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THE SCHWARZIAN DERIVATIVE IN SEVERAL COMPLEX VARIABLES (II)****

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Abstract

The Schwarzian derivative of holomorphic mapping on classical domain $\mathbb{R}_I$ is zero iff it is linear fractional.

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§1.

Let $Z = (z_{ij})_{1\le i,j\le n} \in \mathbb{C}^{n\times n}$ denote a square complex matrix. The classical domain of type one R_I is a domain in $\mathbb{C}^{n\times n}$ such that $I - Z\bar{Z}' > 0$, where I denotes the $n\times n$ idendity matrix, $\bar{Z}'$ denotes the conjugate transpose of Z, > 0 means positive definite. Let $\Lambda = (\lambda_{ij})_{1\le i,j\le n} \in \mathbb{C}^{n\times n}, \lambda = (\lambda_{11}, \cdots, \lambda_{1n}, \cdots, \lambda_{n1}, \cdots, \lambda_{nn}) \in \mathbb{C}^{n^2}, \lambda^{(k)}$ is the k-th Kronecker product of λ. Let $W = W(Z)$ be a holomorphic $n\times n$ matrix mapping of $n\times n$ matrix variable Z. We denote the k-th directional derivative of a holomorphic mapping $W = W(Z)$ in the direction Λ by

$$D^kW(Z) = D^k_\Lambda W(Z) = \lambda^{(k)}\left(\frac{\partial}{\partial Z}\right)' W,$$

where

$$\frac{\partial}{\partial Z} = \left(\frac{\partial}{\partial z_{11}}, \cdots \frac{\partial}{\partial z_{1n}}, \cdots, \frac{\partial}{\partial z_{n1}}, \cdots, \frac{\partial}{\partial z_{nn}}\right).$$

In [1], Gong and FitzGerald defined the Schwarzian derivative of $W = W(Z)$ along the direction Λ by

$$\{W;Z\}_\Lambda = (D^3_\Lambda W)(D_\Lambda W)^{-1} - \frac{3}{2}(D^2_\Lambda W)(D_\Lambda W)^{-1}(D^2_\Lambda W)(D_\Lambda)^{-1},$$

and proved that

$$\{W;Z\}_\Lambda = 0 \tag{1.1}$$

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for all $Z \in R_I$ and all non-zero matrix $\Lambda \in \mathbb{C}^{n\times n}$ (when $\{W;Z\}_\Lambda$ exists), if and only if

$$W = W(Z) = W(0)+(I-L(Z))^{-1}D_ZW(0) \text{ or } W(Z) = W(0)+D_ZW(0)(I-L(Z))^{-1} \quad (1.2)$$

where $L(Z)$ is an $n\times n$ matrix, each entry of $L(Z)$ is a linear homogeneous polynomial of $z_{ij}, i,j = 1,2,\cdots,n, I-L(Z)$ is non-singular when $Z \in R_I$.

In this note we prove that (1.1) may be replaced by $\{W;Z\}_Z = 0$ where $Z \in R_I$.

Theorem 1.1. *Let $W = W(Z) : R_I \to \mathbb{C}^{n\times n}$ be a holomorphic mapping with $J_W(0)$ non-singular. Then*

$$\{W;Z\}_Z = 0 \quad (1.3)$$

for all Z in $\mathbb{R}_I$ (when $\{W;Z\}_Z$ exists) if and only if $W = W(Z)$ is defined by (1.2). *Moreover, $W(Z)$ is biholomorphic in R_I.*

As a consequance, we have

Corollary 1.1. *Let $W = W(Z) : R_I \to \mathbb{C}^{n\times n}$ be a holomorphic mapping with $J_W(0)$ non-singular. Then $\{W;Z\}_Z = 0$ implies $\{W;Z\}_\Lambda = 0$ for all non-zero Λ when $\{W;Z\}_\Lambda$ exists.*

Denote $\{W;Z\}_Z$ by $\{W;Z\}$. By Theorem 1.1, it is reasonable to define it as the Schwarzian derivative of $W(Z)$ at Z.

Of course Theorem 1.1 is still true if we replace R_I by any domain in $\mathbb{C}^{n\times n}$.

§2.

Now we are going to prove Theorem 1.1.

A straighforward calculation shows that (1.3) is true if $W = W(Z)$ is defined by (1.2). The difficult part is that (1.3) implies $W = W(Z)$ is defined by (1.2).

