*Chin. Ann. of Math.*
**18B**: 4(1997),485-494.

# ESSENTIALLY NORMAL + SMALL COMPACT = STRONGLY IRREDUCIBLE

Ji Youqing* Jiang Chunlan* Wang Zongyao**

**Abstract**

Given an essentially normal operator $T$ with connected spectrum and $\text{ind}(\lambda - T) > 0$ for $\lambda$ in $\rho_F(T) \cap \sigma(T)$, and a positive number $\epsilon$, the authors show that there exists a compact $K$ with $||K|| < \epsilon$ such that $T + K$ is strongly irreducible.

**Keywords** Essentially normal operator, Compact operator, Spectral picture
**1991 MR Subject Classification** 41A10, 47A55
**Chinese Library Classification** O177.1, O174.41

## §1. Introduction

Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all linear bounded operators acting on a complex, separable, infinite dimensional Hilbert space $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be strongly irreducible, if it does not commute with any non-trivial idempotent in $\mathcal{L}(\mathcal{H})$ (see [1, 2, 3]). The strong irreducibility is one of the important properties of operators invariant under similiarity. In what follows, $T \in (SI)$ means that $T$ is a strongly irreducible operator on its acting space.

An operator $T$ is essentially normal if the self-commutator $[T^*, T] = T^*T - TT^*$ is compact. We denote $(\mathcal{U}+\mathcal{K})(\mathcal{H}) = \{R \in \mathcal{L}(\mathcal{H}) :$ $R$ is invertible of the form unitary plus compact$\}$, and denote the $\mathcal{U}+\mathcal{K}$ orbit of $T$

$$(\mathcal{U}+\mathcal{K})(T) = \{R^{-1}TR,\ R \in (\mathcal{U}+\mathcal{K})(\mathcal{H})\}.$$

$T \underset{\mathcal{U}+\mathcal{K}}{\simeq} A$ means that $A \in (\mathcal{U}+\mathcal{K})(T)$. It is obvious that $\underset{\mathcal{U}+\mathcal{K}}{\simeq}$ is an equivalent relation. Note that $T$ is essentially normal and $T \underset{\mathcal{U}+\mathcal{K}}{\simeq} A$ imply that $A$ is essentially normal.

D.A. Herrero and C.L. Jiang proved[3] that if $\sigma(T)$, the spectrum of $T$, is connected, then there exists a sequence $\{T_n\}$ of strongly irreducible operators such that $||T_n - T|| \to 0$ $(n \to \infty)$. C. L. Jiang and Z. Y. Wang[4] improved this result and proved that there exists a strongly irreducible operator $A$ such that i) the spectral pictures (i.e., the spectra and index functions) of $T$ and $A$ are equal; ii) $T$ is a limit of operators similar to $A$; iii) if there is another strongly irreducible operator $B$ satisfying i) and ii), then $B$ is also a limit of operators similar to $A$.

Manuscript received May 9, 1995. Revised October 25, 1995.
*Department of Mathematics, Jilin University, Changchun 130023, China.
**Department of Mathematics, East China University of Science and Technology, Shanghai 200237, China.

The following questions were naturally thought by Herreto:

(1) Given $T \in \mathcal{L}(\mathcal{H})$ with connected $\sigma(T)$, does there exist a $K$ compact such that $T+K \in (SI)$?

(2) Given $T \in \mathcal{L}(\mathcal{H})$ with connected $\sigma(T)$ and given $\epsilon > 0$, does there exist $K$ compact such that $||K|| < \epsilon$ and $T+K \in (SI)$?

(3) Given an essentially normal operator $T$ with connected $\sigma(T)$, what are the answers for the above two questions. C. L. Jiang, S. H. Sun and Z. Y. Wang[5] proved that if $T$ is essentially normal with connected $\sigma(T)$, then there exists $K$ compact such that $T+K \in (SI)$. Using the $(\mathcal{U}+\mathcal{K})$ orbit, Q. Y. Ji, C. L. Jiang and Z. Y. Wang[6] proved that if $T$ is essentially normal, $\sigma(T) = \overline{\Omega}$, where $\Omega$ is an analytic Jordan region and $\mathrm{ind}(\lambda - T) = n$ $(\lambda \in \Omega)$, then for each $\epsilon > 0$, there exists $K$ compact such that $||K|| < \epsilon$ and $T+K \in (SI)$.

The following are the main theorem of this article.

**Main Theorem.** *Let $T \in \mathcal{L}(\mathcal{H})$ be essentially normal with connected $\sigma(T)$ and* $\mathrm{ind}(\lambda - T) > 0$ $(\lambda \in \rho_F(T) \cap \sigma(T))$, *then for each $\epsilon > 0$, there exists $K_\epsilon$ compact with $||K_\epsilon|| < \epsilon$ such that $T + K_\epsilon \in (SI)$, where $\rho_F(T) = \{\lambda \in C;\ \lambda - T$ is Fredholm$\}$.*

## §2. Preparation

**Lemma 2.1.** *Let $B \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{M} = [ker(B-\lambda)^*]^{\perp}$, where $\lambda \in \rho_l(B) \cap \sigma(B)$ and $n$ is a natural number. Then $B|_{\mathcal{M}} \sim B$ and $P_{\mathcal{M}} B^*|_{\mathcal{M}} \sim B^*$, where $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$, $\rho_l(B) = \{\lambda \in C;\ \lambda - B$ is left invertible$\}$.*

**Proof.** Since $\mathcal{M} = \mathrm{Ran}(\lambda - B)$ and since $\ker(\lambda - B) = \{0\}$, $A_1 := (\lambda - B) \in \mathcal{L}(\mathcal{H}, \mathcal{M})$ is invertible. Set $B_1 = B|_{\mathcal{M}}$. Then $A_1^{-1} B_1 A_1 = B$, i.e., $B|_{\mathcal{M}} \sim B$. The second conclusion is a direct consequence of the first.

