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# ON A PAIR OF NON-ISOMETRIC ISOSPECTRAL DOMAINS WITH FRACTAL BOUNDARIES AND THE WEYL-BERRY CONJECTURE***

Sleeman, B. D.* Chen Hua**

**Abstract**

This paper is divided into two parts. In the first part the authors extend Kac's classical problem to the fractal case, i.e., to ask: Must two isospectral planar domains with fractal boundaries be isometric? It is demonstrated that the answer to this question is no, by constructing a pair of disjoint isospectral planar domains whose boundaries have the same interior Bouligand-Minkowski dimension but are not isometric. In the second part of this paper the authors give the exact two-term asymptotics for the Dirichlet counting functions associated with the examples given here and obtain sharp two sided estimates for the second term of the counting functions. The first result in the second part of the paper shows that the coefficient of the second term is an oscillatory function of $\lambda$, which implies that the Weyl-Berry conjecture, for the examples given here, is false. The second result implies that the weaker form of the Weyl-Berry conjecture, for these examples, is true. This in turn means that the interior Bouligand-Minkowski dimension of the examples is a spectral invariant.

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## §1. Introduction

Let $(M, g)$ be a compact Riemann manifold with boundary. Then $M$ has a Laplace operator $\Delta$ defined by $\Delta f = -\mathrm{div}\,(\mathrm{grad} f)$ that acts on functions defined on $M$. The spectrum of $M$ is a sequence of eigenvalues of $\Delta$. Two Riemannian manifolds are isospectral if their spectra coincide (counting multiplicities).

A fundamental question concerning the interplay of analysis and geometry is: must two isospectral Riemannian manifolds actually be isometric? If $M$ is a domain in the Euclidean plane then the Dirichlet eigenvalues of $\Delta$ are essentially the frequencies produced by a drumhead shaped like M. In this case Kac[14] rephrased the question in poetic terms: can one hear the shape of a drum?

The problem of uncovering geometric information about $M$ from a knowledge of the spectrum has a long history and originates with the work of Weyl[21,22] who proved that the

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*School of Mathematics, University of Leeds, Leeds LS2 9JT, England.
**Department of Mathematics, Wuhan University, Wuhan 430072, China.
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area of a plane domain is determined by the spectrum. Since Weyl's pioneering work the subject has developed along two main fronts. These are

(i) The structure of isospectral domains in $\mathbf{R}^n$,

(ii) The spectral asymptotics of the counting function for domains in $\mathbf{R}^n$.

On the former theme Milnor[18] constructed a pair of isospectral, non isometric sixteen dimensional tori. There then followed other isospectral pairs of Riemann surfaces by Vigneras[20], Buser[5], Brookes[2,3], Brooks-Tse[4]; pairs of lens spaces by Ikeda[13], pairs of domains in $\mathbf{R}^4$ by Urakawa[19] and continuous families of isospectral metrics on solvmanifolds by Gordon-Wilson[11], De Turck-Gordon[9]. Kac's question concerning planar domains was finally answered in the negative by Gordon, Webb and Wolpert[12].

Regarding the latter theme, one studies the asymptotics of the Dirichlet counting function

$$N(\lambda) \equiv N(\lambda, \Delta, M) = \#\{k \mid \lambda_k < \lambda\}. \tag{1.1}$$

That is, $N(\lambda)$ is the number of eigenvalues of the Dirichlet Laplacian $\Delta$ defined by $M$ less than a given number $\lambda$.

As $\lambda \to \infty$ Weyl[21] established the asymptotic estimate

$$N(\lambda) \sim (2\pi)^{-n}\omega_n |M|_n \lambda^{n/2}, \tag{1.2}$$

where $\omega_n$ is the volume of the unit ball in $\mathbf{R}^n$, $|\cdot|_n$ denotes the $n$-dimensional Lebesgue measure and $f(\lambda) \sim g(\lambda)$ as $\lambda \to \infty$ means $\lim\limits_{\lambda\to\infty} f(\lambda)/g(\lambda) = 1$.

Following this work Weyl[22] conjectured that

$$N(\lambda) = (2\pi)^{-n}\omega_n |M|_n \lambda^{n/2} + O(\lambda^{(n-1)/2}) \tag{1.3}$$

as $\lambda \to \infty$, which has stimulated active and intensive research for most of this century. We now know that, under a variety of geometrical and regularity conditions, for sufficiently smooth boundaries

$$N(\lambda) = (2\pi)^{-n}\omega_n |M|_n \lambda^{n/2} - C_n' |\partial M|_{n-1} \lambda^{(n-1)/2} + o(\lambda^{(n-1)/2}), \tag{1.4}$$

as $\lambda \to \infty$, where $C_n' = [4(4\pi)^{(n-1)/2}\Gamma(1+\frac{n-1}{2})]^{-1}$ (cf. References in [16]).

