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THE RADICALS OF HOPF MODULE ALGEBRAS

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Abstract

The characterization of H-prime radical is given in many ways. Meantime, the relations between the radical of smash product $A\#H$ and the H-radical of Hopf module algebra A are obtained.

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§0. Introduction

Let k be a commutative associative ring with unit, H be an algebra with unit and comultiplication $\triangle : H \to H \otimes H$, A be an algebra over k (A can be without unit). A is called an H-module algebra if the following conditions hold:

(i) A is a unital left H-module (i.e., $1_H \cdot a = a$ for any $a \in A$);

(ii) $h \cdot ab = \sum(h_1 \cdot a)(h_2 \cdot b)$ for any $a, b \in A$, $h \in H$, where $\triangle h = \sum h_1 \otimes h_2$.

An H-module algebra A is called a unital H-module algebra if A has an identity element 1_A such that $h \cdot 1_A = \epsilon(h)1_A$ for any $h \in H$.

For any ideal I of A, set $(I : H) := \{x \in A \mid h \cdot x \in I \text{ for all } h \in H\}$. I is called an H-ideal, if $h \cdot I \subseteq I$ for any $h \in H$. Let I_H denote the maximal H-ideal of A in I. A is called H-semiprime, if there are no non-zero nilpotent H-ideals in A. A is called H-prime if $IJ = 0$ implies $I = 0$ or $J = 0$ for any H-ideals I and J of A. An H-ideal I is called an H-(semi)prime ideal of A if A/I is H-(semi)prime. A left A-module M is called an A-H-module if M is also a left unital H-module with $h \cdot (am) = \sum(h_1 \cdot a)(h_2 m)$ for all $h \in H, a \in A, m \in M$. An A-H-module M is called an irreducible A-H-module if there are no non-trivial A-H-submodules in M. An algebra homomorphism $\phi : A \to B$ is called an H-homomorphism if $\phi(h \cdot a) = h \cdot \phi(a)$ for any $h \in H, a \in A$. Let r_b and r_j denote the Baer radical and the Jacobson radical of algebras, respectively.

J. R. Fisher[7] built up the general theory of H-radicals for H-module algebras, studied H-Jacobson radical $r_{Hj}(A) := \cap\{(0 : M)_A \mid M \text{ is an irreducible } A\text{-}H\text{-module}\}$, and obtained

$$r_j(A\#H) \cap A = r_{Hj}(A) \tag{0.1}$$

for any irreducible Hopf algebra H (see [7, Theorem 4]). J. R. Fisher[7] asked when is

$$r_j(A\#H) = r_{Hj}(A)\#H \tag{0.2}$$

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and asked if

$$r_j(A\#H) \subseteq (r_j(A) : H)\#H. \tag{0.3}$$

If $H = (kG)^*$, then relation (0.2) holds (see [6, Theorem 4.1]).

In this paper, we show that the H-Baer radical $r_{Hb}(A)$ of A consists of H-m-nilpotent elements in A, and give some sufficient conditions for (0.2) and (0.3), respectively. We also show that (0.1) holds for any Hopf algebra H and give the formulae similar to (0.1), (0.2) and (0.3) for H-prime radical.

We denote the H-ideal I of A by $I \triangleleft_H A$. The smash product is the same as in [7] and the other notations are the same as in [11] and [12].

§1. The H-Special Radicals for H-Module Algebras

J. R. Fisherte[7] built up the general theory of H-radicals for H-module algebras. We can easily give the definitions of the H-upper radical and the H-lower radical for H-module algebras as in [12]. In this section, we obtain some properties of H-special radicals for H-module algebras. We omit all of the proofs, because they are similar to those in [13].

Proposition 1.1. *A is H-semiprime iff $(H\cdot a)A(H\cdot a) = 0$ always implies $a = 0$ for any $a \in A$.*

A is H-prime iff $(H\cdot a)A(H\cdot b) = 0$ always implies $a = 0$ or $b = 0$ for any $a, b \in A$.

Proposition 1.2. *If $I \triangleleft_H A$ and I is an H-semiprime module algebra, then*

(i) *$I \cap I^* = 0$,*

(ii) *$I_r = I_l = I^*$,*

(iii) *$I^* \triangleleft_H A$,*

where $I_r = \{a \in A \mid I(H\cdot a) = 0\}$, $I_l = \{a \in A| \ (H\cdot a)I = 0\}$, $I^* = \{a \in A| \ (H\cdot a)I = 0 = I(H\cdot a)\}$.