**Lemma 2.2.** *Let $B_n \in \mathcal{B}_1(\Omega_n)$ and $\lambda_n \in \Omega_n$, $(n = 1, 2, \cdots)$, $\mathcal{B}_1(\Omega_n)$ is the set of Cowen-Douglas operators of index 1[4] and $\{\Omega_n\}$ is a sequence of uniformly bounded, connected open subsets of complex plan $\mathcal{C}$ $(n = 1, 2, \cdots)$. Set $T = \bigoplus_{n=1}^{\infty} B_n$ on $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathcal{H}_n$, and assume that $P_m$ is the orthogonal projection on to $\mathcal{M} = \bigoplus_{n=1}^{m} \{\ker(\lambda_n - B_n)^m \bigoplus 0$. Then $P_m^{\perp} T|_{P_m^{\perp} T \mathcal{H}} \sim T$, where $P_m^{\perp} = I - P_m$.*

**Proof.** Denote $\mathcal{M}_n = \ker(\lambda_n - B_n)^m$ $(n = 1, 2, \cdots, m)$. We have

$$P_m^{\perp} T|_{P_m^{\perp} \mathcal{H}} = \Big( \bigoplus_{n=1}^{m} P_{m_n}^{\perp} B_n|_{P_{m_n}^{\perp} \mathcal{H}_n} \Big) \bigoplus_{n=m+1}^{\infty} B_n,$$

where $P_{m_n}$ is the orthogonal projection on $\mathcal{M}_n$. Then by Lemma 2.1, there exists $X_n$ invertible such that

$$X_n \Big( \bigoplus_{n=1}^{m} P_{m_n}^{\perp} B_n|_{P_{m_n}^{\perp} \mathcal{H}_n} \Big) X_n^{-1} = B_n \quad (n = 1, 2, \cdots, m).$$

Set $X = \bigoplus_{n=1}^{m} X_n \bigoplus \big( \bigoplus_{n=m+1}^{\infty} I_n \big)$. Then $X (P_m^{\perp} T|_{P_m^{\perp} \mathcal{H}}) X^{-1} = T$. Here $I_n$ is the identity operator on $\mathcal{H}_n$.

**Lemma 2.3.** *Let $A \in \mathcal{L}(\mathcal{H})$. If $\lambda \in \rho_F(A)$ such that* $\dim\ker(\lambda - A) = \mathrm{ind}(\lambda - A) = 1$

*and* $e_0 \notin \mathrm{Ran}(\lambda - A)^*$, *then*

$$A \sim \begin{pmatrix} \lambda & 1 \bigotimes e_0 \\ 0 & A \end{pmatrix} \ \text{on } \mathcal{C} \bigoplus \mathcal{H}.$$

**Proof.** Denote $\mathcal{M} = \ker(\lambda - A)^{\perp}$. Then

$$A = \begin{array}{c} \begin{pmatrix} \lambda & C \\ 0 & (A^*|_{\mathcal{M}})^* \end{pmatrix} \end{array} \begin{array}{l} \ker(\lambda - A) \\ \ker(\lambda - A)^{\perp} \end{array}$$

and $A^*|_{\mathcal{M}} \sim A^*$ by Lemma 2.1 or $(A^*|_{\mathcal{M}})^* \sim A$. Thus

$$A \sim \begin{pmatrix} \lambda & C_1 \\ 0 & A \end{pmatrix} \begin{array}{l} \mathcal{C} \\ \mathcal{H} \end{array}, \tag{1.1}$$

where $C_1$ is an operator of rank one, i.e., $C_1 = 1 \bigotimes f$ for some $f \in \mathcal{H}$. Since $e_0 \notin \mathrm{Ran}(\lambda - A)^*$, $f = ae_0 + (A - \lambda)^* f_1$, for some complex number $\alpha$. If $\alpha = 0$, then

$$f = (A - \lambda)^* f_1 \quad \text{or} \quad C_1 = 1 \bigotimes (A - \lambda)^* f_1.$$

From (1.1), $A - \lambda \sim \begin{pmatrix} 0 & C_1 \\ 0 & A - \lambda \end{pmatrix}$. Since

$$\begin{pmatrix} 1 & -1 \bigotimes f_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & C_1 \\ 0 & A - \lambda \end{pmatrix} \begin{pmatrix} 1 & -1 \bigotimes f_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & A - \lambda \end{pmatrix},$$

$$A - \lambda \sim \begin{pmatrix} 0 & 0 \\ 0 & A - \lambda \end{pmatrix}.$$

This contradicts the condition $\dim\ker(\lambda - A) = 1$. Thus $\alpha \neq 0$.

Set $X = \begin{pmatrix} \alpha^{-1} & -1 \bigotimes f_1 \\ 0 & 1 \end{pmatrix}$. Then

$$X \begin{pmatrix} \lambda & C_1 \\ 0 & A \end{pmatrix} X^{-1} = \begin{pmatrix} \lambda & 1 \bigotimes e_0 \\ 0 & A \end{pmatrix}.$$

Thus $A \sim \begin{pmatrix} \lambda & 1 \bigotimes e_0 \\ 0 & A \end{pmatrix}$.

