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ON INTERIOR POINTS OF THE JULIA SET $J(R)$ FOR RANDOM DYNAMICAL SYSTEM R**

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Abstract

The authors consider the random iteration of serval functions. Denote by $J(R)$ the Julia set for the random iteration dynamical system formed by a set of complex functions $R = \{R_1, R_2, \cdots, R_M\}$. Some sufficient conditions are given for $J(R)$ to have no interior points. Also some conditions are given for $J(R)$ to have interior points but fail to be the extended plane. In addition, $J(az^n, bz^n)$ $(n \geq 2, ab \neq 0)$ and $J(z^2+c_1, z^2+c_2)$ are investigated and some interesting results are obtained.

Keywords Random iteration, Dynamical system, Julia set
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§1. Introduction

Earlier in 1900s Fatou and Julia independently develeped iteration theory of a complex function, and great achievements have been made in recent years. Let R be a meromorphic function, denote by R^n the nth iteration of R. Under the iteration of a meromorphic function R, the Riemann sphere decomposes into two completely invarient sets; one of them is Fatou Set, on which the family $\{R^n\}$ is normal, and the other is Julia set, the complement of the Fatou set. Fatou set and Julia set are basic objects studied in the iteration of a function as a dynamical system, which may be called classical dynamical system, in contrast with random dynamical system. In this paper, we concern ourselves with the random dynamical system concerning the random iterations of several functions.

For introduction and surveys of iteration theory of a single function as a dynamical system, we refer to [1, 2, 3]. As an extention of the classical dynamical system, one may consider the iteration of a finite number of functions. To be more precise, let $R = \{R_1, R_2, \ldots, R_M\}$ be a set of meromorphic functions, and $\Sigma_R = \{(j_1, j_2, \cdots, j_n, \cdots) : j_i \in \{1, 2, \cdots, M\}, \text{for all } i \in N\}$. For each $\sigma = (j_1, j_2, \cdots, j_n, \cdots) \in \Sigma_R$, define W_σ^n as the composition of $R_{j_1}, R_{j_2}, \ldots, R_{j_n}$, that is,

$$W_\sigma^1(z) = R_{j_1}(z), \quad W_\sigma^{n+1}(z) = R_{j_n} \circ W_\sigma^n(z).$$

A point z is said to be a stable point if there exists a neighborhood U of z such that $\{W_\sigma^n\}$ is normal on U for each $\sigma \in \Sigma_R$. The set of stable points is called Fatou set of the system

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R, denoted by $F(R)$ or to be more precise by $F(R_1,\cdots,R_M)$, and its complement, $J(R)$ or $J(R_1,\cdots,R_M)$, is called the Julia set of the system R.

As in the classical case, the Fatou set $F(R)$ of a system $R=\{R_1,R_2,\cdots,R_M\}$ is open and the Julia set $J(R)$ is closed. For more information about the random dynamical system, we refer to [4], where we can see that the random dynamical system and the classical dynamical system behave differently in many ways, although in some aspects the two dynamical systems have similar properties. For example, it is well known that the classical Julia set of a meromorphic function has no interior points except for the case when the Julia set is the extended plane. But this is not the case for the random dynamical system, that is to say, it is possibile for $J(R)$ to have interior points but fail to be the extended plane if R is a system formed by a number of functions. A simple example for this case is

$$\hat{C}\neq J(z^2,(z-1)^2+1)\supset\{|z|\le 1\}\cup\{|z-1|\le 1\},$$

which will be proved in §7. On the other hand, the dynamicial properties of the random iterations of a number of functions is much more complicated than that of the classical case. At last, we should poind out that, as shown in [4], Fatou set and Juila set for random dynamical system are not completely invariant, although the former is forward invariant and the latter backforward invariant.

In this paper, however, we will mainly devote ourselves to the investigations of the Julia set $J(f_1,f_2,\cdots,f_n)$, where $f_1,f_2,\ldots,f_n$ are meromorphic functions. In §3, we give some sufficient conditions for $J(f_1,f_2)$ $(J(f_1,f_2,\cdots,f_n))$ to have interior points but fail to be the whole plane. Also, in §4 some sufficient conditions for $J(f_1,f_2)$ $(J(f_1,f_2,\cdots,f_n))$ to have no interior points are given. On the other hand it is proved that if $|c_1-c_2|\ge 4\sqrt{|c_2|}(|c_2|\ge |c_1|)$, then $J(z^2+c_1,z^2+c_2)$ has no interior points. Furthermore, the detailed studies of $J(az^n,bz^n)$ $(n\ge 2, ab\ne 0)$ are available in this paper (see §6).