Definition 1.1. *$\mathcal{K}$ is called an H-(weakly) special class if*

(S1) *$\mathcal{K}$ consists of H-(semiprime) prime module algebras.*

(S2) *For any $A \in \mathcal{K}$, if $0 \neq I \triangleleft_H A$ then $I \in \mathcal{K}$.*

(S3) *A is an H-module algebra. If $B \triangleleft_H A$ and $B \in \mathcal{K}$, then $A/B^* \in \mathcal{K}$, where $B^* = \{a \in A \mid (H\cdot a)B = 0 = B(H\cdot a)\}$.*

It is clear that (S3) may be replaced by one of the following conditions:

(S3') If B is an essential H-ideal of A (i.e., $B \cap I \neq 0$ for any non-zero H-ideal I of A) and $B \in \mathcal{K}$, then $A \in \mathcal{K}$.

(S3") If there exists an H-ideal B of A with $B^* = 0$ and $B \in \mathcal{K}$, then $A \in \mathcal{K}$.

It is easy to check that if $\mathcal{K}$ is an H-special class, then $\mathcal{K}$ is an H-weakly special class.

Theorem 1.1. *If $\mathcal{K}$ is an H-weakly special class, then $r^{\mathcal{K}}(A) = \cap\{I \triangleleft_H A \mid A/I \in \mathcal{K}\}$, where $r^{\mathcal{K}}$ denotes the H-upper radical determined by $\mathcal{K}$.*

Definition 1.2. *If r is a hereditary H-radical (i.e., if A is an r-H-module algebra and B is an H-ideal of A, then so is B) and any nilpotent H-module algebra is an r-H-module algebra, then r is called a supernilpotent H-radical.*

Proposition 1.3. *If r is a supernilpotent H-radical, then r is H-strongly hereditary, i.e., $r(I) = r(A) \cap I$ for any $I \triangleleft_H A$.*

Theorem 1.2. *If $\mathcal{K}$ is an H-weakly special class, then $r^{\mathcal{K}}$ is a supernilpotent H-radical.*

Definition 1.3. *For $a \in A, \{a_n\}$ is called an H-m-sequence in H-module algebra A with beginning a if there exist $h_n, h'_n \in H$ such that $a_1 = a$ and $a_{n+1} = (h_n.a_n)b_n(h'_n.a_n)$ for any natural number n.*

Proposition 1.4. *A is H-semiprime iff for any $0 \neq a \in A$, there exists an H-m-sequence $\{a_n\}$ in A with $a_1 = a$ such that $a_n \neq 0$ for all n.*

§2. *H*-Baer Radical

In this section, we give the characterization of H-Baer radical (H-prime radical) in many ways. We omit all of the proofs because they are similar to the proofs in [13].

Theorem 2.1. *If $\mathcal{K} = \{A \mid A$ is an H-prime module algebra$\}$, then $\mathcal{K}$ is an H-special class and $r^{\mathcal{K}}(A) = \cap\{I \mid I$ is an H-prime ideal of $A\}$, which is called H-prime radical or H-Baer radical, written as r_{Hb}.*

Theorem 2.2. *A is an r_{Hb}-H-module algebra iff every non-zero H-homomorphic image of A contains a non-zero nilpotent H-ideal.*

Theorem 2.3. *Let $\mathcal{E} = \{A \mid A$ is a nilpotent H-module algebra$\}$, then $r_{\mathcal{E}} = r_{Hb}$, where $r_{\mathcal{E}}$ denotes the H-lower radical determined by $\mathcal{E}$.*

Proposition 2.1. *A is H-semiprime if and only if $r_{Hb}(A) = 0$.*

Definition 2.1. *We define an H-ideal N_α in H-module algebra A for every ordinal number α as follows:*

(i) *$N_0 = 0$. Let us assume that N_α is already defined for $\alpha \prec \beta$.*

(ii) *If $\beta = \alpha + 1, N_\beta/N_\alpha$ is the sum of all nilpotent H-ideals of A/N_α*

(iii) *If β is a limit ordinal number, $N_\beta = \sum\limits_{\alpha \prec \beta} N_\alpha$.*

By set theory, there exists an ordinal number τ such that $N_\tau = N_{\tau+1}$.