**Lemma 2.4.** *Given* $A \in \mathcal{L}(\mathcal{H})$, *let* $T = \begin{pmatrix} F & C \\ 0 & A \end{pmatrix} \in \mathcal{L}(\mathcal{C}^n \bigoplus \mathcal{H})$, *where* $F$ *is an* $n \times n$ *matrix satisfying that* $\sigma(F) \subset \sigma(A) \cap \rho_F(A)$ *and* $\dim\ker(\lambda - A) = \mathrm{ind}(\lambda - A) = 1$ *for each* $\lambda \in \sigma(F)$. *Then for* $\epsilon > 0$ *there exists* $K$ *compact with* $||K|| < \epsilon$ *such that*

$$T + K \sim A.$$

**Proof.** When $n = 1$, $T = \begin{pmatrix} \lambda & C \\ 0 & A \end{pmatrix}$, where $C$ is an operator of rank 1, i.e., $C = a \bigotimes f$ for some $f$ in $\mathcal{H}$. If $f \notin \mathrm{Ran}(\lambda - A)^*$, $T \sim A$ by Lemma 2.3. If $f \in \mathrm{Ran}(A - \lambda)^*$, choose $e_0 \in \ker(\lambda - A)$, thus $f + e_0 \notin \mathrm{Ran}(\lambda - A)^*$. Set

$$K = \begin{pmatrix} 0 & \epsilon(\alpha \bigotimes e_0) \\ 0 & 0 \end{pmatrix}.$$

Thus $T + K \sim A$ by Lemma 2.3. Assume that the conclusion of the lemma is true for $n \leq k - 1$. We shall prove that lemma is true for $n = k$. Let $\lambda_0 \in \sigma(F)$. Then there exists $U$ unitary such that

$$U \begin{pmatrix} F & C \\ 0 & A \end{pmatrix} U^* = \begin{pmatrix} \lambda_0 & C_1 & C_2 \\ 0 & F_1 & C_3 \\ 0 & 0 & A \end{pmatrix} \begin{array}{l} \mathcal{C} \\ \mathcal{C}^{n-1} \\ \mathcal{H} \end{array} = A_1.$$

By the induction assumption, there exist $K_1'$ compact with $||K_1'|| < \epsilon/4$ and $X_1$ invertible such that

$$X_1 \left[ \begin{pmatrix} F_1 & C_3 \\ 0 & A \end{pmatrix} + K_1' \right] X_1^{-1} = A.$$

Thus

$$\begin{pmatrix} 1 & 0 \\ 0 & X_1 \end{pmatrix} (A_1 + K_1) \begin{pmatrix} 1 & 0 \\ 0 & X_1^{-1} \end{pmatrix} = \begin{pmatrix} \lambda_0 & C' \\ 0 & A \end{pmatrix}_{\mathcal{H}}^{\mathcal{C}},$$

where $K_1 = \begin{pmatrix} 0 & 0 \\ 0 & K_1' \end{pmatrix}_{\mathcal{C}^{n-1} \bigoplus \mathcal{H}}^{\mathcal{C}}$. The proof of the part "$n = 1$" implies that the conclusion of the lemma is true.

**Lemma 2.5.** *Given $A \in \mathcal{B}_1(\Omega)$ and $\lambda_0 \in \Omega$, then there exist an ONB $\{e_n\}_{n=1}^{\infty}$ of $\mathcal{H}$ and an $r > 0$ such that*

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \cdots \\ 0 & \lambda_0 & a_{23} & \cdots \\ & & \lambda_0 & \\ & & & \ddots \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{matrix}$$

*and $|a_{k\ k+1}| > r > 0$ $(k = 1, 2, \cdots)$.*

**Proof.** Assume that $e_1 \in \ker(A - \lambda_0)$ with $||e_1|| = 1$. Let $B$ be the right inverse of $A - \lambda_0$. Since $\ker B^* = \{0\}$ $(k = 1, 2, \cdots)$ and since $e_1 \notin \mathrm{Ran} B$, $\{e_1,\ Be_1, B^2 e_1, \cdots\}$ is linearly independent. Since $B^{k-1} e_1 \in \ker(A - \lambda_0)^k$ and since $\dim\ker(A - \lambda_0)^k = k$,

$$\ker(A - \lambda_0)^k = \bigvee \{e_1,\ Be_1, \cdots, B^{k-1} e_1\}\ (k = 1, 2, \cdots).$$

Since $\bigvee\{\ker(A - \lambda_0)^k :\ k = 1, 2, \cdots\} = \mathcal{H}$,

$$\bigvee \{B^k e_1,\ k = 0, 1, 2, \cdots\} = \mathcal{H}.$$

Let $\{e_k\}_{k=1}^{\infty}$ be the Gram-schmidt orthonormalization of $\{B^k e_1\}_{k=0}^{\infty}$. Then $A$ is an upper triangular matrix representation

$$A = \begin{pmatrix} \lambda_0 & a_{12} & a_{13} & \cdots \\ 0 & \lambda_0 & a_{23} & \cdots \\ & & \lambda_0 & \\ & & & \ddots \end{pmatrix}$$

with respect to the ONB $\{e_k\}_{k=1}^{\infty}$. Note that $(A - \lambda_0)|_{\ker(A-\lambda_0)}$ is bounded from below, thus there exists $r > 0$ such that $||(A - \lambda_0) y|| \geq r||y||$ for each $y \in [e_1]^{\perp}$. Set $x_k = a_{k\ k+1} e_k$ and $x_k' = -\sum\limits_{i=1}^{k-1} a_{k\ k+1} e_k$ $(k = 1, 2, \cdots)$. Since $A - \lambda_0$ is onto, there is a vector $y_k \in \bigvee\{e_i,\ i = 2, 3, \cdots, k\}$ such that $-x_k' = (A - \lambda_0) y_k$. Thus $x_k = (A - \lambda_0)(e_{k+1} + y_k)$ and

$$\begin{aligned} |a_{k\ k+1}| &= ||x_k|| = ||(A - \lambda_0)(e_{k+1} + y_k)|| \geq r||e_{k+1} + y_k|| \\ &= r\sqrt{||e_{k+1}||^2 + ||y_k||^2} \geq r \quad (k = 1, 2, \cdots). \end{aligned}$$

**Lemma 2.6.** *Let $M$ be an almost normal operator on Hilbert space $\mathcal{H}$ with connected and perfect spectrum $\sigma(M) (= \sigma)$. And let $\{\mu_n\}_{n=1}^{\infty}$ be a dense subset of $\sigma$. Then for each $\delta > 0$ there exists $K$ compact with $||K|| < \delta$ satisfying*

