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# ON THE MINIMAL PERIOD FOR PERIODIC SOLUTION PROBLEM OF NONLINEAR HAMILTONIAN SYSTEMS**

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**Abstract**

The author proves a sharper estimate on the minimal period for periodic solutions of autonomous second order Hamiltonian systems under precisely Rabinowitz' superquadratic condition.

**Keywords** Hamiltonian systems, $\mathbf{Z}_2$-symmetry, Index iteration inequality, Minimal period, Even solution, Superquadratic condition, Non-convexity
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## §1. The Main Results

In this short note, we consider the existence of non-constant periodic solutions with prescribed minimal period for the following autonomous second order Hamiltonian systems

$$\ddot{x}+V'(x)=0, \qquad \forall x\in\mathbf{R}^n, \tag{1.1}$$

where $n$ is a positive integer. $V:\mathbf{R}^n\to\mathbf{R}$ is a function, and $V'$ denotes its gradient. In his pioneer work [6] in 1978, P. Rabinowitz posed a conjecture of whether (1.1) possesses a non-constant solution with any prescribed minimal period under superquadratic conditions. Since then, a large amount of contributions on this minimal period problem have been made by many mathematicians. We refer to [1] for discussions and references before 1990 on this problem. In a recent paper[4] under precisely Rabinowitz' superquadratic condition, the author proved that for every $T>0$ there exists a $T$-periodic even non-constant solution $x$ of (1.1) with minimal period $T/k$ for some integer $k$ satisfying $1\le k\le n+2$ (cf. also [2, 3, 5]). In this paper we further improve this estimate for the integer $k$ to $1\le k\le n+1$. Our main results are the following theorems.

**Theorem 1.1.** *Suppose $V$ satisfies the following conditions.*

(V1) $V\in C^2(\mathbf{R}^n,\mathbf{R})$.

(V2) *There exist constants $\mu>2$ and $r_0>0$ such that*

$$0<\mu V(x)\le V'(x)\cdot x, \qquad \forall |x|\ge r_0.$$

(V3) $V(x)\ge V(0)=0, \quad \forall x\in\mathbf{R}^n$.

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(V4) $V(x) = o(|x|^2)$, *at* $x = 0$.

*Then for every* $T > 0$*, the system* (1.1) *possesses a non-constant* $T$*-periodic even solution with minimal period* $T/k$ *for some integer* $k$ *satisfying* $1 \le k \le n+1$.

**Theorem 1.2.** *Suppose* $V$ *satisfies conditions* (V1)–(V3) *and the following condition.*

(V5) *There exist constants* $\omega > 0$ *and* $r_1 > 0$ *such that*

$$V(x) \le \frac{\omega}{2}|x|^2, \qquad \forall |x| \le r_1.$$

*Then for every positive* $T < \frac{2\pi}{\sqrt{\omega}}$*, the conclusion of Theorem* 1.1 *holds.*

Our proof depends on a new inequality (Theorem 2.1) of iterated Morse indices for the functional corresponding to (1.1) defined on even function spaces, and the approach used in [4].

## §2. The Proofs

As in [4] let $E_T = W^{1,2}(S_T, \mathbf{R}^n)$, where $S_T = \mathbf{R}/(T\mathbf{Z})$, with the norm

$$\|x\|_T = \Big(\int_0^T |\dot{x}|^2 dt + T|x(0)|^2\Big)^{1/2}, \qquad \forall x \in E_T.$$

The functional corresponding to the system (1.1) is defined by

$$\psi_T(x) = \int_0^T \Big(\frac{1}{2}|\dot{x}|^2 - V(x)\Big)dt, \qquad \forall x \in E_T. \tag{2.1}$$

Define

$$SE_T = \{x \in E_T \,|\, x(-t) = x(t), \ \forall t \in \mathbf{R}\}.$$

In [4] it is proved that the critical points of $\psi_T$ restricted to $SE_T$ are one-to-one correspondent to $T$-periodic even solutions of the system (1.1).

For any critical point $x$ of $\psi_T|_{SE_T}$ the following bilinear form is defined by $\psi_T''(x)$ on $SE_T$:

$$\phi_T(y,z) = \int_0^T \{\dot{y}\cdot\dot{z} - A(t)y\cdot z\}dt, \qquad \forall y, z \in SE_T. \tag{2.2}$$

Denote by $\mathcal{L}_s(\mathbf{R}^n)$ the space of symmetric $n\times n$ real matrices. $A(t) = V''(x(t))$ satisfies the following condition.

(AS) $A \in C(S_T, \mathcal{L}_s(\mathbf{R}^n))$ *and* $A(t)$ *is even about* $t = 0$.

Note that $\phi_T$ corresponds to the following linear second order Hamiltonian system

$$\ddot{y} + A(t)y = 0, \qquad \forall y \in \mathbf{R}^n. \tag{2.3}$$

It is proved in [4] that under (AS), $SE_T$ possesses a $\phi_T$-orthogonal decomposition $SE_T = SE_T^+ \oplus SE_T^0 \oplus SE_T^-$ according to $\phi_T$ being positive, null, and negative definite respectively.

If $x$ is a non-constant critical point of $\psi_T$ in $SE_T$, then $\dot{x}$ is a nontrivial solution of (2.3) with $A(t) = V''(x(t))$, and $\dot{x} \in E_T$ is odd about $t = 0$. Thus we define the space of such odd solutions of (2.3) by $OE_T^0 = \{y \in E_T \,|\, y \text{ is an odd solution of } (2.3)\}$.

**Definition 2.1.** *Define* $si_T = \dim SE_T^-$ *and* $o\nu_T = \dim OE_T^0$.

The following estimate on the iterated Morse indices is the main result in this paper.