We expand $W(Z)$ at $Z = 0$

$$W(Z) = W(0) + D_ZW(0) + C_2(Z) + C_3(Z) + C_4(Z) + \cdots,$$

where $C_2, C_3, C_4, \cdots$ are $n\times n$ matrices, all entries of $C_2(Z), C_3(Z), C_4(Z), \cdots$ are homogeneous polynomials of entries of Z of degree 2, 3, 4,..., respectively.

Fix $Z \in R_I$, if $Z, D_ZW(0)$ are non-singular. Let $Q(t) = W(tZ), t \in [0,1]$. Then

$$\begin{aligned}\frac{dQ}{dt} &= D_ZW(tZ) = \frac{1}{t}D_{tZ}W(tZ)\\ &= D_ZW(0) + 2tC_2(Z) + 3t^2C_3(Z) + \cdots,\end{aligned}$$

and

$$\begin{aligned}\frac{d^2Q}{dt^2} &= D_Z^2W(tZ) = \frac{1}{t^2}D_{tZ}^2W(tZ)\\ &= 2C_2(Z) + 6tC_3(Z) + \cdots.\end{aligned}$$

If we take $\epsilon > 0$ sufficiently small, such that $\frac{dQ(t)}{dt}$ is non-singular when $t \in [0,\epsilon]$, then we may define

$$A(t) = \frac{d^2Q}{dt^2}\Big(\frac{dQ}{dt}\Big)^{-1} = D_Z^2W(tZ)(D_ZW(tZ))^{-1}$$

for $t \in [0,\epsilon]$. It is easy to verify that

$$\frac{dA}{dt} - \frac{1}{2}A^2 = 0,$$

since $\{W; Z\} = 0$.

Consider a C^∞ mapping $G(t) : [0, \epsilon] \to \mathbb{C}^{n\times n}$ which is the solution of the following initial value problem

$$\begin{cases} \dfrac{dG}{dt} = -\dfrac{1}{2}GA, \\ G(0) = I. \end{cases} \tag{2.1}$$

It is known that the solution exists and is unique (cf. [2]).

By (2.1), we have

$$\frac{d^2G}{dt^2} = -\frac{1}{2}G\left(\frac{dA}{dt} - \frac{1}{2}A^2\right) = 0.$$

That means $\frac{dG}{dt}$ is independent of t. Thus

$$\frac{dG(t)}{dt} = -\frac{1}{2}G(t)A(t) = -\frac{1}{2}G(0)A(0) = -\frac{1}{2}A(0). \tag{2.2}$$

Integrating both sides of (2.2) with respect to t from 0 to t, we have

$$G(t) - G(0) = \frac{t}{2}A(0);$$

that is,

$$G(t) = I - \frac{t}{2}A(0). \tag{2.3}$$

Substituting it into (2.2), we obtain

$$-\frac{1}{2}\Big(I - \frac{t}{2}A(0)\Big)A(t) = -\frac{1}{2}A(0).$$

Hence

$$\Big(I - \frac{t}{2}A(0)\Big)A(t) = A(0).$$

By the definition of $A(t)$, the preceding equation is

$$\Big(I - \frac{t}{2}A(0)\Big)\frac{d^2Q(t)}{dt^2} = A(0)\frac{dQ(t)}{dt}.$$

It is the same as

$$\frac{d}{dt}\left[\Big(I - \frac{t}{2}A(0)\Big)\frac{dQ(t)}{dt}\right] = \frac{1}{2}A(0)\frac{dQ(t)}{dt}.$$

Integrating both sides of the preceding equation with respect to t from 0 to t, we have

$$\Big(I - \frac{t}{2}A(0)\Big)\frac{dQ}{dt} - D_ZW(0) = \frac{1}{2}A(0)(Q(t) - W(0)),$$

since $Q(0) = W(0)$, $\frac{dQ}{dt}(0) = D_ZW(0)$. Thus

$$\frac{d}{dt}\left[(I - \frac{t}{2}A(0))Q(t)\right] = D_ZW(0) - \frac{1}{2}A(0)W(0)$$

holds. Integrating both sides of the preceding equation with respect to t from 0 to t, we have

$$\Big(I - \frac{t}{2}A(0)\Big)Q(t) - W(0) = t\Big(D_ZW(0) - \frac{1}{2}A(0)W(0)\Big). \tag{2.4}$$