(i) *$N = M + K \in (SI)$ and $\sigma(N) = \sigma(M) = \sigma$;*

(ii) *$\bigvee\{\ker(\lambda - N)^m;\ \lambda \in \sigma_p(N);\ m \geq 1\} = \mathcal{H}$ and $\{\mu_n\}_{n=1}^{\infty} \subset \sigma_p(N)$.*

**Proof.** Denote $G_n = \{z;\ \mathrm{dist}(z,\sigma) < \epsilon^2/2^{n+1}\}$. Then $G_n$ is a connected open set, $G_n \supset G_{n+1}$ $(n = 1, 2, \cdots)$ and $\bigcap\limits_{n=1}^{\infty} G_n = \sigma$. Choose a smooth route $r_1(t)$ of $G_1$ such that $r_1(0) = r_1(1) = \mu_1$, $r_1(t)$ passes through $\mu_2$ and choose $m_1$ points $s_1^1,\ s_2^1, \cdots, s_{m1}^1$ on $r_1(t)$ satisfying

(a) $s_1^1 = \mu_1,\ \mu_2 \in \{s_1^1,\ s_2^1, \cdots, s_{m1}^1\}$,

(b) $0 < |s_j^1 - s_{j-1}^1| < \epsilon^2/4;\ 0 < |s_{m_1}^1 - s_1^1| < \epsilon^2/4$.

Choose pairwise distinct points $\{\lambda_j^1\}_{j=1}^{m_1}$ in $\sigma$ such that

$$|s_j^1 - \lambda_j^1| < \frac{3}{2}\,\mathrm{dist}\,(s_j^1, \sigma); \quad \lambda_1^1 = s_1^1 = \mu_1.$$

Similarly, for each natural number $n$, we can find a smooth route $r_n(t)$ of $G_n$ such that $r_n(0) = r_n(1) = \mu_1$, $r_n(t)$ passes through $\{\lambda_j^{n-1};\ 1 \le j \le m_{n-1}\}$ and $\mu_n$. Choose $m_n$ points $s_1^n,\ s_2^n, \cdots, s_{m_n}^n$ on $r_n(t)$ satisfying

$$s_1^n = \mu_1;\ \{\lambda_j^{n-1},\ j = 1, 2, \cdots, m_{n-1}\} \cup \{\mu_1,\ \mu_2, \cdots, \mu_n\} \subset \{s_j^n,\ 1 \le j \le m_n\}$$

and

$$0 < |s_j^n - s_{j-1}^n| < \frac{\epsilon^2}{2(n+1)}, \quad 0 < |s_{m_n}^n - \mu_1| < \frac{\epsilon^2}{2(n+1)}.$$

Choose pairwise distinct points $\{\lambda_j^n\}_{j=1}^{m_n}$ in $\sigma$ such that

$$|s_j^n - \lambda_j^n| < \frac{3}{2}\mathrm{dist}(s_j^n, \sigma); \quad \lambda_1^n = s_1^n = \mu_1.$$

It is easy to see that $\{\lambda_j^n :\ j = 1, \cdots, m_n,\ n = 1, 2, \cdots\}$ is dense in $\sigma$ and $\{\mu_n\}_{n=1}^{\infty} \subset \{\lambda_j^n : j = 1, \cdots, m_n,\ n = 1, 2, \cdots\}$. Denote

$$\lambda_1 = \lambda_1^1,\ \lambda_2 = \lambda_2^1, \cdots, \lambda_{m_1} = \lambda_{m_1}^1,\ \lambda_{m_1+1} = \lambda_1^2, \cdots, \lambda_{m_1+m_2} = \lambda_{m_2}^2, \cdots.$$

Thus

(i) $\{\lambda_k\}_{k=1}^{\infty}$ is dense in $\sigma$ (without loss of generality, we can assume that $0 \notin \{\lambda_k\}_{k=1}^{\infty}$),

(ii) $\lim\limits_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0$,

(iii) $\mathrm{card}\{n;\ \lambda_k = \lambda_n\} = \infty,\ k = 1, 2, \cdots$.

Let $\{e_k\}_{k=1}^{\infty}$ be an ONB of $\mathcal{H}$. Define $De_k = \lambda_k e_k$. Then $\sigma(D) = \sigma(M) = \sigma$. By Theorem 1 of [6], there exists $K_1$ compact with $||K_1|| < \delta/2$ such that $X(M + K_1)X^{-1} = D$ for some $X \in (\mathcal{U} + \mathcal{K})(\mathcal{H})$. Thus it is sufficent to prove that for each $\epsilon > 0$, there exists a compact $K_2$ with $||K_2|| < \epsilon$ such that $D_2 + K_2 \in (SI)$. Define $K_2 e_{k-1} = \alpha_k e_k$, where $\alpha_k = \sqrt{|\lambda_{k+1} - \lambda_k|}$ $(k = 1, 2, \cdots)$. Since $\lim \alpha_k = 0$, $K_2$ is compact. If

$$P = \begin{array}{c} \begin{pmatrix} x_1 & x_{12} \cdots\cdots\cdots \\ x_{21} & x_2 \cdots\cdots\cdots \\ x_{31} & x_{32} \quad x_3 \quad \cdots \\ \cdots\cdots & \cdots\cdots\cdots\cdots \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \\ e_3 \\ \vdots \end{array} \in \mathcal{A}'(D + K_2)$$

is idempotent, since $P(D + K_2) = (D + K_2)P$, $\lambda_1 x_1 + \alpha_1 x_{21} = x_1 \lambda_1$; or $x_{21} = 0$. Since $\lambda_2 x_{21} + \alpha_2 x_{31} = x_{21}\lambda_1$, $x_{31} = 0, \cdots$, we can prove that $x_{k1} = 0$ $(k = 2, 3, \cdots)$. Similarly, $x_{ij} = 0$ $(i > j)$, i.e., $P$ admits an upper triangular matrix representation with respect to the ONB $\{e_k\}_{k=1}^{\infty}$. Since $P^2 = P$, $x_k = 0$ or 1. Since $P(D + K_2) = (D + K_2)P$ and $P$ is an upper trangular matrix,

$$x_{k\ k+1} = \frac{x_k - x_{k+1}}{\lambda_{k+1} - \lambda_k}\alpha_k\ (k = 1, 2, 3, \cdots).$$