In 1979, Berry[1] proposed an extension of Weyl's conjecture to the case of domains with fractal boundaries, namely

$$N(\lambda) = (2\pi)^{-n}\omega_n |M|_n \lambda^{n/2} - C_{n,H} H(\partial M) \lambda^{H/2} + o(\lambda^{H/2}), \tag{1.5}$$

as $\lambda \to \infty$, where $H \in (n-1, n)$ is the Hausdorff dimension of the boundary $\partial M, H(\partial M)$ is the $H$-dimensional Hausdorff measure of $\partial M$ and $C_{n,H}$ is a positive constant depending only on $n$ and $H$. Berry's conjecture turns out to be false in general but has attracted considerable attention over the passed decade. Indeed it has been proved that

$$N(\lambda) = (2\pi)^{-n}\omega_n |M|_n \lambda^{n/2} + O(\lambda^{\delta/2}) \tag{1.6}$$

as $\lambda \to \infty$, where $\delta$ is the interior Bouligand-Minkowski dimension of the boundary $\partial M$ (or the interior Minkowski dimension for simplicity).

For an up-to-date account of these developments we cite Lapidus[16], Chen-Sleeman[7,8], Kigami-Lapidus[15], Fleckinger-Vassiliev[10].

In this paper we extend Kac's question to the fractal case, i.e., to ask : Must two isospectral planar domains with fractal boundaries be isometric?

We demonstrate that the answer to this question is no, by constructing a pair of disjoint isospectral planar domains whose fractal boundaries have the same interior Bouligand-Minkowski dimension but are not isometric. This pair is a sample of a wide class of isospectral non-isometric fractal domains which may be constructed using the methods described here.

The plan of this paper is as follows: In Section 2 we introduce some basic ideas and notions required for the construction of our examples. Section 3 describes the examples and Section 4 gives precise asymptotic estimates for the Dirichlet counting functions associated with the examples. We also prove that the so-called weaker form of the Weyl-Berry conjecture holds for the examples.

## §2. Concepts

### 2.1. Interior Bouligand-Minkowski Dimension and Measure

Given $\epsilon \geq 0$, define

$$M_\epsilon^i = \{x \in M \mid d(x, \partial M) < \epsilon\}, \tag{2.1}$$

where $d(x, \partial M)$ denotes the Euclidean distance of $x$ to the boundary $\partial M$. The set $M_\epsilon^i$ is called the interior $\epsilon$-neighbourhood of $\partial M$. For $l \geq 0$, let

$$\mu^*(l, \partial M) = \lim_{\epsilon \to 0+} \sup \epsilon^{-(n-l)} |M_\epsilon^i|_n, \quad \mu_*(l, \partial M) = \lim_{\epsilon \to 0+} \inf \epsilon^{-(n-l)} |M_\epsilon^i|_n. \tag{2.2}$$

The interior Minkowski dimension of $\partial M$ is defined as

$$\delta = \inf\{l \in \mathbf{R}_+ \mid \mu^*(l, \partial M) = 0\},$$

or

$$\delta = \sup\{l \in \mathbf{R}_+ \mid \mu^*(l, \partial M) = +\infty\}. \tag{2.3}$$

Observe that $\delta \in [n-1, n]$ and $\mu^*(\delta, \partial M) \in [0.\infty]$. On the other hand, if $H$ is the Hausdorff dimension of $\partial M$ then we know that $H \leq \delta$. Further, if $\delta$ is the interior Minkowski dimension of $\partial M$ and

$$0 < \mu_*(\delta, \partial M) = \mu^*(\delta, \partial M) < \infty, \tag{2.4}$$

we say that $\partial M$ is interior $\delta$-Minkowski measurable and denote by

$$\mu(\delta, \partial M) = \lim_{\epsilon \to 0+} \epsilon^{-(n-\delta)} \left|M_\epsilon^i\right|_n \tag{2.5}$$

the interior $\delta$-Minkowski measure of $\partial M$.

In the light of (1.6) and the above notions conjecture (1.5) (i.e., the so-called Weyl-Berry conjecture) has been modified to

$$N(\lambda) = (2\pi)^{-n} \omega_n |M|_n \lambda^{n/2} - C_{n,\delta} \mu(\delta, \partial M) \lambda^{\delta/2} + o(\lambda^{\delta/2}), \tag{2.6}$$

as $\lambda \to +\infty$ where $\delta \in (n-1, n)$ and $C_{n,\delta}$ is a positive constant depending only on $\delta$ and $n$.