Throughout this paper, we shall assume that the functions to be discussed are neither constants nor rational functions of degree one. And as usual, the rational functions and entire functions are considered special cases of the meromorphic functions.

§2. Some Lemmas

In order to prove our theorems, the following Lemmas are necessary.

Lemma 2.1.[4] *Let $R=\{R_1,R_2,\ldots,R_M\}$ be a set of rational functions, $E(R)$ be the exceptional set of R. If $z\notin E(R)$, then $J(R)$ is contained in the set of accumulation points of the full backward orbit of z for every $\sigma\in\Sigma_R$. That is*

$$J(R)\subset\Big\{\textit{accumulation points of }\Big[\bigcup_{\sigma\in\Sigma_R}\bigcup_{n\ge 0}W_\sigma^{-n}(z)\Big]\Big\}.$$

Lemma 2.2.[4] *Let $J(R_i)$ be the classical Julia set of rational functions R_i for $i=1,2,\cdots,M$. Then*

$$J(R)=\textit{closure of }\Big\{\bigcup_{\sigma\in\Sigma_R}\bigcup_{n\ge 0}W_\sigma^{-n}\Big(\bigcup_{i=1}^{M}J(R_i)\Big)\Big\}.$$

Lemma 2.3. *If there exists a domain Ω such that $\Omega\cap F\ne\emptyset$, and $R_i^{-1}(\Omega\cap F)\subset\Omega\cap F$ for $i=1,2,\ldots,M$, where $F=F(R)$, then the Julia set $J(R)$ of $R=\{R_1,R_2,\cdots,R_M\}$ has* *no interior points.*

Proof. Let z be a point in $J(R)$. Take a point z_0 in $\Omega \cap F$ such that $z_0 \notin E(R)$, where $E(R)$ denotes the exceptional set of R. Let U be any neighborhood of z. Then there must exist $\sigma \in \Sigma_R$ and $n \in N$ such that

$$U \bigcap \{\bigcup_{n\geq 0} W_\sigma^{-n}(z_0)\} \neq \emptyset$$

by Lemma 2.1. From the assumption, it is easy to show $W_\sigma^{-n}(z_0) \subset F$ by induction and so U contains stable points. This means that z is not an interior point of $J(R)$. The proof is complete.

Lemma 2.4. *Let $f = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ ($a_n \neq 0$) and denote by $D(f)$ the diameter of J(f). Then there exists a constant $C(n)$ depending only on n such that for any fixed point Q of f we have $|f(z) - Q| \geq 4|z - Q|$, provided $|z - Q| \geq C(n)D(f)$. In fact we may take $C(n) = \frac{n-1}{\ln(4/3)}$.*

Proof. For any fixed point Q of f, let $h(z) = z - Q$. Then the origin is a fixed point of $h \circ f \circ h^{-1}$. And so we may assume $Q = 0$ without loss of generality. Under the assumption, we have $a_0 = 0$ and hence

$$f(z) = z(a_n z^{n-1} + a_{n-1}z^{n-2} + \cdots + a_1).$$

Consider equation

$$a_n z^{n-1} + a_{n-1}z^{n-2} + \cdots + a_1 = 0.$$

Denoting by $x_1, x_2, \ldots, x_{n-1}$ the roots of the equation, we have

$$\frac{a_i}{a_n} = (-1)^{n-i} \sum_{j_1\neq j_2 \cdots \neq j_{n-i} \neq j_1} x_{j_1} x_{j_2} \cdots x_{j_{n-i}}, \quad i = 1, 2, \ldots, n-1.$$

Let $D = D(f)$. Then

$$\Big|\frac{a_i}{a_n}\Big| \leq \binom{n-i}{n-1} D^{n-i}, \quad i = 1, 2, \cdots, n-1.$$