Theorem 2.4. *$N_\tau = r_{Hb}(A) = \cap\{I \mid I$ is an H-semiprime ideal of $A\}$.*

Definition 2.2. *Let $a \in A$. If for every H-m-sequence $\{a_n\}$ with $a_1 = a$, there exists a natural number k such that $a_k = 0$, then a is called an H-m-nilpotent element, written as $W_H(A) = \{a \in A \mid a$ is an H-m-nilpotent element$\}$.*

Theorem 2.5. *$r_{Hb}(A) = W_H(A)$.*

Definition 2.3. *Let $\Phi \neq L \subseteq H$. An H-m-sequence $\{a_n\}$ in A is called an L-m-sequence with beginning a if $a_1 = a$ and $a_{n+1} = (h_n.a_n)b_n(h'_n.a_n)$ such that $h_n, h'_n \in L$ for all n. If for every L-m-sequence $\{a_n\}$ with $a_1 = a$, there exists a natural number k such that $a_k = 0$, then a is called an L-m-nilpotent element, written as $W_L(A) = \{a \in A \mid a$ is an L-m-nilpotent element$\}$.*

Proposition 2.2. *If $L \subseteq H$ and $H = kL$, then*

(i) *A is H-semiprime iff $(L.a)A(L.a) = 0$ always implies $a = 0$ for any $a \in A$.*

(ii) *A is H-prime iff $(L.a)A(L.b) = 0$ always implies $a = 0$ or $b = 0$ for any $a, b \in A$.*

(iii) *A is H-semiprime if and only if for any $0 \neq a \in A$, there exists an L-m-sequence $\{a_n\}$ with $a_1 = a$ such that $a_n \neq 0$ for all n.*

(iv) *$W_H(A) = W_L(A)$.*

§3. The *H*-Module Theoretical Characterization of *H*-Special Radicals

In this section, let k be a commutative ring with unit, H be a Hopf algebra over k and

A be an H-module algebra over k (A can be without unit). We shall characterize H-Baer radical r_{Hb}, H-locally nil radical r_{Hl}, H-Jacobson radical r_{Hj} and H-Brown-McCoy radical r_{Hbm} by A-H-modules. We omit most of the proofs because they are similar to the proofs in [5].

Lemma 3.1. *If M is an A-H-module, then M is an $A\#H$-module. In this case,* $(0 : M)_{A\#H} \cap A = (0 : M)_A$ *and* $(0 : M)_A$ *is an H-ideal of A.*

Proof. Let γ be a map from H to $A\#H$ by $\gamma(h) = 1\#h$ for any $h \in H$. It is clear that γ is invertible with convolution inverse $\gamma^{-1} : h \mapsto 1\#S(h)$ and $h\cdot a = \sum \gamma(h_1)a\gamma^{-1}(h_2)$ for any $h \in H, a \in A$, where S is the antipode of H. Obviously, $(0 : M)_A = (0 : M)_{A\#H} \cap A$. For any $h \in H, a \in (0 : M)_A$, we see that $(h \cdot a)M = \sum \gamma(h_1)a\gamma^{-1}(h_2)M \subseteq \sum \gamma(h_1)aM = 0$. Thus $h \cdot a \in (0 : M)_A$, which implies that $(0 : M)_A$ is an H-ideal of A.

Definition 3.1. *An A-H-module M is called an A-H-prime module if for M the following conditions are fulfilled:*

(i) $AM \neq 0$,

(ii) *If x is an element of M and I is an H-ideal of A, then $I(Hx) = 0$ always implies $x = 0$ or $I \subseteq (0 : M)_A$.*

Definition 3.2. *We associate to every H-module algebra A a class $\mathcal{M}_A$ of A-H-modules. Then the class $\mathcal{M} = \cup\mathcal{M}_A$ is called an H-special class of modules if the following conditions are fulfilled:*

(M1) *If $M \in \mathcal{M}_A$, then M is an A-H-prime module.*

(M2) *If I is an H-ideal of A and $M \in \mathcal{M}_I$, then $IM \in \mathcal{M}_A$.*

(M3) *If $M \in \mathcal{M}_A$ and I is an H-ideal of A with $IM \neq 0$, then $M \in \mathcal{M}_I$.*

(M4) *Let I be an H-ideal of A and $\bar{A} = A/I$. If $M \in \mathcal{M}_A$ and $I \subseteq (0 : M)_A$, then $M \in \mathcal{M}_{\bar{A}}$. Conversely, if $M \in \mathcal{M}_{\bar{A}}$, then $M \in \mathcal{M}_A$.*

Let $\mathcal{M}(A)$ denote $\cap\{(0 : M)_A \mid M \in \mathcal{M}_A\}$.