In the literature cited above, it has been shown that even (2.6) is false in general (see [7,10]) and that the conjecture should be further refined. Indeed our isospectral examples constructed in this paper run counter to (2.6).

### 2.2. Isospectral Planar Domains

The isospectral non-isometric domains constructed by Gordon, Webb and Wolpert[12] are simply connected domains of rather complex shape. Since the announcement of their work, other examples have been constructed which, although not simply connected, are quite elementary.

A pair of domains which form the basis of our constructions is due to Chapman[6] and are shown in Figure 1.

Figure 1a Figure 1b

From [6] or directly using the method of separation of variables it is fairly straight forward to show that these configurations are isospectral.

In order to discuss the asymptotics of the counting function for domains with fractal boundaries and so investigate the Weyl-Berry conjecture a basic tool is the idea of tessellation of domains whereby $M$ is approximated by a finer and finer Whitney covering with a sequence of smaller and smaller disjoint and non-overlapping cubes (see [7,16]). This idea also motivates the constructions of this paper. The essential difference is that instead of using cubes we choose a tessellation built up of "tiles" formed from a combination of the domains illustrated in Figure 1. Specifically we consider tiles of the form shown in Figure 2.

Figure 2a Figure 2b

It is clear that since the domains in Figure 1 are isospectral the same is true of the tiles in Figure 2.

## §3. Isospectral domains with Fractal Boundaries

In this section we describe the pair of isospectral domains which form the basis of our discussion. The first domain is constructed as follows.

Let $Q_0$ be the central tile shown in Figure 2a. On each of the four sides of length 2 we erect a similar tile $Q_1$ of size $s$ times smaller. Thus there are four tiles $Q_1$. Next, on the three exposed faces of a tile $Q_1$ we erect three tiles $Q_2$ again scaled by $s$. The process is repeated to obtain the domain $M_1$ shown in Figure 3a.

Thus $M_1$ is the union of disjoint non-overlapping tiles $Q_k$ which are copies of $Q_0$ scaled by $s^{-k}$. Clearly there are

$$n_k = 4 \times 3^{k-1}, \quad k \geq 1, \quad n_0 = 1, \tag{3.1}$$

tiles $Q_k$.

Figure 3a Domain $M_1$

In order to show that $M_1$ is a domain with fractal boundary we need to determine the interior Bouligand-Minkowski dimension $\delta$ of $\partial M_1$. Furthermore we need to show that the length of $\partial M_1$ is infinite and also determine a condition on $s$ to ensure no overlapping of tiles. Now the length of $\partial M_1$ is

$$|\partial M_1|_1 = 4(8+2\sqrt{2})\Big\{1+\frac{4}{3}\sum_{k=1}^{\infty}\Big(\frac{3}{s}\Big)^k\Big\},$$

which is infinite if $s < 3$. To prevent overlapping of tiles it is easy to see that $s > 1+\sqrt{2}$. Thus we have the condition

$$1+\sqrt{2} < s < 3. \tag{3.2}$$

Next we calculate the "area" of interior $\epsilon$-neighbourhood of $\partial M_1$. This is seen to be

$$|M^i_{1,\epsilon}|_2 = \sum_{k=0}^{K} n_k(2\sqrt{2}(1+2\sqrt{2})\epsilon s^{-k} - (7+2\sqrt{2})\epsilon^2) + 3\sum_{k=K+1}^{\infty} n_k s^{-2k}, \tag{3.3}$$

where $K$ is such that

$$s^{-(K+1)} < \frac{7+2\sqrt{2}}{2\sqrt{2}}\cdot\frac{\epsilon}{1+2\sqrt{2}} < s^{-K}. \tag{3.4}$$

A simple calculation (see [25, Proposition 3.2, p. 42]) then shows that

$$\delta = \frac{\ln 3}{\ln s}. \tag{3.5}$$

Let us now turn to the construction of a fractal drum $M_2$ isospectral to $M_1$. This time we take $Q_0$ to be the central tile of Figure 2b and develop $M_2$ in precisely the same manner as for $M_1$ and using precisely the same scale factor $s$. The domain is shown in Figure 3b.

By analysing this domain along the lines of that for $M_1$ we again find that the length of $\partial M_2$ is infinite provided $s < 3$, is non-overlapping if $s > 1+\sqrt{2}$, and furthermore the interior Bouligand-Minkowski dimension of $\partial M_2$ is again $\delta = \ln 3/\ln s$.

