)$ satisfying one of the following conditions:

(i) $n = 4$ and $(a_1, a_2) = (0, 0)$,

(ii) $n = 5$ and $(a_1, a_2, a_3) \neq (1, 1, 1)$,

(iii) $n = 6$ and $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$.

In the following we show that for the above matrix labelling posets the algebras $\mathcal{A}$ are $\mathcal{F}(\Delta)$–infinite.

a) In case $n = 4$ and $(a_1, a_2) = (0, 0)$, the infiniteness of $\mathcal{F}(\Delta_{\mathcal{A}})$ follows from [6, Proposition 3.4].

b) In case $n = 5$ and $(a_1, a_2, a_3) = (0, 0, 0)$, $(0, 0, 1)$ or $(1, 0, 0)$, the infiniteness of $\mathcal{F}(\Delta_{\mathcal{A}})$ follows from a), Theorems 1.1 and 2.1.

c) In case $n = 5$ and $(a_1, a_2, a_3) = (0, 1, 1)$, the algebra $\mathcal{A}$ is given by the quiver

$$1 \underset{\alpha'}{\overset{\alpha}{\leftrightarrows}} 2 \underset{\beta'}{\overset{\beta}{\leftrightarrows}} 3 \underset{\gamma'}{\overset{\gamma}{\leftrightarrows}} 4 \underset{\delta'}{\overset{\delta}{\leftrightarrows}} 5$$

with relations

$$\alpha\alpha' = \delta\delta' = 0, \quad \beta\beta' = \gamma'\gamma \ \text{ and } \ \gamma\gamma' = \delta'\delta.$$

By the reduction in [6], the categories $\mathcal{F}(\Delta_{\mathcal{A}})$ and $\breve{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(P_C(2), -))$ have the same representation type, where C is the algebra given by the following quiver

$$2 \underset{\beta'}{\overset{\beta}{\leftrightarrows}} 3 \underset{\gamma'}{\overset{\gamma}{\leftrightarrows}} 4 \underset{\delta'}{\overset{\delta}{\leftrightarrows}} 5$$

with relations

$$\delta\delta' = 0, \quad \beta\beta' = \gamma'\gamma \ \text{ and } \ \gamma\gamma' = \delta'\delta$$

and where $P_C(2)$ is the projective C–module corresponding to the vertex 2. Consider the indecomposable C–module

$$M = (M_2, M_3, M_4, M_5; \beta, \beta', \gamma, \gamma', \delta, \delta') \in \mathcal{F}(\Delta_C)$$

given by

$$\left(k^4, k^3, k^2, k; \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1 \\ 0&1 \\ 0&0 \end{bmatrix}, \begin{bmatrix} 0&1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

A computation shows that $\mathrm{Hom}_C(P_C(2), M)$ considered as right $\mathrm{End}_C(M)$–module is not uniserial, thus with $\breve{\mathcal{U}}(\mathcal{F}(\Delta_C), \mathrm{Hom}_C(P_C(2), -))$ also $\mathcal{F}(\Delta_{\mathcal{A}})$ is infinite.

d) In case $n = 5$ and $(a_1, a_2, a_3) = (0, 1, 0)$, an argument similar to c) shows that the algebra $\mathcal{A}$ is $\mathcal{F}(\Delta)$–infinite.

e) In case $n = 5$ and $(a_1, a_2, a_3) = (1, 1, 0)$, the algebra $\mathcal{A}$ is given by the quiver

$$1 \underset{\alpha'}{\overset{\alpha}{\leftrightarrows}} 2 \underset{\beta'}{\overset{\beta}{\leftrightarrows}} 3 \underset{\gamma'}{\overset{\gamma}{\leftrightarrows}} 4 \underset{\delta'}{\overset{\delta}{\leftrightarrows}} 5$$

with relations

$$\alpha\alpha' = \beta'\beta, \quad \beta\beta' = \gamma'\gamma \ \text{ and } \ \gamma\gamma' = \delta\delta' = 0.$$