From this we get

$$\begin{aligned} |f(z)| &= |a_n z^n| \Big|1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_1}{a_n z^{n-1}}\Big| \\ &\geq |a_n z^n| \Big[2 - \Big(1 + \frac{D}{|z|}\Big)^{n-1}\Big] = |z| m\Big(\frac{D}{|z|}\Big), \end{aligned}$$

where

$$m(x) = \frac{|a_n| D^{n-1}[2 - (1+x)^{n-1}]}{x^{n-1}}.$$

We want to show that if $x > 0$ is small enough then we have $m(x) \geq 4$. For this purpose, let $x = \frac{y}{n-1}$, then $(1+x)^{n-1} < e^y$. If

$$x = \frac{D}{|z|} = \frac{y}{n-1} \leq \frac{\ln(4/3)}{n-1},$$

then we can get

$$y \leq \sqrt[n-1]{\frac{1}{6}} \cdot \sqrt[n-1]{|a_n|} \cdot (n-1) \cdot D, \tag{2.1}$$

using the next lemma. It is not hard to show that (2.1) implies $m(x) \geq 4$. Take $C(n) = \frac{n-1}{\ln(4/3)}$. Then obviously $|z| \geq C(n)D(f)$ implies $|f(z)| \geq 4|z|$, as required.

Lemma 2.5. *Let* $f = a_n z^n + \cdots + a_1 z + a_0$ $(a_n \neq 0)$. *Then the diameter* $D(f)$ *of* $J(f)$ *is lager than* $2 \cdot \sqrt[n-1]{|a_n|}$

Proof. Take $a = \sqrt[n-1]{a_n}$, let $\gamma(z) = az$. Then $\gamma \circ f \circ \gamma^{-1}$ is a monic polynomial, and clearly $J(f) = \gamma^{-1}(\gamma \circ f \circ \gamma^{-1})$. Our lemma follows from the assertion that a monic polynomial has the Julia set of diameter greater than or equal to 2, which was conjectured by Y. E. Arumaraj and proved by Yin Yongcheng in [5].

Lemma 2.6. *Let*

$$f = \frac{z^{n+2} + a_{n+1}z^{n+1} + \cdots + a_1 z + a_0}{b_k z^k + \cdots + b_1 z + b_0} = \frac{Q(z)}{P(z)},$$

where $b_k \neq 0, 0 \leq k \leq n$. *Suppose that* $P(z)$ *and* $Q(z)$ *do not have common factors. Denote by* $D(f)$ *the diameter of* $J(f)$. *Then there exists a constant* $C(n)$ *depending only on* n *such that for any fixed point* $\tilde{Q}$ *of* f *we have* $|f(z) - \tilde{Q}| \geq 4|z - \tilde{Q}|$, *provided* $|z - \tilde{Q}| \geq C(n)D(f)$. *In fact one may take*

$$C(n) = \frac{n+1}{\ln\left(1 + \frac{1}{16n+28}\right)}.$$

Proof. We can suppose that $P(z)$ is not a constant, for this case is contained in Lemma 2.4. Also, as before we may assume $\tilde{Q} = 0$ without loss of generality. Because otherwise we may consider $h \circ f \circ h^{-1}$ instead of f, where $h(z) = z - \tilde{Q}$. Denote by $K(f)$ the filled Julia set of f, that is the set consisting of all points with bounded forward orbit under f. By hypothesis above, we have

$$f(z) = \frac{z(z^{n+1} + a_{n+1}z^n + \cdots + a_1)}{b_k z^k + \cdots + b_1 z + b_0}.$$

Consider equation

$$z^{n+1} + a_{n+1}z^n + \cdots + a_1 = 0,$$

and we get as in the proof of Lemma 2.4,

$$|Q(z)| \geq |z|^{n+2}\Big[2 - \Big(1 + \frac{D}{|z|}\Big)^{n+1}\Big]. \tag{2.2}$$

By assumption we can take a point $\omega \in K(f)$ such that $|\omega| \geq \frac{D(f)}{2}$, and consider equation

$$\frac{z^{n+2} + a_{n+1}z^{n+1} + \cdots + a_1 z}{b_k z^k + \cdots + b_1 z + b_0} = \omega,$$

that is,

$$z^{n+2} + a_{n+1}z^{n+1} + \cdots + (a_k - \omega b_k)z^k + \cdots + (a_1 - \omega b_1)z - \omega b_0 = 0.$$

Let $D = D(f)$. It is easy to see $|b_0\omega| \leq D^{n+2}$ and thus $|b_0| \leq 2D^{n+1}$. In general, we have

$$|a_i - \omega b_i| \leq \binom{n+2-i}{n+2} D^{n+2-i}, \quad i = 1, 2, \cdots, k,$$

and so

$$|b_i| \leq 2\Big[\binom{n+2-i}{n+1} + \binom{n+2-i}{n+2}\Big] D^{n+1-i}, \quad i = 1, 2, \cdots, k,$$

which implies

$$\begin{aligned}|P(z)| &= |b_k z^k + \cdots + b_1 z + b_0| \\ &= |z|^k \Big| b_k + \frac{b_{k-1}}{z} + \cdots + \frac{b_{k-i}}{z^i} + \cdots + \frac{b_0}{z^k} \Big| \\ &\le 2|z|^{n+1} \cdot \frac{|z|}{D} \cdot \Big[\Big(1 + \frac{D}{|z|}\Big)^{n+1} + \Big(1 + \frac{D}{|z|}\Big)^{n+2} - (2n+3) \cdot \frac{D}{|z|} - 2 \Big].\end{aligned}$$