From our constructions we have actually established the main result of this section.

Figure 3b Domain $M_2$

**Theorem 3.1.** *There exist non-isometric, isospectral planar domains with fractal boundaries.*

**Remark 3.1.** The domains constructed in this paper are relatively simple examples of a wide class of non-isometric isospectral fractal domains and it is clear how to generalise and explore many other domains not only in $\mathbf{R}^2$ but in higher dimensions as well.

**Remark 3.2.** It would be very interesting to construct fractal domains which are simply connected. One way of proceeding would be to open up sufficiently small cuts on adjacent sides of connecting elements as in [8, 10]. We shall consider these problems in a forthcoming paper.

Figure 4 Fleckinger-Vassiliev Example

An example of a domain in $\mathbf{R}^2$ with fractal boundary which is closely related to those constructed above is due to Fleckinger and Vassiliev[10]. In this example the basic tile $Q_0$ is the unit square. Thus $M$ is the union of disjoint open squares. The construction proceeds in precisely the same manner as the above examples in that we append four squares $Q_1$ of side $s^{-1}$ to the sides of $Q_0$. Then on each of the exposed sides of $Q_1$ we append squares $Q_2$ of sides $s^{-2}$ and so on. The resulting planar domain is shown in Figure 4.

For the same reasons as above we require $1+\sqrt{2} < s < 3$. Furthermore the interior Minkowski dimension of $\partial M$ is again $\delta = \frac{\ln 3}{\ln s}$.

This example is important in that $0 < \mu_*(\delta, \partial M) < \mu^*(\delta, \partial M) < \infty$ and that the modified Weyl-Berry conjecture (2.6) is also false. It does however satisfy what we have introduced in [7] as the "weaker form" of the modified Weyl-Berry conjecture. That is, there exist two positive constants $C_\delta^*$, $C_{*,\delta}$ depending only on $\delta$ such that the counting function $N(\lambda)$ satisfies the two sided inequality

$$\begin{aligned} C_{*,\delta}\mu_*(\delta,\partial M)\lambda^{\delta/2} + o(\lambda^{\delta/2}) &\le (2\pi)^{-2}\omega_2|M|_2\lambda - N(\lambda) \\ &\le C_\delta^*\mu^*(\delta,\partial M)\lambda^{\delta/2} + o(\lambda^{\delta/2}) \quad \text{as} \quad \lambda\to\infty. \end{aligned} \tag{3.6}$$

## §4. Two Term Asymptotics for Dirichlet Counting Functions and the Weyl-Berry Conjecture

We now turn to the study of the asymptotics of the Dirichlet counting functions associated with the isospectral fractal domains $M_1$ and $M_2$. Clearly it is sufficient to consider the counting function for one of these fractal domains; we choose $M_2$. Since $M_2$ is the union of disjoint squares and triangles we have

$$N(\lambda,\Delta,M_2) = \sum_{k=0}^{\infty} n_k N(\lambda,\Delta,Q_k), \tag{4.1}$$

where $n_k$ is given by (3.1) and $N(\lambda,\Delta,Q_k)$ can be represented by

$$N(\lambda,\Delta,Q_k) = 4N_1(\lambda,\Delta,Q_{1k}) + 4N_2(\lambda,\Delta,Q_{2k}), \tag{4.2}$$

where $N_1(\lambda,\Delta,Q_{1k})$ is the Dirichlet counting function for a square of side $s^{-k}$ and $N_2(\lambda, \Delta, Q_{2k})$ is the Dirichlet counting function for a right triangle of the form shown in Figure 1a but with sides $2s^{-k}, 2s^{-k}$ and $\sqrt{8}s^{-k}$.

It is well known that

$$N_1(\lambda,\Delta,Q_{1k}) = \#\Big\{(q_1,q_2)\in\mathbf{N}^2 \;\big|\; q_1^2+q_2^2 < \frac{\lambda}{\pi^2}s^{-2k}\Big\}, \tag{4.3}$$

and from a result of Makai[17] we know that the Dirichlet eigenvalues for the right triangle with sides $2s^{-k}, 2s^{-k}$ and $\sqrt{8}s^{-k}$ are

$$\lambda_{m,n} = \Big(\frac{m\pi}{2s^{-k}}\Big)^2 + \Big(\frac{n\pi}{2s^{-k}}\Big)^2, \quad (m,n)\in\mathbf{N}^2, \quad m>n. \tag{4.4}$$

Thus we deduce that

$$N_2(\lambda,\Delta,Q_{2k}) = \frac{1}{2}\#\Big\{(q_1,q_2)\in\mathbf{N}^2 \;\big|\; q_1^2+q_2^2 < \Big(\frac{2s^{-k}}{\pi}\Big)^2\lambda\Big\} - \frac{1}{2}\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big], \tag{4.5}$$

where $[x]$ denotes the integer part of $x$.