By (2.3), $G(t)$ is non-singular when t is sufficiently small, solving $Q(t)$ from (2.4),

$$Q(t) = W(tZ) = t\Big(I - \frac{t}{2}A(0)\Big)^{-1}D_ZW(0) + W(0).$$

Expanding $W(tZ)$ and $t(I-\frac{t}{2}A(0))^{-1}D_ZW(0)+W(0)$ with respect to t at a neighborhood of $t=0$, we have

$$W(tZ)=W(0)+tD_ZW(0)+t^2C_2(Z)+t^3C_3(Z)+t^4C_4(Z)+\cdots, \tag{2.5}$$

where $C_2(Z),C_3(Z),C_4(Z),\cdots$ are $n\times n$ matrices, all entries of $C_2(Z),C_3(Z),C_4(Z),\cdots$ are homogeneous polynomails of entries of Z of degree 2, 3, 4, $\cdots$ respectively.

On the other hand, in a neighborhood of $t=0$, we have

$$\begin{aligned}&W(0)+t\Big(I-\frac{t}{2}A(0)\Big)^{-1}D_ZW(0)\\&=W(0)+tD_ZW(0)+\frac{t^2}{2}A(0)D_ZW(0)+\frac{t^3}{4}A^2(0)D_ZW(0)+\cdots.\end{aligned} \tag{2.6}$$

Comparing the corresponding coefficeints of t^2,t^3 in (2.5) and (2.6), we get

$$C_2(Z)=\frac{1}{2}A(0)D_ZW(0),\qquad C_3(Z)=\frac{1}{4}A^2(0)D_ZW(0).$$

Since $D_ZW(0)$ is non-singular, we have

$$C_3(Z)=\frac{1}{4}A(0)D_ZW(0)(D_ZW(0))^{-1}A(0)D_ZW(0)=C_2(Z)(D_ZW(0))^{-1}C_2(Z). \tag{2.7}$$

Let $\xi=(\xi_{ij})_{1\le i,j\le n}=D_ZW(0)$. Because $J_W(0)$ is non-singular, we may express $z_{ij},i,j=1,2,\cdots,n$ as homogeneous polynomials of the entries of ξ of degree one; that is,

$$Z=P(\xi)=(p_{ij}(\xi))_{1\le i,j\le n},$$

each $p_{ij},i,j=1,2,\cdots,n$, is a homogeneous polynomial of the entries of ξ of degree one. By (2.7), we have

$$C_3(P(\xi))=C_2(P(\xi))\xi^{-1}C_2(P(\xi)).$$

Let $C_3(P(\xi))=B_3(\xi)$, $C_2(P(\xi))=B_2(\xi)$. Then all entries of $B_2(\xi)$, $B_3(\xi)$ are homogeneous polynomials of the entries of ξ of deree 2, degree 3, respectively. From $B_3(\xi)=B_2(\xi)\xi^{-1}B_2(\xi)$, and Theorem 2 of [1], we have $B_2(\xi)=\xi L_0(\xi)$ or $B_2(\xi)=L_0(\xi)\xi$, where $L_0(\xi)$ is an $n\times n$ matrix, each entry is a homogeneous polynomial of the entries of degree one. Thus we have $C_2(Z)=D_ZW(0)L_0(D_ZW(0))$ or $C_2(Z)=L_0(D_ZW(0))D_ZW(0)$. We get

$$C_2(Z)=D_ZW(0)L(Z),\qquad\text{or}\qquad C_2(Z)=L(Z)D_ZW(0).$$

If $L(Z)+L_0(D_ZW(0))$, obviously each entry of $L(Z)$ is a homogeneous polynomial of the entries of Z of degree one.

If $C_2(Z)=D_ZW(0)L(Z)$, then

$$\begin{aligned}C_3(Z)&=C_2(Z)(D_ZW(0))^{-1}C_2(Z)\\&=D_ZW(0)L(Z)(D_ZW(0))^{-1}D_ZW(0)L(Z)\\&=D_ZW(0)(L(Z))^2.\end{aligned}$$

Similarly, we have $C_4(Z)=D_ZW(0)(L(Z))^3,\cdots$. Substituting all these results into (2.5), we obtain

$$\begin{aligned}W(tZ)=&W(0)+tD_ZW(0)+t^2D_ZW(0)L(Z)+t^3D_ZW(0)(L(Z))^2\\&+t^4D_ZW(0)(L(Z))^3+\cdots\\=&tD_ZW(0)(I-tL(Z))^{-1}.\end{aligned} \tag{2.8}$$

Similarly, in the case $C_2(Z) = L(Z)D_Z W(0)$, we obtain

$$W(tZ) = W(0) + t(I - tL(Z))^{-1} D_Z W(0). \tag{2.9}$$

Thus (2.8) or (2.9) holds true at a neighborhood of $t = 0$. No doubt, $W(tZ)$ is an analytic function of t when $tZ \in R_I$. It implies (2.8) or (2.9) holds true when $tZ \in \mathbb{R}_I$.