Combining this with (2.2) we get

$$|f(z)| \ge |z| \cdot \frac{D}{|z|} \cdot \frac{2 - \Big(1 + \frac{D}{|z|}\Big)^{n+1}}{2\Big[\Big(1 + \frac{D}{|z|}\Big)^{n+1} + \Big(1 + \frac{D}{|z|}\Big)^{n+2} - (2n+3) \cdot \frac{D}{|z|} - 2\Big]} = |z| \rho\Big(\frac{D}{|z|}\Big),$$

where

$$\rho(x) = \frac{x[2 - (1+x)^{n+1}]}{2[(1+x)^{n+1} + (1+x)^{n+2} - (2n+3)x - 2]}.$$

What remains to prove is to show that if x is small enough then $\rho(x) \ge 4$, that is,

$$\theta(x) = 2x - x(1+x)^{n+1} - 8(1+x)^{n+1} - 8(1+x)^{n+2} + 8(2n+3)x + 16 \ge 0. \tag{2.3}$$

Let $x = \frac{y}{n+1}$. We can prove that if $y < \ln\big(1 + \frac{1}{16n+28}\big)$, then (2.3) holds, which implies $\rho(x) \ge 4$. Take $C(n) = \frac{n+1}{\ln(1+\frac{1}{16n+28})}$. Then clearly $|z| \ge C(n)D$ implies

$$|f(z)| \ge |z| \rho\Big(\frac{D}{|z|}\Big) \ge 4|z|.$$

The proof is complete.

§3. Sufficient Conditions for $J(f_1, f_2)$ to Have Interior Points

In the classical iteration theory of a single function, it is well-knowen that if the Julia set $J(f)$ of a meoromorphic function f has interior points, then $J(f) = \hat{C}$. But this is not true for the random iteration theory of finitely many functions. As we will see in §6, $J(z^2, 4z^2)$ has interior points but it is not the whole plane. In fact, we have in general the following assertions.

Theorem 3.1. *Let f_1, f_2 be two meromorphic functions in the complex plane C. Suppose that f_1, f_2 have a common attractive (superattractive) fixed point, and that there exists a superattractive periodic point P_1 of f_1 such that $P_1 \in \Gamma \subset J(f_2)$, where Γ is a Jordan arc. Then $J(f_1, f_2)$ has interior points but $J(f_1, f_2) \ne \hat{C}$.*

Corollary 3.1. *Assume $f_1, f_2, \cdots, f_n$ to be meromorphic functions in C. Suppose that $f_1, f_2, \cdots, f_n$ have a common attractive (superattractive) fixed point, and that there exists a superattractive periodic point P_i of f_i for some i such that $P_i \in \Gamma \subset J(f_j)$ for some $j \ne i$, where Γ is a Jordan arc. Then $J(f_1, f_2, \cdots, f_n)$ has interior points but $J(f_1, f_2, \cdots, f_n) \ne \hat{C}$.*

Proof of Corollary 3.1. Note that P_i is a stable point and so $F(f_1, f_2, \cdots, f_n) \ne \emptyset$. On the other hand, $J(f_1, f_2, \cdots, f_n) \supset J(f_i, f_j)$, the corollary follows.

Corollary 3.2. *Let f_1, f_2 be two meromorphic functions in C. Suppose that f_1, f_2 have a common attractive (superattractive) fixed point, and that $J(f_2)$ is connected (or locally* *connected). If f_1 has a superattractive periodic point P_1 of f_1 such that $P_1 \in J(f_2)$, then $J(f_1, f_2)$ has interior points but $J(f_1, f_2) \neq \hat{C}$.*

Proof of Corollary 3.2. The Proof is similar to the proof for Theorem 3.1.