If we denote by $N_2(r)$ the number of positive lattice points within a disc of radius $r$, i.e., $N_2(r) = \#\{(q_1, q_2) \in \mathbf{N}^2 \mid q_1^2 + q_2^2 < r^2\}$, then

$$N(\lambda, \Delta, M_2) = 4\sum_{k=0}^{\infty} n_k N_2\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) + 2\sum_{k=0}^{\infty} n_k\Big\{N_2\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) - \Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big]\Big\}. \tag{4.6}$$

Denoting $\phi(\lambda, M_i) = \frac{1}{4\pi}|M_i|_2\lambda\,(i = 1, 2)$ as the first term (i.e., the Weyl term) of $N(\lambda, \Delta, M_i)$, then

$$\begin{aligned}\phi(\lambda, M_2) - N(\lambda, \Delta, M_2) &= 4\sum_{k=0}^{\infty} n_k\Big[\frac{\lambda}{4\pi}s^{-2k} - N_2\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big)\Big] \\ &\quad + 2\sum_{k=0}^{\infty} n_k\Big[\frac{\lambda}{\pi}s^{-2k} - N_2\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big)\Big] + 2\sum_{k=0}^{\infty} n_k\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big].\end{aligned} \tag{4.7}$$

Finally, if we define $h(r) = \frac{\pi}{4}r^2 - N_2(r)$, then (4.7) gives

$$\phi(\lambda, M_2) - N(\lambda, \Delta, M_2) = 4\sum_{k=0}^{\infty} n_k h\Big(\frac{\lambda}{\pi}s^{-k}\Big) + 2\sum_{k=0}^{\infty} n_k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) + 2\sum_{k=0}^{\infty} n_k\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big]. \tag{4.8}$$

Thus we have the following result:

**Theorem 4.1.** *Using the above notation, for $i = 1, 2$, we have as $\lambda \to \infty$*

$$\begin{aligned}\frac{|M_i|_2}{4\pi}\lambda - N(\lambda, \Delta, M_i) &= F_1\Big(\frac{\ln\lambda - 2\ln\pi}{2\ln s}\Big)\lambda^{\delta/2} + 2^{\delta-1}F_1\Big(\frac{\ln(4\lambda) - 2\ln\pi}{2\ln s}\Big)\lambda^{\delta/2} \\ &\quad + F_2\Big(\frac{\ln(2\lambda) - 2\ln\pi}{2\ln s}\Big)\lambda^{\delta/2} + O(\sqrt{\lambda}),\end{aligned} \tag{4.9}$$

*where*

$$F_1(y) = \frac{16}{3}\pi^{-\delta}\sum_{k=-\infty}^{\infty} s^{\delta(k-y)}h(s^{y-k}), \tag{4.10}$$

$$F_2(y) = \frac{8}{3}\Big(\frac{\sqrt{2}}{\pi}\Big)^{\delta}\sum_{k=-\infty}^{\infty} s^{\delta(k-y)}[s^{y-k}] \tag{4.11}$$

*are two well defined, positive, bounded, 1-periodic and left continuous functions and their sets of points of discontinuity are dense in* $\mathbf{R}$.

**Proof.** From (4.8) we have

$$\begin{aligned}&\frac{|M_i|_2}{4\pi}\lambda - N(\lambda, \Delta, M_i) \\ &= \frac{16}{3}\sum_{k=-\infty}^{\infty} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) + \frac{8}{3}\sum_{k=-\infty}^{\infty} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) + \frac{8}{3}\sum_{k=-\infty}^{\infty} 3^k\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big] \\ &\quad - \frac{16}{3}\sum_{k=-\infty}^{-1} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) - \frac{8}{3}\sum_{k=-\infty}^{-1} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) - \frac{8}{3}\sum_{k=-\infty}^{-1} 3^k\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big].\end{aligned}$$

Observe that $3^k = s^{\delta k}$, and so if we write $y_1 = \frac{\ln\lambda - 2\ln\pi}{2\ln s}$, then $\frac{\sqrt{\lambda}}{\pi}s^{-k} = s^{y_1-k}$ and $\lambda^{-\delta/2} = \pi^{-\delta}s^{-\delta y_1}$. Consequently

$$\frac{16}{3}\sum_{k=-\infty}^{\infty} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) = \frac{16}{3}\pi^{-\delta}\sum_{k=-\infty}^{\infty} s^{\delta(k-y_1)}h(s^{y_1-k})\lambda^{\delta/2} = F_1(y_1)\lambda^{\delta/2}. \tag{4.12}$$