Theorem 3.1. (i) *If $\mathcal{M}$ is an H-special class of modules and $\mathcal{K} = \{ A \mid$ there exists a faithful A-H-module $M \in \mathcal{M}_A\}$, then $\mathcal{K}$ is an H-special class and $r^{\mathcal{K}}(A) = \mathcal{M}(A)$.*

(ii) *If $\mathcal{K}$ is an H-special class and $\mathcal{M}_A = \{ M \mid M$ is an A-H-prime module and $A/(0 : M)_A \in \mathcal{K}\}$, then $\mathcal{M} = \cup\mathcal{M}_A$ is an H-special class of modules and $r^{\mathcal{K}}(A) = \mathcal{M}(A)$.*

Theorem 3.2. *Let $\mathcal{M}_A =\{ M \mid M$ is an A-H-prime module$\}$ for any H-module algebra A and $\mathcal{M} = \cup\mathcal{M}_A$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(A) = r_{Hb}(A)$.*

Theorem 3.3. *Let $\mathcal{M}_A =\{ M \mid M$ is an irreducible A-H-module$\}$ for any H-module algebra A and $\mathcal{M} = \cup\mathcal{M}_A$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(A) = r_{Hj}(A)$, where r_{Hj} denotes the H-Jacobson radical defined by J. R. Fisher[7].*

Let $r_b, r_k, r_l, r_j, r_{bm}$ denote the common prime radical, nil radical, locally nilpotent radical, the Jacobson radical, the Brown-McCoy radical for algebras, respectively. J. R. Fisher (see [7, Proposition 2]) constructed an H-radical r_H by a common hereditary radical r for algebras. Thus we can get H-radicals $r_{bH}, r_{kH}, r_{lH}, r_{jH}, r_{bmH}$. Let $r_{Hl} = r_{lH}$ and $r_{Hk} = r_{kH}$ for convenience.

Definition 3.3. *An A-H-module M is called an A-H-BM-module, if for M the following conditions are fulfilled:*

(i) $AM \neq 0$.

(ii) *If I is an H-ideal of A and $I \not\subseteq (0:M)_A$, then there exists an element $u \in I$ such that $m = um$ for all $m \in M$.*

Theorem 3.4. *Let $\mathcal{M}_A = \{\ M \mid M$ is an A-H-BM-module$\}$ for every H-module algebra A and $\mathcal{M} = \cup \mathcal{M}_A$. Then $\mathcal{M}$ is an H-special class of modules and $r_{Hbm}(A) = \mathcal{M}(A)$, where r_{Hbm} denotes the H-upper radical determined by $\{A \mid A$ is an H-simple module algebra (i.e., A has no non-trivial H-ideal and $A^2 \neq 0$) with unit $\}$.*

Theorem 3.5. *H is a finite-dimensional semisimple Hopf algebra with $t \in \int_H^l$ and $\epsilon(t) = 1$. Let $G_t(a) = \{z \mid z = x + (t.a)x + \sum (x_i(t.a)y_i + x_iy_i)$ for all $x_i, y_i, x \in A\}$. A is called an r_{gt}-H-module algebra, if $a \in G_t(a)$ for all $a \in A$. Then r_{gr} is an H-radical property of module algebras and $r_{gt}(A) = r_{Hbm}(A)$ for any unital H-module algebra A.*

Proof. It is similar to the proof of [9, Theorem 9.3.1] and [9, Theorem 9.5.5].

Definition 3.4. *Let I be an H-ideal of H-module algebra A, N be an A-H-submodule of A-H-module M. N and I are said to have "L-condition", if for any finite subset $F \subseteq I$, there exists a positive integer k such that $F^kN = 0$.*

Definition 3.5. *An A-H-module M is called an A-H-L-module, if for M the following conditions are fulfilled:*

(i) $AM \neq 0$.

(ii) *For every non-zero A-H-submodule N of M and every H-ideal I of A, if N and I have "L-condition", then $I \subseteq (0:M)_A$.*

Theorem 3.6. *Let $\mathcal{M}_A = \{\ M \mid M$ is an A-H-L-module$\}$ for any H-module algebra A and $\mathcal{M} = \cup \mathcal{M}_A$. Then $\mathcal{M}$ is an H-special class of modules and $\mathcal{M}(A) = r_{Hl}(A)$.*

§4. The Relations Between the Radical of $A\#H$ and the H-Radical of A

In this section, k is a field, H is a Hopf algebra over k, A is an H-module algebra (A can be without unit).