Theorem 3.2. *Let f_1, f_2 be two polynomials. Suppose that f_1 has a superattractive periodic point P_1 such that $P_1 \in \Gamma \subset J(f_2)$, where Γ is a Jordan arc. Then $J(f_1, f_2)$ has interior points but $J(f_1, f_2) \neq \hat{C}$.*

Corollary 3.3. *Let $f_1, f_2, \cdots, f_n$ be polynomials. Assume that there exists a superattractive periodic point P_i of f_i for some i such that $P_i \in \Gamma \subset J(f_j)$ for some $j \neq i$, where Γ is a Jordan arc. Then $J(f_1, f_2, \cdots, f_n)$ has interior points but $J(f_1, f_2, \cdots, f_n) \neq \hat{C}$.*

Proof of Corollary 3.3. Since ∞ is a common superattractive fixed point of $f_1, f_2, \cdots, f_n$.

Proof of Theorem 3.1. For simplicity, we will often write F for $F(f_1, f_2, \cdots, f_n)$ and J for $J(f_1, f_2, \cdots, f_n)$ if no ambiguous meaning yields. In addition, in what follows we always let R denote the set of functions as a random system to discuss.

Clearly, the common superattractive fixed point of f_1 and f_2 is a stable point, and hence $J \neq \hat{C}$. On the other hand, without loss of generality, we may suppose 0 is a superattractive fixed point of f_1 contained in a Jordan arc $\Gamma \subset J(f_2)$. That is $f_1(0) = f_1'(0) = 0$. Then there exists a neighborhood U of 0 and a conformal mapping $\phi : D = \{|z| < 1\} \to U$ such that

$$\phi(0) = 0, \quad \phi^{-1} \circ f_1 \circ \phi(z) = az^n \quad (n \geq 2).$$

It is clear that there exists a 'cut sector'

$$S = \{z : r_1 < |z| < r_2, \theta_1 < \arg z < \theta_2\} \subset D$$

such that $\phi(S) \subset F$. If n is large enough, ${f_1}^n(\phi(S))$ is a doubly-connected domaim, of which the complement has two components, and the bounded one contains 0. Obviously, $\operatorname{dist}({f_1}^n(\phi(S)), 0)$ tends to zero as n tends to ∞. Therefore, for sufficiently large n,

$${f_1}^n(\phi(S)) \cap J(f_2) \supset {f_1}^n(\phi(S)) \cap \Gamma \neq \emptyset;$$

this contradicts the fact that ${f_1}^n(\phi(S)) \subset F(f_1, f_2)$, since the Fatou set for the random iteration system is forward invariant.

Proof of Theorem 3.2. Notice that ∞ is a common superattractive fixed point of f_1 and f_2. The proof for Theorem 3.1 also works for Theorem 3.2 with few modifications.

§4. Sufficient Conditions for $J(f_1, f_2)$ to Have no Interior Points

In this section, we are going to show the following assertions.

Theorem 4.1. *Let f_1, f_2 be two rational functions of the form:*

$$f_1 = \frac{z^{n+2} + a_{n+1}z^{n+1} \cdots + a_0}{b_k z^k + \cdots + b_1 z + b_0} = \frac{Q_1}{P_1} \ (n \geq k),$$
$$f_2 = \frac{z^{m+2} + b_{m+1}z^{m+1} \cdots + b_0}{b_l z^l + \cdots + b_1 z + b_0} = \frac{Q_2}{P_2} \ (m \geq l).$$

Suppose that P_i and Q_i have no common factors, $i = 1, 2$. If

$$\text{dist } (J(f_1), J(f_2)) \geq C(n)D(f_1) + C(m)D(f_2),$$

then $J((f_1),(f_2))$ *has no interior points, where* $C(n)$ *(respectively* $C(m)$*) is the constant appeared in Lemma* 2.4 *or Lemma* 2.6 *according as* f_1 *(respectively* f_2*) is a polynomial (i.e.,* P_1 *is a constant) or not.*

Note. Since $C(n), C(m) \geq 3$, we have

$$D(f_1)+D(f_2)<\frac{1}{2}\cdot \text{dist}(J(f_1),J(f_2)).$$

In particular, we have

Corollay 4.1. *Let*

$$f_1 = a_n z^n + \cdots + a_1 z + a_0,$$
$$f_2 = b_m z^m + \cdots + b_1 z + b_0.$$

If dist $(J(f_1), J(f_2)) \geq \frac{n-1}{\ln(4/3)}D(f_1)+\frac{m-1}{\ln(4/3)}D(f_2)$, *then* $J((f_1),(f_2))$ *has no interior points.*