Let $t \to 1$. We have

$$W(Z) = W(0) + (I - L(Z))^{-1} D_Z W(0)$$

or $W(Z) = W(0) + D_Z W(0)(I - L(Z))^{-1}$ when $D_Z W(0)$ is non-singular. Since $W(Z)$ is holomorphic in $\mathbb{R}_I$, we know that $I - L(Z)$ is non-singular when Z is non-singular.

If $D_Z W(0)$ is non-singular at a point $Z \in R_I$, then there exists a neighborhood of Z, such that $D_Z W(0)$ is non-singular at this neighborhood.

By the uniqueness theorem of holomorphic mapping, we conclude that

$$W(Z) = W(0) + (I - L(Z))^{-1} D_Z W(0)$$

or $W(Z) = W(0) + D_Z W(0)(I_L(Z))^{-1}$ for all $Z \in \mathbb{R}_I$, and hence $I - L(Z)$ is non-singular when $Z \in \mathbb{R}_I$. Thus we have proved that $W = W(Z)$ is defined by (1.2) if (1.3) is true in the case that $J_W(0)$ is non-singular.

§3.

Now, we prove that $W = W(Z)$ which is defined by (1.2) is biholomorphic in $\mathbb{R}_I$.

If it is not biholomorphic, then there exist two points $Z_1, Z_2 \in R_I$, such that $W_1 = W(Z_1) = W_2 = W(Z_2)$. We consider $W(Z) = W(0) + D_Z W(0)(I - L(Z))^{-1}$ at first. R_I is a convex domain, $(1-t)Z_1 + tZ_2 \in R_I$, if $t \in [0, 1]$. Let

$$\begin{aligned}
Q(t) &= D_{(1-t)Z_1+tZ_2} W(0)(I - L((1-t)Z_1 + tZ_2))^{-1} - D_Z W(0)(I - L(Z_1))^{-1} \\
&= (D_{Z_1} W(0) + t(D_{Z_2} W(0) - D_{Z_1} W(0))[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1} \\
&\quad - D_{Z_1} W(0)(I - L(Z_1))^{-1}.
\end{aligned}$$

Then

$$Q(0) = Q(1) = 0,$$

and

$$\begin{aligned}
\frac{dQ}{dt} &= \{(D_{Z_2} W(0) - D_{Z_1} W(0) + (D_{Z_1} W(0) + t(D_{Z_2} W(0) - D_{Z_1} W(0)))[I - L(Z_1) \\
&\quad - t(L(Z_2) - L(Z_1))]^{-1}(L(Z_2) - L(Z_1))\}[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1},
\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}
\frac{d^2Q}{dt^2} &= 2\{(D_{Z_2} W(0) - D_{Z_1} W(0) + (D_{Z_1} W(0) + t(D_{Z_2} W(0) - D_{Z_1} W(0)))[I - L(Z_1) \\
&\quad - t(L(Z_2) - L(Z_1))]^{-1}(L(Z_2) - L(Z_1))\}[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1} \\
&\quad \times (L(Z_2) - L(Z_1))[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1} \\
&= 2\frac{dQ}{dt}(L(Z_2) - L(Z_1))[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1}.
\end{aligned} \tag{3.2}$$

Let

$$A(t) = 2(L(Z_2) - L(Z_1))[I - L(Z_1) - t(L(Z_2) - L(Z_1))]^{-1}. \tag{3.3}$$

Then

$$\frac{dA(t)}{dt} = 2(L(Z_2) - L(Z_1))[I - L(Z_1) - tL(L(Z_2) - L(Z_1)]^{-1} \times (L(Z_2) - L(Z_1))[I - L(Z_1) - t(L(Z_2) - L(Z_1)]^{-1}.$$

It follows that

$$\frac{dA(t)}{dt} - \frac{1}{2}A^2(t) = 0, \tag{3.4}$$

when $t \in [0,1]$. If $G(t), t \in [0,1]$ is the solution of the initial value problem

$$\begin{cases} \dfrac{dG}{dt} = -\dfrac{1}{2}AG, \\ G(0) = I, \end{cases} \tag{3.5}$$

where $A(t)$ is defined by (3.3), we have

$$\frac{d^2(QG)}{dt^2} = \frac{d^2Q}{dt^2}G + 2\frac{dQ}{dt}\frac{dG}{dt} + Q\frac{d^2G}{dt^2}.$$