Proposition 4.1. *If r is a hereditary common radical for algebras, then*

$$r_H(A) = (r(A))_H = (r(A) : H).$$

Furthermore, $r_H(A) \subseteq r(A)$.

Proposition 4.2. *If B is an H-ideal of A, then $(A\#H)/(B\#H) \cong (A/B)\#H$ (as algebras).*

Proposition 4.3. *Let $\bar{\mathcal{M}} = \cup \bar{\mathcal{M}}_A$ be a common special class of modules and satisfy the condition: $M \in \bar{\mathcal{M}}_A$ and $A \overset{\psi}{\cong} B$ (as algebras) imply $M \in \bar{\mathcal{M}}_B$ (defined by $\psi(a)x = ax$). If let $\mathcal{M}_A = \{M \mid M \in \bar{\mathcal{M}}_{A\#H}\}$ for every H-module algebra A and $\mathcal{M} = \cup \mathcal{M}_A$, then $\mathcal{M}$ is an H- special class of modules.*

Proof. It is easy to check that $\mathcal{M}$ satisfies (M1), (M2) and (M3) in Definition 3.3. Using the assumption, we see that $\mathcal{M}$ satisfies (M4).

Let

$\bar{r}_{Hj}(A) = \cap\{(0:M)_A \mid M$ is an irreducible $A\#H$- module $\} = r_j(A\#H) \cap A$;

$\bar{r}_{Hb}(A) = \cap\{(0:M)_A \mid M$ is an $A\#H$-prime module$\}= r_b(A\#H) \cap A$;

$\bar{r}_{Hl}(A) = \cap\{(0:M)_A \mid M$ is an $A\#H$-L- module$\}= r_l(A\#H) \cap A$;

$\bar{r}_{Hbm}(A) = \cap\{(0:M)_A \mid M$ is an $A\#H$-BM-module$\}= r_{bm}(A\#H) \cap A$.

Then $\bar{r}_{Hj}$, $\bar{r}_{Hb}$, $\bar{r}_{Hl}$ and $\bar{r}_{Hbm}$ are H-radicals and H-special radicals by Proposition 4.3.

Proposition 4.4. $r_{Hb}(A) \subseteq r_b(A\#H) \cap A \subseteq r_{bH}(A)$.

Proof. If A is a nilpotent H-module algebra, then $r_b(A\#H) = A\#H$, which implies $\bar{r}_{Hb}(A) = A$. Since $\bar{r}_{Hb}$ is an H-radical property for H-module algebras, we have $\bar{r}_{Hb} \geq r_{Hb}$ by Theorem 2.3, i.e., $r_{Hb}(A) \subseteq r_b(A\#H) \cap A$. $r_b(A\#H) \cap A \subseteq r_{bH}(A)$ can be showed by m-nilpotent element.

Theorem 4.1. $r_{Hb}(A)\#H \subseteq r_b(A\#H)$.

Proof. $r_{Hb}(A)\#H \subseteq r_b(A\#H)$ since $r_{Hb}(A) \subseteq r_b(A\#H)$ by Proposition 4.4.

Theorem 4.2. *If $r_{bH}(A)$ is nilpotent, then*

$$r_{Hb}(A) = r_{bH}(A) = r_b(A\#H) \cap A = \bar{r}_{Hb}(A).$$

Proof. By Theorem 2.3 and Proposition 4.4, $r_{Hb}(A) = r_{bH}(A) = \bar{r}_{Hb}(A)$.

Theorem 4.3. *Let H be a finite-dimensional Hopf algebra and A be a unital H-module algebra. Let H^* act on $A\#H$ by $f \cdot (a\#h) = \sum a\#f(h_2)h_1$ for any $f \in H^*, h \in H, a \in A$. If A is semiprime, then $A\#H$ is H^*-semiprime, where H^* is the dual space of H.*

Proof. By duality theorem $(A\#H)\#H^* \cong M_n(A)$, which implies that $(A\#H)\#H^*$ is semiprime. It follows from Theorem 4.1 that $A\#H$ is H^*-semiprime.