Proof of Theorem 4.1. Let d =dist $(J(f_1), J(f_2))$,

$$\Omega_1=\Big\{z:\text{dist}(z,K(f_1))\leq \frac{d}{2}\Big\},\quad \Omega_2=\Big\{z:\text{dist}(z,K(f_2))\leq \frac{d}{2}\Big\},$$

where $K(f_i)$ denote the filled Julia sets of f_i $(i=1,2)$. First of all, we show that

$$J(f_1,f_2)\subset \Omega_1\bigcup\Omega_2. \tag{4.1}$$

For this purpose, we establish the following assertion

$$f_1{}^{-1}(\Omega_i)\subset\Omega_1,\ \ f_2{}^{-1}(\Omega_i)\subset\Omega_2\quad (i=1,2). \tag{4.2}$$

In fact, if $z\in f_1{}^{-1}(\Omega_1)$ but $z\notin\Omega_1$, we will get a contradiction. Let $\tilde{Q}_1$ be a fixed point of f_1. Then we can prove by Lemma 2.6 that for any point $\tilde{P}_1\in K(f_1)$,

$$|f_1(z)-\tilde{P}_1|\geq|f_1(z)-\tilde{Q}_1|-|\tilde{Q}_1-\tilde{P}_1|>\frac{3d}{2}.$$

This means $f_1(z)\notin\Omega_1$, a contradiction, that is to say, $f_1{}^{-1}(\Omega_1)\subset\Omega_1$. A similar argument can show

$$f_1{}^{-1}(\Omega_2)\subset\Omega_1\ \ \text{and}\ \ f_2{}^{-1}(\Omega_i)\subset\Omega_2\quad (i=1,2),$$

and hence we come to assertion (4.2).

Take a point ω_0 in $\Omega_1\cup\Omega_2$, which is not an exceptional point. We get from (4.2) by induction that $W_\sigma^{-n}(\omega_0)\subset\Omega_1\bigcup\Omega_2$ for any $\sigma\in\Sigma_R$ and $n\in N$, from which we arrive at assertion (4.1) by Lemma 2.1.

Our theorem follows from Lemma 2.3 together with the following assertion

$$f_i{}^{-1}((\Omega_1\bigcup\Omega_2)\bigcap F)\subset(\Omega_1\bigcup\Omega_2)\bigcap F,\quad i=1,2,$$

which can be proved by using (4.1) and (4.2). The proof is complete.

§5. Discussions of $J(z^2+c_1, z^2+c_2)$

Theorem 5.1. *If* $|c_1-c_2|\geq 4\sqrt{|c_2|}$ $(|c_2|\geq|c_1|)$, *then* $J(z^2+c_1,z^2+c_2)$ *has no interior points.*

Remark. Obviously, $f_1=z^2+c_1, f=z^2+c_2$ fail to satisfy the conditions of Theorem 4.1.

Corollary 5.1. *If* $|c|\geq 16$, *then* $J(z^2, z^2+c)$ *has no interior points.*

Proof of Theorem 5.1. Let $f_1 = z^2 + c_1$, $f_2 = z^2 + c_2$. We may as well suppose $c_1 \neq c_2$. First we show

$$\Omega_0 = \{z : |z| > \sqrt{2|c_2|}\} \subset F. \tag{5.1}$$

By assumption it is easy to see $2|c_2| \geq |c_2 - c_1| \geq 4\sqrt{|c_2|}$, so that if $|z| > \sqrt{2|c_2|}$, then

$$|z^2 + c_1| > 2|c_2| - |c_1| \geq |c_2| \geq 2\sqrt{|c_2|},$$
$$|z^2 + c_2| > 2|c_2| - |c_2| = |c_2| \geq 2\sqrt{|c_2|}.$$

That is to say $f_1(z), f_2(z) \in \Omega_0$, and furthermore $f_1{}^n(z), f_2{}^n(z) \in \Omega_0$ for all $n \in N$ by induction. This implies (5.1) by Montel's principle. Secondly, let $\Omega = \{z : |z| < 2\sqrt{|c_2|}\}$, we prove

$$f_i{}^{-1}(\Omega \bigcap F) \subset \Omega \bigcap F, \quad i = 1, 2. \tag{5.2}$$

Suppose that $z \in \Omega \cap F$. We want to show $f_1{}^{-1}(z), f_2{}^{-1}(z) \subset \Omega \bigcap F$. We need only to prove $f_1{}^{-1}(z) \subset \Omega \cap F$, the proof for $f_2{}^{-1}(z) \subset \Omega \cap F$ is similar. Let $\omega_1 \in f_1{}^{-1}(z)$. Since $|z| < 2\sqrt{|c_2|}$, we have

$$|\omega_1{}^2| = |z - c_1| < 2\sqrt{|c_2|} + |c_1| < 4|c_2|$$

and hence $|w_1| < 2\sqrt{|c_2|}$. This means $w_1 \in \Omega$.