By Theorem 1.1, we consider the vectorspace category

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}}, \mathrm{Ext}^1_{\mathcal{A}}(-, P_{\mathcal{A}}(5))),$$

where $\mathcal{A}_0$ is the factor algebra $\mathcal{A}/\mathcal{A}e_5\mathcal{A}$. Consider the indecomposable $\mathcal{A}_0$–module

$$N = (N_1, N_2, N_3, N_4; \alpha, \alpha', \beta, \beta', \gamma, \gamma')$$

given by

$$\left(k^4, k^3, k^2, k; \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 1&0&0 \\ 0&0&1 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0 \\ 0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1 \\ 0&1 \\ 0&0 \end{bmatrix}, \begin{bmatrix} 0&1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

This module belongs to $\mathcal{F}(\Delta_{\mathcal{A}_0})$ and one can prove that the space $\mathrm{Ext}^1_{\mathcal{A}_0}(N, P_{\mathcal{A}}(5))$ considered as right module over $\mathrm{End}_{\mathcal{A}_0}(N)$ is not uniserial, thus the category

$$\breve{\mathcal{U}}(\mathcal{F}(\Delta_{\mathcal{A}_0})^{\mathrm{op}}, \mathrm{Ext}^1_{\mathcal{A}}(-, P_{\mathcal{A}}(5)))$$

is infinite. By Theorem 1.1, the category $\mathcal{F}(\Delta_{\mathcal{A}})$ is then infinite.

f) In case $n = 5$ and $(a_1, a_2, a_3) = (1, 0, 1)$, the infiniteness of $\mathcal{F}(\Delta_{\mathcal{A}})$ can be proved in a way similar to e).

g) In case $n = 6$ and $(a_1, a_2, a_3, a_4) = (1, 1, 1, 1)$, the infiniteness of $\mathcal{F}(\Delta_{\mathcal{A}})$ is proved in [5].

This finishes the proof of the proposition.

References

[1] Bernstein, I. N., Gelfand, I. M. & Gelfand, S. I., A category of $\mathcal{G}$–modules, *Funct. Anal. Appl.,* **10**(1976), 67–92.

[2] Cline, E., Parshall, B. & Scott, L., Finite dimensional algebras and highest weight categories, *J. Reine Angew. Math.,* **391**(1988), 85–99.

[3] Dyer, M. J., Kazhdan–Lusztig–Stanley polynomials and quadratic algebras I, preprint, 1993.

[4] Xi, C. C., Endomorphism algebras of $\mathcal{F}(\Delta)$ over quasi-hereditary algebras, *J. algebra,* **175**(1995), 966–978.

[5] Dlab, V. & Ringel, C. M., The module theoretical approach to quasi–hereditary algebras, LMS Lecture Notes Series 168, "Representations of algebras and related topics", ed. H. Tachikawa and S.Brenner, 1992, 200–224.

[6] Deng, B. M. & Xi, C. C., Quasi–hereditary algebras which are dual extensions of algebras, *Comm. in Algebra,* **22**(1994), 4717–4735.

[7] Deng, B. M. & Xi, C. C., Quasi–hereditary algebras which are twisted double incidence algebras of posets, *Contributions to Algebra and Geometry,* **36**(1995), 37–71.

[8] Erdmann, K., Symmetric groups and quasi–hereditary algebras, preprint, 1993.

[9] Green, J. A., Polynomial representations of GL_n, Springer LNM 830, 1980.

[10] Gabriel, P. & Roiter, A. V., Representations of finite–dimensional algebras, *Encyclopaedia of Math. Sci.,* **73**, Algebra VIII, (1992).

[11] Irving, R. S., BGG–algebras and the BGG reciprocity principle, *J. Algebra,* **135**(1990), 363–380.

[12] Kleiner, M. M., Partially ordered sets of finite type. Investigations on representation theory, *Engl. transl.: J. Soviet Math.,* **23**(1975), 585–607.

[13] Nazarova, L. A. & Roiter, A. V., Representations of biinvolutive partially ordered sets, *Transl. of Preprint Inst. Math. Ukr. Ac. Sc.,* **91–34**(1991), 1–31, by V.Dmitrenko.

[14] Nazarova, L. A. & Roiter, A. V., Representations of biinvolutive partially ordered sets II, preprint, Kiev–1994.

[15] Ringel, C. M., The category of modules with good filtrations over a quasi–hereditary algebra has almost split sequences, *Math. Zeit.,* **208**(1991), 209–223.

[16] Xi, C. C., Quasi–hereditary algebras with a duality, *J. Reine Angew. Math.,* **449**(1994), 201–215.

[17] Ringel, C. M., Tame algebras, Springer LNM 831 (1980), 137–287.