Proposition 4.5. *If the action of H on A is inner (defined in* [2]*) and r is a hereditary common radical for algebras, then $r(A) = r_H(A)$. Furthermore,*

$$r_b(A) = r_{Hb}(A) = r_{bH}(A) = r_b(A\#H) \cap A$$

and

$$r_{bm}(A) = r_{Hbm}(A) = r_{bmH}(A).$$

Proof. Since the action of H on A is inner, every ideal of A is an H-ideal, which implies $r(A) = r_H(A)$. By the same reason, A is H-prime iff A is prime. A is H-simple iff A is simple. Then the others hold.

Theorem 4.4. *If H is a finite-dimensional semisimple Hopf algebra and A is a unital H-module algebra, then*

$$r_b(A\#H) = r_{Hb}(A)\#H$$

in the following four cases:

(1) *k is a perfect field and H is cocommutative.*

(2) *H is irreducible cocommutative.*

(3) *The action of H on A is inner.*

(4) *$H = (kG)^*$, where G is a finite group.*

Proof. By [6, Theorem 5.3], (4) holds. Now we show that (1) and (2) and (3) hold. Considering Theorem 4.1, we only need to show that

$$r_b(A\#H) \subseteq r_{Hb}(A)\#H.$$

Since

$$(A\#H)/(r_{Hb}(A)\#H) \cong (A/r_{Hb}(A))\#H \text{ (as algebras)}$$

by Proposition 4.2 and $A/r_{Hb}(A)\#H$ is semiprime by [3, Theorem 2], or [3, Corollary 1], or [10, Theorem 7.4.7] and Proposition 4.5, we have $r_b(A\#H) \subseteq r_{Hb}(A)\#H$.

Proposition 4.6. *If r is a hereditary common radical for algebras and $H = kG$ is a group algebra, then $r_H(A) = r(A)$.*

Proof. For any $g \in G$, define a map α_g from A to A by $\alpha_g(a) = g \cdot a$ for any $a \in A$. We easily check that α_g is an algebra epimorphism, then $g \cdot r(A) \subseteq r(A)$, which implies $r_H(A) = r(A)$ by Proposition 4.1.

Theorem 4.5. *If A is a unital H-module algebra, then $r_{Hj}(A) = r_j(A\#H) \cap A$.*

Proof. It follows from [7, Lemma 1] that $\{(0 : M)_{A\#H} \cap A \mid M$ is an irreducible $A\#H$-module $\} = \{(0 : M)_A \mid M$ is an irreducible A-H-module $\}$. Thus $r_{Hj}(A) = r_j(A\#H) \cap A$.

Theorem 4.6. $r_j(A\#H) \cap A \subseteq r_{jH}(A)$. *Furthermore, if A is a unital H-module algebra, then $r_{Hj}(A) \subseteq r_{jH}(A)$ and $r_{Hj}(A)\#H \subseteq r_j(A\#H)$.*

Proof. For any $a \in r_j(A\#H) \cap A$, there exists $u = \sum a_i\#h_i \in A\#H$ such that $a+u+au = 0$. Using $(id \otimes \epsilon)$, we get

$$a + \Big(\sum a_i\epsilon(h_i)\Big) + a\Big(\sum a_i\epsilon(h_i)\Big) = 0.$$

Thus a is right quasi-regular in A. Considering $r_j(A\#H) \cap A$ is an H-ideal of A, we have $r_j(A\#H) \cap A \subseteq r_{jH}(A)$. By Theorem 4.5, $r_{Hj}(A) \subseteq r_j(A\#H)$. Thus

$$r_{Hj}(A)\#H = (r_{Hj}(A)\#1)(1\#H) \subseteq r_j(A\#H).$$

Theorem 4.7. *Let A be a unital H-module algebra. If H is a finite-dimensional semisimple Hopf algebra and the action of H on A is inner, then $r_j(A\#H) \subseteq r_{jH}(A)\#H$.*

Proof. By Proposition 4.5, $r_{jH}(A) = r_j(A)$. Since

$$(A\#H)/(r_{jH}(A)\#H) \cong (A/r_{jH}(A))\#H$$

and $A/r_{jH}(A)$ is semiprimitive , we have $(A\#H)/(r_{jH}(A)\#H)$ is semiprimitive by [10, Corollary 7.4.3] and $r_j(A\#H) \subseteq r_{jH}(A)\#H$.