On the other hand, $f_2(w_1) = z - c_1 + c_2$, and so

$$|f_2(\omega_1)| \geq |c_2 - c_1| - |z| > |c_2 - c_1| - 2\sqrt{|c_2|} \geq 2\sqrt{|c_2|},$$

and thus $f_2(\omega_1) \in F$ by (5.1). Since $f_1(\omega_1) = z \in F$, we have $\omega_1 \in F$. The argument above shows $f_1{}^{-1}(z) \subset \Omega \cap F$, as required.

Now it is easy to see that Theorem 5.1 follows from Lemma 2.3 and assertion (5.2), since (5.1) implies that there are stable points in Ω. The proof is complete.

§6. Discussion of $J(az^n, bz^n)$

In this section, we are to prove some interesting results.

Theorem 6.1. *$J(az^2, bz^2)$ has interior points for $|a| \neq |b|$ $(ab \neq 0)$. If $|a| < |b|$, then*

$$J(az^2, bz^2) = \Big\{\frac{1}{|b|} \leq |z| \leq \frac{1}{|a|}\Big\}.$$

Remark. If $|a| = |b|$, then $J(az^2, bz^2) = J(az^2) = \{|z| = \frac{1}{|a|}\}$.

Theorem 6.2. *If $|a| < |b|, n \geq 3$, then $J(az^n, bz^n)$ has no interior points. As a matter of fact, we have*

$$J(az^n, bz^n) = \Big\{z : |z| = |a|^{-\frac{1}{n-1}} \cdot \Big(\Big|\frac{a}{b}\Big|\Big)^{\sum\limits_{i=1}^{\infty} \frac{t_i}{n^i}}, \quad t_i = 0, 1.\Big\}. \tag{6.1}$$

It is easy to see that Theorem 6.1 and Theorem 6.2 suggest some differences between $J(az^2, bz^2)$ and $J(az^n, bz^n)$ for $n \geq 3$. But, as a generalization of Theorem 6.1, we still have

Theorem 6.3. *If $|b| > 1$, $n \geq 2$, then*

$$J(z^n, bz^n, \cdots, b^{n-1}z^n) = \Big\{\frac{1}{|b|} \leq |z| \leq 1\Big\},$$

and if $|b| < 1$, then

$$J(z^n, bz^n, \cdots, b^{n-1}z^{n-1}) = \Big\{1 \leq |z| \leq \frac{1}{|b|}\Big\}.$$

Proof of Theorem 6.1. First we prove

$$J(z^2, \lambda z^2) = \Big\{ \frac{1}{|\lambda|} \leq |z| \leq 1 \Big\} \tag{6.2}$$

for $|\lambda| > 1$. By Montel's principle, it is trival to see

$$\Big\{ |z| < \frac{1}{|\lambda|} \Big\} \subset F, \quad \{|z| > 1\} \subset F. \tag{6.3}$$

On the other hand, all circles

$$\Big\{ z : |z| = |\frac{1}{\lambda}|^{\frac{1}{2^{n_1}} + \frac{1}{2^{n_2}} + \cdots + \frac{1}{2^{n_k}}} \Big\},$$

where $1 \leq n_1 < n_2 < \cdots < n_k$, are preimages of $\{|z| = 1\} \subset J(z^2, \lambda z^2)$ under W_σ^n for some $\sigma \in \Sigma_R$ and $n \in N$ and so are contained in $J(z^2, \lambda z^2)$.

We notice that any real number in $[0, 1]$ can be represented as infinity series

$$\frac{t_1}{2} + \frac{t_2}{2^2} + \cdots + \frac{t_k}{2^k} + \cdots ,$$

where $t_k = 0, 1$, and since the Julia set is closed, we obtain (6.2) from (6.3).

Now we are able to deal with the general case by use of transformation $\gamma : z \mapsto az$. It is simple to verify

$$J(f_1, f_2) = \gamma^{-1}(J(\gamma \circ f_1 \circ \gamma^{-1}, \gamma \circ f_2 \circ \gamma^{-1})),$$

and therefore we get by (6.2),

$$J(az^2, bz^2) = \gamma^{-1}\Big(J\Big(z^2, \frac{bz^2}{a}\Big)\Big) = \Big\{ \frac{1}{|b|} \leq |z| \leq \frac{1}{|a|} \Big\}.$$

This is what we want to prove.