Since $x_k = 0$ or 1, $|x_k - x_{k+1}| = 0$ or 1. Since $\lim_{k\to\infty} |\lambda_{k+1} - \lambda_k| = 0$ and since $\alpha_k = \sqrt{|\lambda_{k+1} - \lambda_k|}$, when $m$ is big enough $x_k = x_m$ for $k \geq m$. Assume that $x_m = 0$ (if $x_m = 1$, consider $I - P$) and $m$ is the smallest number satisfying $x_k = 0$, when $k \geq m$, i.e., $x_{m-1} \neq 0$. Set

$$D_1 = \begin{pmatrix} \lambda_1 & \alpha_1 & & & 0 \\ 0 & \lambda_2 & \alpha_2 & & \\ 0 & 0 & \lambda_3 & \ddots & \\ & & & \ddots & \alpha_{m-1} \\ & & & & \lambda_{m-1} \end{pmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_{m-2} \\ e_{m-1} \end{matrix} \quad \text{on} \quad \bigvee_{k=1}^{m-1} \{e_k\} = \mathcal{H}_1,$$

$$D_2 = \begin{pmatrix} \lambda_m & \alpha_m & & \\ 0 & \lambda_{m+1} & & \\ & \ddots & & \\ & & & \ddots \end{pmatrix} \begin{matrix} e_m \\ e_{m+1} \\ \vdots \\ \vdots \end{matrix} \quad \text{on} \quad \bigvee_{k=m}^{\infty} \{e_k\} = \mathcal{H}_2,$$

and

$$Y = \begin{pmatrix} 0 & 0 & \dots \\ \dots & \dots & \dots \\ \lambda_{m-1} & 0 & \dots \end{pmatrix} \in \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1).$$

Thus

$$D + K_2 = \begin{pmatrix} D_1 & Y \\ 0 & D_2 \end{pmatrix},$$

and $P = \begin{pmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{pmatrix}$ with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$. Since $P_{22}^2 = P_{22},\ x_k = 0;\ k \geq m,\ P_{22} = 0$. Thus

$$P_{11} = \begin{pmatrix} x_1 & x_2 & \dots & x_{1\ m-1} \\ 0 & x_2 & \dots & x_{2\ m-1} \\ & & \ddots & \\ & & & x_{m-1} \end{pmatrix}.$$

Suppose that

$$P_{12} = \begin{pmatrix} l_{11} & l_{12} & l_{13} & \dots \\ l_{21} & l_{22} & l_{23} & \dots \\ \dots & \dots & \dots & \dots \\ l_{m-1\ 1} & l_{m-1\ 2} & l_{m-1\ 3} & \dots \end{pmatrix}.$$

Since

$$\begin{aligned} &P(D + K_2) = (D + K_2)P, \quad P_{12}D_2 + P_{11}Y = D_1P_{12}, \\ &x_{m-1}\alpha_{m-1} + l_{m-1,\ 1}\lambda_m = \lambda_{m-1}l_{m-1,\ 1}, \\ &l_{m-1,m+j-1}\alpha_{m+j-1} + l_{m-1,m+j}(\lambda_{m+j} - \lambda_{m-1}) = 0\ (j \geq 1). \end{aligned}$$

. Since card$\{n : \lambda_j = \lambda_0\} = \infty$, there is a natural number $j_0$ such that $\lambda_{m+j_0} = \lambda_{m-1}$. This shows that $l_{m-1,m+j_0-1} = 0$. By induction, we have $l_{m-1,1} = 0$. Thus $x_{m-1} = 0$. The contradiction implies that $m = 1$, i.e., $P = 0$ and $D + K_2 \in (SI)$.

Suppose $\{\Omega_i\}_{i=1}^{\infty}$ is a sequence of connected, uniformly bounded open subsets of $\mathcal{C}$, and

$\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$. Let $B_i \in \mathcal{B}_1(\Omega_i)$ on $\mathcal{H}_i$ $(i = 1, 2, \cdots)$, $\sigma(B_i) = \overline{\Omega_i}$, and

$$T_i = \bigoplus_{k=1}^{n_i} B_i,\ 1 \leq n_i \leq +\infty, i = 1, 2, \cdots .$$

Suppose

$$A = \bigoplus_{i=1}^{l} T_i \in \mathcal{L}\Big(\bigoplus_{i=1}^{l}\bigoplus_{k=1}^{n_i} \mathcal{H}_i\Big),\ 1 \leq l \leq +\infty.$$

Let $F \in \mathcal{L}(\mathcal{C}^m)$ with $\sigma(F) \subset \sigma(A) \cap \rho_F(A)$. Then the following lemma holds.