By (3.3), (3.4) and (3.5),

$$\frac{d^2G}{dt^2} = -\frac{1}{2}\frac{dA}{dt}G - \frac{1}{2}A\frac{dG}{dt} = -\frac{1}{2}\frac{dA}{dt}G + \frac{1}{4}A^2G = 0.$$

By (3.2), (3.5),

$$2\frac{dQ}{dt}\frac{dG}{dt} + \frac{d^2Q}{dt^2}G = -\frac{dQ}{dt}AG + \frac{d^2Q}{dt^2}G = 0.$$

We have

$$\frac{d^2(QG)}{dt^2} = 0 \tag{3.6}$$

when $t \in [0,1]$.

Since $Q(t)$ is an analytic function of t when $t \in [0,1], G(t) \in C^\infty$, (3.6) is true for $t \in [0,1]$.

The matrix $QG\bar{G}'\bar{Q}'$ is a semi-positive definite Hermitian matrix, and hence

$$\mathrm{tr}\,(QG\bar{G}'\bar{Q}') \geq 0.$$

By (3.6),

$$\frac{d^2}{dt^2}\mathrm{tr}\,(QG\bar{G}'\bar{Q}') = \mathrm{tr}\Big(\frac{d^2}{dt^2}(QG\bar{G}'\bar{Q}')\Big) = \mathrm{tr}\Big(\frac{d(QG)}{dt}\frac{d(\overline{QG}')}{dt}\Big) \geq 0$$

for $t \in [0,1]$. That means $\mathrm{tr}\,(QG\bar{G}'\bar{Q}')$, as a function of t in [0,1], is a concave contineous function. We know that $QG\bar{G}'\bar{Q}' = 0$, at $t = 0$ and $t = 1$. It implies $\mathrm{tr}\,(QG\bar{G}'\bar{Q}') \equiv 0$, when $t \in [0,1]$, that is, $QG\bar{G}'\bar{Q}' \equiv 0, t \in [0,1]$. Hence $QG \equiv 0, t \in [0,1]$. There is a neighborhood N of $t = 0$ such that $G(t)$ is non-singular for $t \in N$, since $G(0) = I$, It follows that $Q(t) \equiv 0$, when $t \in N$. It is impossible.

We have proved that $W(Z) = W(0) + D_Z W(0)(I - L(Z))^{-1}$ is biholomorphic in $\mathbb{R}_I$.

Using the similar argument we can prove $W(Z) = W(0) + (I - L(Z))^{-1}D_Z W(0)$ is biholpmorphic in $\mathbb{R}_I$.

§4.

Finally, we would like to make the following remark.

Let us consider the mappings

$$W(Z) = (A(Z) + B)(C(Z) + D)^{-1} \quad \text{or} \quad W(Z) = (C(Z) + D)^{-1}(A(Z) + B), \tag{4.1}$$

where $A(Z), B, C(Z), D \in \mathbb{C}^{n\times n}$, all entries of $A(Z), C(Z)$ are the homogeneous polynomials of entries of Z of degree one, and $C(Z) + D$ is non-singular when $Z \in \mathbb{R}_I$.

We consider $W = W(Z) = (C(Z) + D)^{-1}(A(Z) + B)$ at first. Fix $0 \neq \Lambda \in \mathbb{C}^{n\times n}$. We have

$$D_\Lambda W(Z) = (C(Z) + D)^{-1} \left[A(\Lambda) - C(\Lambda)(C(Z) + D)^{-1}(A(Z) + B)\right], \tag{4.2}$$

$$\begin{aligned} D^2_\Lambda W(Z) &= -2(C(Z) + D)^{-1}C(\Lambda)(C(Z) + D)^{-1}[A(\Lambda) - C(\Lambda)(C(Z) + D)^{-1}(A(Z) + B)] \\ &= -2(C(Z) + D)^{-1}C(\Lambda)D_\Lambda W(Z), \end{aligned}$$

and

$$\begin{aligned} D^3_\Lambda W(Z) = 6&(C(Z) + D)^{-1}C(\Lambda)(C(Z) + D)^{-1}C(\Lambda)(C(Z) + D)^{-1} \\ &\times [A(\Lambda) - C(\Lambda)(C(Z) + D)^{-1}(A(Z) + B)] \\ = 6&(C(Z) + D)^{-1}C(\Lambda)(C(Z) + D)^{-1}C(\Lambda)D_\Lambda W(Z). \end{aligned}$$

It follows that

$$\{W; Z\}_\Lambda = 0$$

for $Z \in \mathbb{R}_I$ except a lower dimension manifold of Z such that $A(\Lambda) - C(\Lambda)(C(Z) + D)^{-1}(A(Z) + B)$ is singular.