Proof of Theorem 6.2. Let $f_1 = az^n, f_2 = bz^n$. We may as well assume that $a = 1, |b| > 1$ (see the proof of Theorem 6.1). It is easy to see

$$D_1 = \Big\{ |z| < \sqrt[n-1]{\Big|\frac{1}{b}\Big|} \Big\} \subset F, \quad D_2 = \{|z| > 1\} \subset F, \tag{6.4}$$

since

$$W_\sigma^n(D_1) \subset D_1, W_\sigma^n(D_2) \subset D_2$$

for any $\sigma \in \Sigma_R$ and $n \in N$ by induction. From (6.4), we can prove easily that

$$D = \Big\{ \sqrt[n]{\Big|\frac{1}{b}\Big|} < |z| < \sqrt[n]{\sqrt[n-1]{\Big|\frac{1}{b}\Big|}} \quad \Big\} \subset F. \tag{6.5}$$

Now, let $\Omega = \{ \sqrt[n-1]{|\frac{1}{b}|} < |z| < 1\}$. We are able to show

$$f_i^{-1}(\Omega \cap F) \subset \Omega \cap F, \;\; i = 1, 2. \tag{6.6}$$

Evidently, $f_i^{-1}(\Omega) \subset \Omega (i = 1, 2)$. Let $z \in \Omega \cap F, \omega_1 \in f_1^{-1}(z)$. Then $f_1(\omega_1) = \omega_1{}^n = z \in F$, and hence

$$|f_2(\omega_1)| = |b\omega_1{}^n| = |bz| \geq |b| \cdot \sqrt[n-1]{\Big|\frac{1}{b}\Big|} > 1.$$

By (6.4), $f_2(\omega_1) \in F$, and then $\omega_1 \in F$. The argument above shows $f_1^{-1}(\Omega \cap F) \subset \Omega \cap F$, and in the same way we can show $f_2{}^{-1}(\Omega \cap F) \subset \Omega \cap F$.

It is clear that Lemma 2.3 and assertion (6.6) prove that $J(az^n, bz^n)$ has no interior points, as there are stable points in Ω by (6.5). To prove (6.1), we note that all circles

$$\Big\{z : |z| = \Big|\frac{1}{\lambda}\Big|^{\frac{1}{n^{t_1}}+\frac{1}{n^{t_2}}+\cdots+\frac{1}{n^{t_k}}}\Big\},$$

where $1 \le t_1 < t_2 < \cdots < t_k$, are preimages of $\{|z| = 1\} \subset J(az^n, bz^n)$ under W_σ^n for some $\sigma \in \Sigma_R$ and $n \in N$ and so are contained in J. This observation completes our proof.

Proof of Theorem 6.3. The proof of this theorem is similar to that of Theorem 6.1, we omit the details.

§7. An Example and a Problem

Let $f_0 = z^2, f_1 = (z-1)^2 + 1$. As pointed out in §1, $J(f_0, f_1)$ has interior points. What is more, it is easy to see

$$J(f_0, f_1) \supset K(f_0) \bigcup K(f_1) = \{|z| \le 1\} \bigcup \{|z-1| \le 1\},$$

where $K(f_i)$ denote the filled Julia sets of $f_i (i = 0, 1)$. In fact, if on the contrary, we may as well suppose that the unit disc $\{|z| \le 1\}$ contains an open set $U \subset F(f_0, f_1)$. Furthermore, one can assume that U contains a 'cut sector'

$$S = \{z : r_1 < |z| < r_2, \theta_1 < \arg z < \theta_2\}.$$

For a natural number n large enough, ${f_0}^n(S)$ is an annulus with the origin as its center, and then ${f_0}^n(S) \cap J(f_1) \ne \emptyset$, which contradicts ${f_0}^n(S) \subset F(f_0, f_1)$. In general, we can study the dynamical system of $R = \{f_0, f_t\}$, where $f_0 = z^2, f_t = (z-t)^2$. By Theorem 4.1, if $|t| > 2 + \frac{4}{\ln(4/3)}$, then $J(f_0, f_t)$ has no interior points. Also it is trival that if $t = 0$, then $J(f_0, f_t) = J(f_0)$ has no interior points. It would be interesting but seems difficult to determine the set of t for which the Julia sets $J(f_0, f_t)$ have interior points. Unfortunately, we even do not know what will happen if t is suffciently small other than 0.

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