**Lemma 2.7.** *For each $\epsilon > 0$, there exists a compact $K$ with $\|K\| < \epsilon$ such that*

$$\begin{pmatrix} F & G \\ 0 & A \end{pmatrix} + K \ \sim A,$$

*where $G \in \mathcal{L}(\bigoplus\limits_{i=1}^{l}\bigoplus\limits_{k=1}^{n_i} \mathcal{H}_i, \mathcal{C}^m)$.*

**Proof.** Using Lemma 2.4 repeatedly, we come to the conclusion immediately.

**Lemma 2.8.**[6, Theorem 1] *Let $\sigma$ be a perfect compact subset of $\mathcal{C}$ and $\{\lambda_k\}_{k=1}^{\infty}$ be a dense subset of $\sigma$. Assume that* card$\{n :\ \lambda_k = \lambda_n\} = \infty$ *and $D =$*diag$\{\lambda_1, \cdots, \}$ *on $\mathcal{H}$. Let* $T = \begin{pmatrix} F_m & G_m \\ 0 & D \end{pmatrix}$ *on $\mathcal{C}^m \bigoplus \mathcal{H}$, where $F_m \in \mathcal{L}(\mathcal{C}^m)$ with $\sigma(F_m) \subset \sigma$ and $G_m \in \mathcal{L}(\mathcal{H}, \mathcal{C}^m)$. Then for each $\epsilon > 0$, there exists $K$ compact with $||K|| < \epsilon$ such that $T + K \underset{\mathcal{U}+\mathcal{K}}{\simeq} D$.*

**Lemma 2.9.**[5, Lemma 2.5] *Let $\Omega$ be a bounded connected open subset of $\mathcal{C}$ and $n$ be a natural number. Let $\delta, \epsilon$ be two positive numbers. Then there exist an essentially normal operator $B \in \mathcal{B}_1(\Omega)$ and a co-subnormal operator $\overline{B}$, and compact $K,\ K_1,\ K_2$ with $||K|| < \delta,\ ||K_1|| < \epsilon,\ ||K_2|| < \epsilon$ such that*

(i) $B = \overline{B} + K$*;*

(ii) $\sigma(B) = \overline{\Omega},\ \sigma(B) \cap \rho_F(B) = \Omega$*;*

(iii) $\partial\overline{\Omega}^0 \cap \sigma_p(B) = \emptyset$*;*

(iv) $T = B^{(n)} + K_1 \in \mathcal{B}_n(\Omega);\ T \in (SI)$*;*

(v) ker$\tau_{B,T} = \{0\};\ K_2 \notin$Ran$\tau_{T,B}$*.*

*Here $\mathcal{B}_n(\Omega)$ is the set of Cowen-Douglas operators of index $n$; $B^{(n)} = \bigoplus\limits_{1}^{n} B$ on $\bigoplus\limits_{1}^{n} \mathcal{H}$.*

Let $T \in \mathcal{L}(\mathcal{H})$ be an essentially normal operator with connected $\sigma(T)$. Assume that ind$(\lambda - T) > 0$ $(\lambda \in \rho_F(T) \cap \sigma(T))$. Let $\{\Omega_i\}_{i=1}^{l}$ $(1 \leq l \leq \infty)$ be the connected components of $\rho_F^+(T)$ $(\rho_F^+(T) = \{\lambda \in \mathcal{C};$ ind$(\lambda - T) > 0\})$ and let $\sigma = \sigma(T) \setminus \cup\overline{\Omega}_i^0$. Denote $n_i =$ind$(T - \lambda)$ $(\lambda \in \Omega_i)$. Let $B_i \in B_1(\Omega_i)$. By Lemma 2.9, $\sigma_p(B_i) \cap \partial\overline{\Omega}_i^0 = \emptyset$. Assume that $\{\lambda_j\}_{j=1}^{\infty}$ is a dense subset of $\sigma$ satisfying card$\{k :\ \lambda_n = \lambda_k\} = \infty$. Denote $D =$diag$\{\lambda_1, \lambda_2, \cdots, \}$ and $G = \bigoplus\limits_{1}^{l} T_i \bigoplus D$. Then we have

**Lemma 2.10.** (i) $\Lambda(G) = \Lambda(T)$*;*

(ii) *For each $\epsilon > 0$, there exists $K$ compact with $||K|| < \epsilon$ such that $T + K \sim G$, where $\Lambda(G)$ denotes the spectrum pictures of $G$.*

**Proof.** Denote the acting space of $T_i$ $(1 \leq i \leq l)$ and $D$ by $\mathcal{H}_i$ and, respectively, $\mathcal{H}_\infty$. Then it is obvious that $\Lambda(G) = \Lambda(T)$. Thus by B. D. F. Theorem[8] there exists $K_0$ compact and $U$ unitary such that

$$UTU^* = G + K_0.$$

Let $\{e_k^\infty\}_{k=1}^\infty$ be an ONB of $\mathcal{H}_\infty$ satisfying $De_k^\infty = \lambda_k e_k^\infty$. Assume that $\lambda_k^i \in \Omega_k$ $(k = 1, 2, \cdots)$ and

$$\mathcal{M}_1 = \bigvee_{k=1}^{m} \ker(T_k - \lambda_k^i)^m, \quad \mathcal{M}_2 = \bigvee\{e_1^\infty,\ e_2^\infty, \cdots, e_m^\infty\},$$

where $m$ is a natural number. Let $P_{m_1}$ and $P_{m_2}$ be the orthogonal projections onto $\mathcal{M}_1$ and respectively, $\mathcal{M}_2$. Set $P_m = P_{m_1} + P_{m_2}$. Then $P_m \overset{\text{SOT}}{\longrightarrow} I$ $(m \to \infty)$. Thus there is a natural number $m_0$ such that $||K_1|| < \epsilon/8$, where $K_1 = P_{m_0} K_0 P_{m_0} - K_0$, and