Similarly if we define $y_2 = \frac{\ln(4\lambda)-2\ln\pi}{2\ln s}$, then

$$\frac{8}{3}\sum_{k=-\infty}^{\infty} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) = 2^{\delta-1}F_1(y_2)\lambda^{\delta/2}. \tag{4.13}$$

Next, by letting $y_3 = \frac{\ln(2\lambda)-2\ln\pi}{2\ln s}$, we have

$$\frac{8}{3}\sum_{k=-\infty}^{\infty} 3^k \Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big] = F_2(y_3)\lambda^{\delta/2}. \tag{4.14}$$

Now we denote by $P_2(r)$ the number of all lattice points within a disc of radius $r$. From a result of Chen Jingrun[23] we have, as $r \to \infty$

$$0 < \pi r^2 - P_2(r) = O\big(r^{\frac{24}{37}+\epsilon}\big), \text{ for any } \epsilon > 0. \tag{4.15}$$

Observe that $4N_2(r) + 4[r] + 1 = P_2(r)$, and so

$$0 < h(r) = r + O\big(r^{\frac{24}{37}+\epsilon}\big), \quad \text{for any } \ \epsilon > 0. \tag{4.16}$$

Since $O(r^{\frac{24}{37}+\epsilon}) \subset O(r^{\frac{2}{3}})$, we have from (4.16)

$$\sum_{k=-\infty}^{-1} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) = \frac{\sqrt{\lambda}}{\pi}\sum_{k=-\infty}^{-1} s^{(\delta-1)k} + O(\lambda^{1/3})\sum_{k=-\infty}^{-1} s^{(\delta-2/3)k},$$

which implies that

$$\sum_{k=-\infty}^{-1} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) = O(\sqrt{\lambda}). \tag{4.17}$$

In a similar manner we find

$$\sum_{k=-\infty}^{-1} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) = O(\sqrt{\lambda}), \quad \sum_{k=-\infty}^{-1} 3^k \Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big] = O(\sqrt{\lambda}). \tag{4.18}$$

By combining (4.12)–(4.18) we see that the formula (4.9) holds.

Since $\mu^*(\delta, \partial M_i) < +\infty$, for $i = 1, 2$, we know from (1.6) that $\frac{|M_i|_2}{4\pi}\lambda - N(\lambda, \Delta, M_i) = O(\lambda^{\delta/2})$, as $\lambda \to \infty$, which implies that $F_j(y)$ $(j = 1, 2)$ is well defined, positive and bounded. Furthermore it is obvious that $F_j(y) = F_j(y+1)$, i.e. $F_j(y)$ is 1-periodic. Finally we prove that $F_j(y)$ is left continuous. Actually we see that the functions $h(s^{y-k})$ and $[s^{y-k}]$ ( and so $F_1(y)$ and $F_2(y)$ ) are left continuous with discontinuity in $y \in \mathbf{R}$, satisfying

$$s^{2(y-k)} = q_1^2 + q_2^2, \quad k \in \mathbf{Z}, \quad q_j \in \mathbf{N}, \quad j = 1,\ 2, \tag{4.19}$$

$$s^{y-k} = m, \quad k \in \mathbf{Z}, \quad m \in \mathbf{N}, \tag{4.20}$$

respectively.

If we take $q_1 = q_2 = q$ in (4.19), then we see that $h(s^{y-k})$ is discontinuous at those points $y \in \mathbf{R}$, where $y = \frac{\ln 2}{2\ln s} + \frac{\ln q}{\ln s} + k$, $\ k \in \mathbf{Z}_+$, $\ q \in \mathbf{N}$. For any given $y_0 \in \mathbf{R}$, choose $k \in \mathbf{Z}$ with $|k|$ sufficiently large so that $\frac{\ln 2}{2\ln s} + k < y_0$. Furthermore, by choosing $q_k \in \mathbf{N}$, the largest positive integer for which

$$y_k = \frac{\ln 2}{2\ln s} + \frac{\ln q_k}{\ln s} + k \le y_0,$$

then we have

$$0 \le y_0 - y_k < \frac{\ln(q_k+1)}{\ln s} - \frac{\ln q_k}{\ln s} = \frac{1}{\ln s}\ln\Big(1 + \frac{1}{q_k}\Big). \tag{4.21}$$

Observe that $q_k \to +\infty$ as $k \to -\infty$ and so (4.21) implies that $y_k \to y_0$ as $k \to -\infty$. That is, the set of points of discontinuity of $F_1(y)$ is dense in $\mathbf{R}$.