By (4.2), we have

$$D_Z W(0) = D^{-1} \left[A(Z) - C(Z)D^{-1}B\right].$$

Using the identity of matrix

$$\begin{pmatrix} A & C \\ B & D \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & D^{-1} \end{pmatrix} \begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} = \begin{pmatrix} A - CD^{-1}B & CD^{-1} \\ 0 & I \end{pmatrix},$$

we have $\det D_Z W(0) \not\equiv 0$ if and only if $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \not\equiv 0$.

Similarly, if $W = W(Z) = (A(Z) + B)(C(Z) + D)^{-1}$, fixing $0 \neq \Lambda \in \mathbb{C}^{n\times n}$, then we have

$$\{W; Z\}_\Lambda = 0$$

for $Z \in \mathbb{R}_I$, except a lower dimension manifold of Z, and $\det D_Z W(0) \not\equiv 0$ if and only if $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \not\equiv 0$. Thus we may rewrite Theorem 1.1 as

Theorem 4.1. *The assumptions are the same as Theorem* 1.1. *Then* $\{W; Z\} = 0$ *holds for* $Z \in R_I$ *if and only if*

$$W = W(Z) = (A(Z) + B)(C(Z) + D)^{-1} \quad \text{or} \quad W = W(Z) = (C(Z) + D)^{-1}(A(Z) + B),$$

where $A(Z), B, C(Z), D \in \mathbb{C}^{n\times n}$, *all entries of* $A(Z), C(Z)$ *are homogeneous polynomials of the entries of* Z *of degree one. Moreover,* $C(Z) + D$ *is non-singular in* $\mathbb{R}_I$, *and* $\det\begin{pmatrix} A(Z) & B \\ C(Z) & D \end{pmatrix} \not\equiv 0$ *in* $\mathbb{R}_I$. *The mappings* $W = W(Z)$ *which was defined by* (4.1) *are biholomorphic in* $\mathbb{R}_I$.

Actually, mappings (1.2) and (4.1) are equivalent to each other. For example,

$$\begin{aligned} W &= W(Z) = (A(Z)+B)(C(Z)+D)^{-1} \\ &= BD^{-1} + (A(Z) - BD^{-1}C(Z))D^{-1}(I + C(Z))^{-1}; \end{aligned}$$

comparing it with (1.2), we get

$$W(0) = BD^{-1}, \quad L(Z) = -C(Z), \quad D_Z W(0) = A(Z)D^{-1} - BD^{-1}C(Z)D^{-1}.$$

Hence $\det D_Z W(0) \not\equiv 0$ and $\det\begin{pmatrix} A(Z) & B \\ C(Z) & D \end{pmatrix} \not\equiv 0$ are equivalent to each other.

Conversely,

$$\begin{aligned} W &= W(Z) = W(0) + D_Z W(0)(I - L(Z))^{-1} \\ &= [W(0)(I - L(Z) + D_Z W(0)](I - L(Z))^{-1}; \end{aligned}$$

comparing it with (4.1), we get

$$A(Z) = D_Z W(0) - W(0)L(Z), \quad B = W(0), \quad C(Z) = -L(Z), \quad D = I.$$

Hence

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det\begin{pmatrix} D_Z W(0) - W(0)L(Z) & W(0) \\ -L(Z) & I \end{pmatrix} = \det\begin{pmatrix} D_Z W(0) & 0 \\ -L(Z) & I \end{pmatrix}.$$

Thus $\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} \not\equiv 0$ and $\det D_Z W(0) \not\equiv 0$ are equivalent to each other.

The conclusion is also true when

$$W(Z) = (C(Z)+D)^{-1}(A(Z)+B) \quad \text{and} \quad W(Z) = W(0) + (I - L(Z))^{-1} D_Z W(0).$$

Theorem 4.1 is also true if we replace $\mathbb{R}_I$ by a convex domain in $\mathbb{C}^{n\times n}$.

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