$$\begin{aligned}
G + K_0 + K_1 &= G + P_{m_0} K_0 P_{m_0} \\
&= \begin{pmatrix} P_{m_{01}}(\bigoplus_1^m T_i)P_{m_{01}} & P_{m_{01}}(\bigoplus T_i)P_{m_{01}}^\perp & 0 & 0 \\ 0 & P_{m_{01}}^\perp(\bigoplus T_i)P_{m_{01}}^\perp & 0 & 0 \\ 0 & 0 & P_{m_{02}} D P_{m_{02}} & 0 \\ 0 & 0 & 0 & P_{m_{02}}^\perp D P_{m_{02}}^\perp \end{pmatrix} \\
&\quad + \begin{pmatrix} P_{m_{01}} K_0 P_{m_{01}} & 0 & P_{m_{01}} K_0 P_{m_{02}} & 0 \\ 0 & 0 & 0 & 0 \\ P_{m_{02}} K_0 P_{m_{01}} & 0 & P_{m_{02}} K_0 P_{m_{02}} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} K_{11} & K_{13} & K_{12} & 0 \\ 0 & P_{m_{01}}^\perp(\bigoplus T_i)P_{m_{01}}^\perp & 0 & 0 \\ K_{21} & 0 & K_{22} & 0 \\ 0 & 0 & 0 & P_{m_{02}}^\perp D P_{m_{02}}^\perp \end{pmatrix} \\
&\cong \begin{pmatrix} K_{11} & K_{12} & K_{13} & 0 \\ K_{21} & K_{22} & 0 & 0 \\ 0 & 0 & P_{m_{01}}^\perp(\bigoplus T_i)P_{m_{01}}^\perp & 0 \\ 0 & 0 & 0 & D \end{pmatrix},
\end{aligned}$$

i.e., there exists $U_1$ unitary such that

$$U_1(G + K_0 + K_1)U_1^* = \begin{pmatrix} L_{11} & L_{12} & 0 \\ 0 & P_{m_{01}}^\perp(\bigoplus T_i)P_{m_{01}}^\perp & 0 \\ 0 & 0 & D \end{pmatrix},$$

where

$$L_{11} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}, \quad L_{12} = \begin{pmatrix} K_{13} \\ 0 \end{pmatrix}.$$

From the upper contintuity of spectrum, $\sigma(L_{11}) \subset \sigma(T)_{\epsilon/8}$. Therefore there exists an operator $L$ with $||L|| < \epsilon/4$ such that $\sigma(L_{11} + L) \subset \sigma(T)$ and

(a) $\overline{X}_2(L_{11} + L)\overline{X}_2^{-1}$ =diag$\{\mu_1, \cdots, \mu_m\} = D_1$ for some $\overline{X}_2$ invertible;

(b) For each $i$, $\mu_i \in \rho_F(\bigoplus T_i) \cap \sigma(\bigoplus T_i)$ or $\mu_i \notin \sigma(\bigoplus T_i)$ and $\mu_i \in \sigma(D_1)$.

Without loss of generality, we can assume that

$$\{\mu_1, \cdots, \mu_p\} \subset \sigma(D), \quad \{\mu_{p+1}, \cdots, \mu_m\} \subset \rho_F(\bigoplus T_i) \cap \sigma(\bigoplus T_i).$$

Set $D_{11}$ =diag$\{\mu_1, \cdots, \mu_p\}$, $D_{22}$ =diag$\{\mu_{p+1}, \cdots, \mu_m\}$. Then $D_1 = D_{11} \bigoplus D_{22}$. By Lemma 2.2, $P_{m_{01}}^\perp(\bigoplus T_i)P_{m_{01}}^\perp \sim T$. Thus there exist $K_2$ compact with $||K_2|| < \epsilon/8$ and $X_1$ invertible such that

$$\begin{aligned}
&X_1U_1(G+K_0+K_1+U_1^*K_2U_1)U_1^*X_1^{-1}\\
&=X_1\left[\begin{pmatrix} L_{11} & L_{12} & 0\\ 0 & P_{m_{01}}^{\perp}(\bigoplus T_i)P_{m_{01}}^{\perp} & 0\\ 0 & 0 & D\end{pmatrix}+K_2\right]X_1^{-1}\\
&=\begin{pmatrix} D_{11} & 0 & \overline{L}_1 & 0\\ 0 & D_{22} & \overline{L}_2 & 0\\ 0 & 0 & \bigoplus T_i & 0\\ 0 & 0 & 0 & D\end{pmatrix}=G_2,
\end{aligned}$$

where $\overline{L}_1$, $\overline{L}_2$ are still compact operators. Therefore it is sufficient to show that for each $\delta>0$, there exists $\overline{K}$ compact with $||\overline{K}||<\delta$ such that $G_2+\overline{K}\sim G=\bigoplus\limits_{i=1}^{l}T_i\bigoplus D$. Since $\{\mu_i\}_{k=1}^{p}\cap\sigma(\bigoplus T_i)=\emptyset$, and Lemma 3.22 of [9], $\tau_{D_{11},\bigoplus T_i}$ is a surjection, there is $Y_1$ such that $D_{11}Y_1-Y_1\bigoplus T_i=\overline{L}_1$. Set

$$X_3=\begin{pmatrix} I & 0 & -Y_1 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{pmatrix},\quad X_3G_2X_3^{-1}=\begin{pmatrix} D_{11} & 0 & 0 & 0\\ 0 & D_{22} & \overline{L}_2 & 0\\ 0 & 0 & \bigoplus T_i & 0\\ 0 & 0 & 0 & D\end{pmatrix}=G_3.$$

By Lemma 2.7, there exist $\overline{K}_4$ compact with $||\overline{K}_4||<\delta/8||X_3||||X_3^{-1}||$ and $\overline{X}_4$ invertible of the form unitary plus compact such that

$$\overline{X}_4\left[\begin{pmatrix} D_{22} & \overline{L}_2\\ 0 & \bigoplus T_i\end{pmatrix}+\overline{K}_4\right]=\bigoplus T_i.$$

Therefore there are $X_4$ invertible, $K_4$ compact with $||K_4||<\delta/8||X_3||||X_3^{-1}||$ such that

$$X_4(G_3+K_4)X_4^{-1}=\begin{pmatrix} D_{11} & 0 & 0\\ 0 & \bigoplus T_i & 0\\ 0 & 0 & D\end{pmatrix}\cong\begin{pmatrix} \bigoplus T_i & 0 & 0\\ 0 & D_{11} & 0\\ 0 & 0 & D\end{pmatrix}\underset{\mathcal{U}+\mathcal{K}}{\simeq}\begin{pmatrix} \bigoplus T_i & 0\\ 0 & D\end{pmatrix}.$$

The last unitary equivalent relation comes from Lemma 2.8.