**Theorem 2.1.** *Suppose the condition* (AS) *holds. Then*

$$si_{kT} \ge k\min\{o\nu_T, 1\}, \qquad \forall k \in \mathbf{N}. \tag{2.4}$$

**Proof.** Suppose $o\nu_T \geq 1$ and fix an integer $k \in \mathbf{N}$. Fix $u \in OE_T^0 \backslash \{0\}$. We define new functions $u_+$ and $u_- \in C(S_{kT}, \mathbf{R}^n)$ by

$$u_+(t) = \begin{cases} u(t), & \text{if } 0 \leq t \leq T/2, \\ 0, & \text{if } T/2 \leq t \leq kT, \end{cases} \qquad u_-(t) = \begin{cases} u(t+T/2), & \text{if } 0 \leq t \leq T/2, \\ 0, & \text{if } T/2 \leq t \leq kT. \end{cases} \tag{2.5}$$

Define a $T/2$-translation operator $\eta : C(S_{kT}, \mathbf{R}^n) \to C(S_{kT}, \mathbf{R}^n)$ by

$$\eta v(t) = v(t - T/2), \qquad \forall v \in C(S_{kT}, \mathbf{R}^n). \tag{2.6}$$

Now we define a sequence of functions $\{u_i\}$ for $1 \leq i \leq k$ based on $u$ by

$$u_i = \eta^{i-1} u_+ - \eta^{2k-i} u_-, \quad \text{if } i \in 2\mathbf{N} - 1, \tag{2.7}$$

$$u_i = -\eta^{i-1} u_- + \eta^{2k-i} u_+, \quad \text{if } i \in 2\mathbf{N}, \tag{2.8}$$

(cf. Figure 1.) and define $N = \mathrm{span}\{u_i \,|\, 1 \leq i \leq k\}$.

Since all the $u_i$'s have mutually nonintersect supports, they are linearly independet. Note that each $u_i \in E_{kT}$ is even, there hold

$$\dim N = k, \qquad N \subset SE_{kT}, \tag{2.9}$$

$$\phi_{kT}(u_i, u_j) = 0, \qquad \text{if} \quad i \neq j. \tag{2.10}$$

Since $u \in OE_T^0 \backslash \{0\}$, we obtain $\dot{u}(0) = \dot{u}(T) \neq 0$ and $\dot{u}(T/2) \neq 0$. Therefore each $u_i$ is not $C^1$, and therefore does not belong to $\ker \phi_T$ in $E_T$. Since $u_k$ is not $C^1$ at $t = kT/2$, any function in $N \backslash \{0\}$ is not $C^1$. Therefore we obtain

$$N \cap \ker \phi_{kT} = \{0\}. \tag{2.11}$$

If $i$ is odd, from the definition of $u_i$, we obtain

$$\begin{aligned} \phi_{kT}(u_i, u_i) &= \int_{(i-1)T/2}^{iT/2} (|\dot{u}_i|^2 - A(t) u_i \cdot u_i) dt + \int_{(2k-i)T/2}^{(2k-i+1)T/2} (|\dot{u}_i|^2 - A(t) u_i \cdot u_i) dt \\ &= \int_0^{T/2} (|\dot{u}_+|^2 - A(t) u_+ \cdot u_+) dt + \int_0^{T/2} (|\dot{u}_-|^2 - A(t+T/2) u_- \cdot u_-) dt \\ &= \int_0^T (|\dot{u}|^2 - A(t) u \cdot u) dt \\ &= 0. \end{aligned} \tag{2.12}$$

If $i$ is even, from the definition of $u_i$, we obtain similarly

$$\begin{aligned} \phi_{kT}(u_i, u_i) &= \int_0^{T/2} (|\dot{u}_-|^2 - A(t+T/2) u_- \cdot u_-) dt + \int_0^{T/2} (|\dot{u}_+|^2 - A(t) u_+ \cdot u_+) dt \\ &= 0. \end{aligned} \tag{2.13}$$

Note that (2.10), (2.12), and (2.13) imply

$$\phi_{kT}(v_1, v_2) = 0 \qquad \forall v_1, v_2 \in N. \tag{2.14}$$

Based upon (2.9), (2.11), and (2.14), we can apply the step 4 in the proof of Theorem 3.10 of [4], and obtain (2.4). The proof is complete.

Note that different from the constructions in Theorem 3.10 of [4], the $u_i$'s may not be $\phi_{kT}$-orthogonal to constant solutions. Theorem 2.1 should be compared with the following iteration inequality of Morse indices, Theorem 3.10 of [4], where $\sigma_{kT}^+ \in [0, n]$ is an integer determined by $A(t)$,

$$si_{kT} \geq (k-1) \min\{o\nu_T, 1\} + \sigma_{kT}^+, \qquad \forall k \in \mathbf{N}.$$

Since the non-constant solution $x$ obtained via the saddle point theorem in the proofs of Theorems 1.1 and 1.2 always possesses Morse indices $si_T \leq n+1$ and $o\nu_{T/k} \geq 1$, where the minimal period of $x$ is denoted by $\tau \equiv T/k$ for some $k \in \mathbf{N}$, we then obtain from Theorem 2.1,

$$n+1 \geq si_T = si_{k\tau} \geq k \min\{o\nu_\tau, 1\} \geq k.$$

This proves Theorems 1.1 and 1.2. For details of the proof we refer to that in the section 4 of [4], where we replace the Theorem 3.10 of [4] by the above Theorem 2.1. Note that in the proof of Theorem 1.2, we use Grownwall's inequality to get the constant $2\pi/\sqrt{\omega}$.

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**Figure 1**. Functions in $N$ defined by (2.7) and (2.8) for the case of $k=4$.