Similarly from (4.20) we know that $[s^{y-k}]$ is discontinuous at those points $y \in \mathbf{R}$ satisfying $y = \frac{\ln m}{lns} + k, k \in \mathbf{Z}, m \in \mathbf{N}$. For any fixed $y_0 \in \mathbf{R}$, we choose $k \in \mathbf{Z}$ so that $k < y_0$ and choose $m_k \in \mathbf{N}$, the largest positive integer, so that $y_k = \frac{\ln m_k}{\ln s} + k \le y_0$. That is,

$$0 \le y_0 - y_k < \frac{1}{\ln s} \ln\Big(1 + \frac{1}{m_k}\Big). \tag{4.22}$$

Since $m_k \to +\infty$ as $k \to -\infty$, (4.22) shows that the set of points of discontinuity of $F_2(y)$ is dense in $\mathbf{R}$. This completes the proof of Theorem 4.1.

From Theorem 4.1 we know that the Weyl-Berry conjecture (2.6) is not true for the domains $M_1$ and $M_2$. This is because in these examples the second term of $N(\lambda)$ is an oscillatory function of $\lambda$. Theorem 4.1 also shows that Conjecture 3 in [16] holds for strictly self-similar fractal drums $M_1$ and $M_2$.

We now go on to prove , for the domains $M_1$ and $M_2$ constructed here, that the weaker form of the Weyl-Berry conjecture (3.6) holds. Actually we can give two sided sharp estimates for the second term of $N(\lambda, \Delta, M_i)$ $(i = 1, 2)$, which demonstrate that the interior Minkowski dimension $\delta$ is a spectral invariant.

**Theorem 4.2.** *Following the above notation, for $i = 1, 2$ we have as $\lambda \to \infty$*

$$c_2(\delta)\lambda^{\delta/2} + o(\lambda^{\delta/2}) \le \frac{|M_i|_2}{4\pi}\lambda - N(\lambda, \Delta, M_i) \le c_1(\delta)\mu^*(\delta, \partial M_i)\lambda^{\delta/2} + o(\lambda^{\delta/2}), \tag{4.23}$$

*where*

$$c_1(\delta) = (1 + \frac{1}{\pi})^{\frac{1}{2}} 2^{\frac{2-\delta}{2}} (2 - \delta)^{\frac{\delta-2}{\delta-1}} (\delta - 1)^{-1} + \frac{1}{4\pi} 2^{\frac{2-\delta}{2}}, \tag{4.24}$$

$$\begin{aligned} c_2(\delta) = {} & \frac{16}{3} c_d \pi^{d-1} (1 + 2^{-d}) \frac{1}{1 - s^{1-d-\delta}} + \frac{4s^{\delta-2}}{\pi(1 - s^{\delta-2})} \\ & + \frac{8\sqrt{2}}{3\pi(1 - s^{1-\delta})} - \frac{8}{3(1 - s^{-\delta})} \end{aligned} \tag{4.25}$$

*are two positive constants and in which $c_d > 0$ is a constant depending only on $d \in (1/3, 1/2]$.*

**Proof.** From a result in [7, Corollary 2.1] we have directly for $i = 1, 2$,

$$\phi(\lambda, M_i) - N(\lambda, \Delta, M_i) \le c_1(\delta)\mu^*(\delta, \partial M_i)\lambda^{\delta/2} + o(\lambda^{\delta/2}) \tag{4.26}$$

as $\lambda \to \infty$, where $c_1(\delta)$ is defined by (4.24).

To obtain a lower bound for $\phi(\lambda, M_i) - N(\lambda, \Delta, M_i)$, we take $L \sim \frac{\ln \lambda}{2\ln s}$ (i.e., $L \to \infty$ if and only if $\lambda \to \infty$), then $k > L$ implies $\frac{2\sqrt{\lambda}}{\pi} s^{-k} < 1$. Furthermore from the definitions we know that for $r \in (0, 1), N_2(r) = 0, h(r) = \frac{\pi}{4} r^2$ and $[r] = 0$.