Proposition 4.7. *If A is a unital H-module algebra, then $r_{Hbm}(A) = r_{bmH}(A)$.*

Proof. It is clear that $\{B \mid B \triangleleft_H A$ and A/B is H-simple with unit $\} = \{I_H \mid I \triangleleft A$ and A/I is simple with unit $\}$. Thus

$$\begin{aligned} r_{bmH}(A) &= (\cap\{I \mid I \triangleleft A \text{ and } A/I \text{ is simple with unit }\})_H \quad \text{by Proposition 4.1} \\ &= \cap\{I_H \mid I \triangleleft A \text{ and } A/I \text{ is simple with unit }\} \\ &= \cap\{B \mid B \triangleleft_H A \text{ and } A/B \text{ is } H\text{-simple with unit}\} \\ &= r_{Hbm}(A) \quad \text{by Theorem 3.4 .} \end{aligned}$$

Proposition 4.8. $r_{Hb} \le r_{Hl}, r_{Hl} \le r_{Hj}$, $r_{bH} \le r_{lH} \le r_{kH} \le r_{jH} \le r_{bmH}$, $r_{jH} \le r_{Hbm}$.

Proof. It is easy to check that $r_{bH} \le r_{lH} \le r_{kH} \le r_{jH} \le r_{bmH}$ by Proposition 4.1 and ring theory. By Theorem 2.3, $r_{Hb} \le r_{Hl}$. Since every irreducible A-H-module is an A-H-L-module, we have $r_{Hl} \le r_{Hj}$ by Theorem 3.6. If A is an H-simple module algebra with unit, then $r_{jH}(A) = 0$. By Theorem 3.4, $r_{jH} \le r_{Hbm}$.

Theorem 4.8. *Let A be a unital H-module algebra. If $r_{jH}(A)$ is nilpotent, then*

$$r_{jH}(A) = r_{Hj}(A) = r_{Hb}(A) = r_{bH}(A) = r_j(A\#H) \cap A = r_b(A\#H) \cap A.$$

Proof. Since $r_{jH}(A)$ is nilpotent, $r_{jH}(A) \subseteq r_{Hb}(A)$ by Theorem 2.3. It is easy to check that

$$r_{Hb}(A) \subseteq r_b(A\#H) \cap A \subseteq r_{bH}(A) \subseteq r_{jH}(A) \quad \text{and} \quad r_{Hb}(A) \subseteq r_{Hj}(A) \subseteq r_{jH}(A)$$

by Proposition 4.4, Proposition 4.8 and Theorem 4.6. Therefore

$$r_{jH}(A) = r_{Hj}(A) = r_{Hb}(A) = r_{bH}(A) = r_j(A\#H) \cap A = r_b(A\#H) \cap A.$$

Theorem 4.9. *Let A be a unital H-module algebra, H be a finite-dimensional semisimple Hopf algebra and the action of H on A be inner. If $r_{jH}(A)$ is nilpotent (Example: A is left Artinian or right Artinian or finite dimensional), then*

$$r_j(A\#H) = r_{jH}(A)\#H = r_b(A\#H)$$

and

$$r_{jH}(A) = r_{Hj}(A) = r_j(A) = r_b(A) = r_{bH}(A) = r_{Hb}(A) = \bar{r}_{Hj}(A) = \bar{r}_{Hb}(A).$$

Proof. By Proposition 4.5, $r_j(A) = r_{jH}(A)$ and $r_b(A) = r_{bH}(A)$. Applying Theorem 4.8, we get

$$r_{jH}(A) = r_{Hj}(A) = r_j(A) = r_b(A) = r_{Hb}(A) = r_{bH}(A) = \bar{r}_{Hb}(A) = \bar{r}_{Hj}(A).$$

By Theorem 4.7 and Theorem 4.6, $r_j(A\#H) = r_{Hj}(A)\#H$. We see that

$$\begin{aligned} r_j(A\#H) &\supseteq r_b(A\#H) \\ &= r_{Hb}(A)\#H \quad \text{by Theorem 4.4} \\ &= r_{Hj}(A)\#H \quad \text{by Theorem 4.8} \\ &= r_j(A\#H). \end{aligned}$$

Thus $r_j(A\#H) = r_b(A\#H)$.

Proposition 4.9. *A is H-semiprime if and only if $aA(H\cdot a) = 0$ always implies $a = 0$ for any $a \in A$, if and only if $(H\cdot a)Aa = 0$ always implies $a = 0$ for any $a \in A$.*

Proof. It is similar to the proof of Lemma 3.1.

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