To summarize, there exist $K$ compact with $||K||<\delta$ and $X$ invertible such that $X(G_2+K)X^{-1}=G=(\bigoplus T_i)\bigoplus D$.

**Lemma 2.11.**[4,Lemma 2.6] *Suppose $A$, $B\in\mathcal{L}(\mathcal{H})$ such that there are $\Lambda_1\subset\sigma_p(B)$ and $\Lambda_2\subset\sigma_p(A)$ satisfying*

(i) $\Lambda_1\cap\sigma_p(A)=\emptyset$,

(ii) $\bigvee\limits_{\lambda\in\Lambda_2}\{\ker(A-\lambda)\}=\mathcal{H}$.

*Then for each $\epsilon>0$, there exists $K$ compact with $||K||<\epsilon$ such that $K\notin\mathrm{Ran}\tau_{A,B}$.*

## §3. Proof of Main Theorem

Assume that $G=\left(\bigoplus\limits_{i=1}^{l}T_i\right)\bigoplus D$ is given in Lemma 2.9 and Lemma 2.10. Thus for each $\epsilon>0$, there exist a compact $K$ with $||K||<\epsilon$ and an invertible $X$ such that $X(T+K)X^{-1}=G$. Thus it is sufficient to show that for each $\delta>0$, there exists a compact $\overline{K}$ with $||\overline{K}||<\delta$ such that $G+\overline{K}\in(SI)$. If $\sigma(T)=\sigma_{lre}(T)$ (i.e., $l=0$), we can obtain Main Theorem by using Lemma 2.6 and Lemma 2.8. If $\sigma_{lre}(T)\neq\sigma(T)$, set $G_1=B_1\bigoplus(\bigoplus\limits_{i=2}^{l}T_i)\bigoplus D$. By

Theorem 2 of [10], we can find a compact $\overline{K}_1$ with $\|\overline{K}_1\| < \delta/3$ such that $G_1 + \overline{K}_1 \in \mathcal{B}_1(\Omega_1)$. Set $A = G_1 + \overline{K}_1$ and then $A$ is strongly irreducible. Thus there exists a compact $K_1$ with $\|K_1\| < \delta/3$ such that

$$G + K_1 = \Big(\bigoplus_{k=1}^{n-1} B_1\Big) \bigoplus A.$$

Similar to the proof of the Main Theorem in [4], we can find a compact $\overline{K}_2$ with $\|\overline{K}_2\| < \delta/3$ such that

$$B = \bigoplus_{k=1}^{n_i-1} B_1 \,+\, \overline{K}_2 \in (SI)$$

and either $\mathrm{Ker}\tau_{B,A} = \{0\}$ or $\mathrm{Ker}\tau_{A,B} = \{0\}$. Without loss of generality, we assume that $\mathrm{Ker}\tau_{A,B} = \{0\}$. Since $\sigma_{lre}(A) \cap \sigma_{lre}(B) \neq$, there is a compact $\overline{K}_3$ with $\overline{K}_3 \notin \mathrm{Ran}\tau_{B,A}$ and $\|\overline{K}_3\| < \delta/3$. Thus we can find a compact $K$ with $\|K\| < \delta$ such that

$$G + K = \begin{pmatrix} B & \overline{K}_3 \\ & A \end{pmatrix}.$$

A simple computation shows that $G + K \in (SI)$.

## References


[1] Jiang, Z. J., A lecture on operator theory, the report in the seminar of functional analysis in Jilin Univ., Changchun, 1979.

[2] Gifeather, F., Strongly reducibility of operators, *Indiana Univ. Math. J.,* **22**(1972).

[3] Herrero, D. A. & Jiang, C. L., Limits of strongly irreducible operators and the Riesz decomposition theorem, *Mich. Math. J.,* **37**(1990), 283–291.

[4] Jiang, C. L. & Wang, Z. Y., The spectral picture and the closure of the similarity orbit of strongly irreducible operator, *Intergral Equation Operator Theory*, **24**(1996), 81–105.

[5] Jiang, C. L., Sun, S. H. & Wang, Z. Y., Essentially normal operator + compact operator = strongly irreducible operator, *Trans. Amer. Math. Soc.,* **349:**1, 217–233.

[6] Ji, Y. Q., Jiang, C. L. & Wang, Z. Y., The $(\mathcal{U}+\mathcal{K})$-orbit of essentially normal operators and compact perturbation of strongly irreducible operators, Functional Analysis in China, Kluwer Academic Publishers, 1996, Math. and Its Appl., Vol. 356, 307–314.

[7] Cowen, M. & Douglas, R. G., Complex geometry and operator theory, *Acta Math.,* **141**(1978), 187–261.

[8] Brown, L. G., Douglas, R. G. & Fillmore, P. F., Unitary equivalent modelo the compact operators and extensions of $C^*$-algebras, Proceedings of conference on operator theory, Halifax, Nova scotia 1973, *Lect. Notes in Math.,* **345** (1973), 98-125.

[9] Herrero, D. A., Approximation of Hilbert space operators, Vol 1 and ed., Pitman Res. Notes Math. 224 1 Longman Sci. Tech. Harlou, Essex, 1990.

[10] Herrero, D. A., Spectral pictures of operators in the Cowen-Douglas class $\mathcal{B}_n(\Omega)$ and its closur, *J. Operator Theory,* **18**(1987), 213–222.