Next, from [24] we know that the positive bounded function $\frac{1}{r}h(r)$ is at most polynomially decreasing as $r \to \infty$, and that there exists a positive constant $c_d > 0$ depending on $d \in (1/3, 1/2]$ such that

$$\inf_{r \ge 1/\pi} \{h(r)r^{d-1}\} \ge c_d > 0. \tag{4.27}$$

This implies that

$$h(r) \ge c_d r^{1-d}, \quad \text{for} \quad r \ge 1/\pi. \tag{4.28}$$

Now we have from (4.8)

$$\begin{aligned}
&\phi(\lambda, M_i) - N(\lambda, \Delta, M_i)\\
&= \frac{16}{3}\sum_{k=0}^{L} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) + \frac{16}{3}\sum_{k=L+1}^{\infty} 3^k \frac{\pi}{4}\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big)^2\\
&\quad + \frac{8}{3}\sum_{k=0}^{L} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) + \frac{8}{3}\sum_{k=L+1}^{\infty} 3^k \frac{\pi}{4}\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big)^2 + \frac{8}{3}\sum_{k=0}^{L} 3^k \Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big].
\end{aligned}$$

Since $3^k = s^{\delta k}$ and $s^L \sim \lambda^{1/2}$, on using the estimate (4.28) we have, for $d \in (1/3, 1/2]$,

$$\sum_{k=0}^{L} 3^k h(\frac{\sqrt{\lambda}}{\pi}s^{-k}) \geq c_d \pi^{d-1}\sum_{k=0}^{L}\lambda^{\frac{1-d}{2}} s^{(\delta-1+d)k} \sim c_d\pi^{d-1}\sum_{k=0}^{L} s^{(1-d-\delta)(L-k)}\lambda^{\delta/2},$$

which implies that

$$\sum_{k=0}^{L} 3^k h\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big) \geq c_d\pi^{d-1}\frac{1}{1-s^{1-d-\delta}}\lambda^{\delta/2} + o(\lambda^{\delta/2}). \tag{4.29}$$

Similarly

$$\sum_{k=0}^{L} 3^k h\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big) \geq c_d\Big(\frac{2}{\pi}\Big)^{1-d}\frac{1}{1-s^{1-d-\delta}}\lambda^{\delta/2} + o(\lambda^{\delta/2}). \tag{4.30}$$

Next

$$\sum_{k=L+1}^{\infty} 3^k\frac{\pi}{4}\Big(\frac{\sqrt{\lambda}}{\pi}s^{-k}\Big)^2 \sim \frac{1}{4\pi}\sum_{k=L+1}^{\infty} s^{(\delta-2)(k-L)}\lambda^{\delta/2} = \frac{1}{4\pi}\frac{s^{\delta-2}}{1-s^{\delta-2}}\lambda^{\delta/2} + o(\lambda^{\delta/2}). \tag{4.31}$$

In a similar manner we have

$$\sum_{k=L+1}^{\infty} 3^k\frac{\pi}{4}\Big(\frac{2\sqrt{\lambda}}{\pi}s^{-k}\Big)^2 = \frac{1}{\pi}\cdot\frac{s^{\delta-2}}{1-s^{\delta-2}}\lambda^{\delta/2} + o(\lambda^{\delta/2}). \tag{4.32}$$

Finally, we know that

$$\sum_{k=0}^{L} 3^k\Big[\frac{\sqrt{2\lambda}}{\pi}s^{-k}\Big] \geq \sum_{k=0}^{L} 3^k\Big(\frac{\sqrt{2\lambda}}{\pi}s^{-k} - 1\Big),$$

and that

$$\sum_{k=0}^{L} 3^k\frac{\sqrt{2\lambda}}{\pi}s^{-k} = \frac{\sqrt{2}}{\pi}\frac{1}{1-s^{1-\delta}}\lambda^{\delta/2} + o(\lambda^{\delta/2}), \quad -\sum_{k=0}^{L} 3^k = -\frac{\lambda^{\delta/2}}{1-s^{-\delta}} + o(\lambda^{\delta/2}). \tag{4.33}$$

Combining (4.29)–(4.33) we obtain the lower bound estimate in (4.23) and so Theorem 4.2 is proved.

**Concluding Remarks.** In this paper we have demonstrated by example that there exist non-isometric, isospectral planar domains with fractal boundaries . The examples and their method of construction allow one to generate a wide class of non-isometric,isospectral domains. It would be of considerable interest to study the problem in relation to simply connected fractal domains and in higher dimensions. Furthermore, regarding the asymptotics of the counting function, our examples show once again that the modified Weyl-Berry conjecture is false in general, but that the weaker form of the conjecture holds. It would therefore be of interest to explore the question of whether there exist non-isometric, isospectral planar domains with fractal boundaries and for which the modified Weyl-Berry conjecture holds.

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