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OSCILLATING MULTIPLIERS ON NONCOMPACT RIEMANNIAN SYMMETRIC SPACE $SL(3,\mathbb{H})/Sp(3)$**

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Abstract

The author studies the oscillating multipliers on Riemannian symmetric space $SL(3, \mathbb{H})/Sp(3)$. The results are analogous to that for Riemannian symmetric spaces of rank one and of complex type.

Keywords Oscillating multipliers, Inverse Abel transform, Riemannian symmetric space
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§0. Introduction

We consider the family $m_{a,b}$ of radial multipliers on noncompact Riemannian symmetric space $SL(3,\mathbb{H})/Sp(3)$, defined by

$$m_{a,b}(\lambda) = (||\lambda||^2 + ||\rho||^2)^{-\frac{b}{2}} e^{i(||\lambda||^2+||\rho||^2)^{\frac{a}{2}}}, \quad \mathrm{Re}\, b \geq o, \quad a > 0.$$

Let $T_{a,b}$ be the associated convolutive operator (see section 1 for its precise definition). The $L^p(\mathbb{R}^n)$ boundedness of Euclidean analogue of $T_{a,b}$ are well-known (cf. [5] for $a \neq 1$ and [11] for $a = 1$).

Guilini and Meda[6] have studied the oscillating multipliers $T_{a,b}$ on noncompact Riemannian symmetric spaces of rank one. The values $a = 1$ and $a = 2$ are of particular interest, since the operators $T_{1,b}$ and $T_{2,b}$ are closely related to the wave and the Schrödinger equations. Recently, Alexopoulos[1] generalizes these results to the connected Lie groups of polynomial growth and Riemannian manifolds of nonnegative Ricci curvature. In presente paper we extend the results of [6] to Riemannian symmetric space $SL(3,\mathbb{H})/Sp(3)$. More precisely, we shall prove

Theorem. *For Riemannian symmetric space $M = SL(3,\mathbb{H})/Sp(3)$, the convolutive operators $T_{a,b}$ associated with the oscillating multipliers $m_{a,b}$ have the properties:*

(1) *when $a > 1$, $T_{a,b}$ is bounded on $L^p(M)$ if and only if $p = 2$;*

(2) *when $a = 1$, $T_{a,b}$ is bounded on $L^p(M)$ if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{\mathrm{Re}\, b}{(n-1)}$, $n = \dim M = 14$;*

(3) *when $a < 1$, $T_{a,b}$ is bounded on $L^p(M)$ if $|\frac{1}{p} - \frac{1}{2}| \leq \frac{\mathrm{Re}\, b}{na}$.*

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§1. Preliminlaries

1.1. Let $\mathbb{H}$ be the division ring of quaternions and $sl(3,\mathbb{H})$ be the real simple Lie algebra (which is written as $\underline{g}$) of all matrices of order 3 over $\mathbb{H}$, with imaginary trace. With respect to Cartan involution θ defined by $\theta(Z) = -{}^t\overline{Z}, Z\in\underline{g}$, we have the Cartan decomposition $\underline{g} = \underline{k}\oplus\underline{p}$. The set $\underline{a}(\subset\underline{p})$ formed by all real diagonal matrices is a maximal abelian subspace of $\underline{p}$. Each $H\in\underline{a}$ can be presented as $H = H_t = H_{(t_1,t_2,t_3)}$ with $t_1+t_2+t_3 = 0$. We take a positive Weyl chamber $\underline{a}^+ = \{H_t\in\underline{a}|t_1 > t_2 > t_3\}$. Then the set of positive roots of $(\underline{g},\underline{a})$ is $\Delta^+ = \{\alpha,\beta,\gamma\}$, where $\alpha(H_t) = t_1 - t_2$, $\beta(H_t) = t_2 - t_3$, $\gamma(H_t) = t_1 - t_3$. For each $\xi\in\Delta^+$, let $\underline{g}^\xi = \{x\in\underline{g}|[H,X] = \xi(H)X, \forall H\in\underline{a}\}$ be the root subspace of ξ. Then $m_\xi = \dim\underline{g}^\xi = 4$. The half-sum of the positive roots is $\rho = \frac{1}{2}\sum\limits_{\xi\in\Delta^+} m_\xi\xi = 2(\alpha+\beta+\gamma) = 4\gamma$ (see [8]).

1.2. Let $\underline{a}^*$ be the real dual of $\underline{a}$ and $\underline{a}^*_C$ be the complexification of $\underline{a}^*$. The Kiling form B of the Lie algebra $\underline{g}$ induces an inner product $\langle .,.\rangle$ and a norm $||.||$ on $\underline{p}$, which are defined respectively by $\langle X,Y\rangle = -B(X,\theta Y)$ and $||X||^2 = \langle X,X\rangle, \forall X,Y\in\underline{p}$. For each $\lambda\in\underline{a}^*$, let H_λ be unique element $\in\underline{a}$ such that $\lambda(H) = \langle H,H_\lambda\rangle$ for every $H\in\underline{a}$. The inner product and the norm on $\underline{a}^*$ are defined respectively by $\langle\lambda,\mu\rangle = \langle H_\lambda,H_\mu\rangle$ and $||\lambda||^2 = ||H_\lambda||^2$ for $\lambda,\mu\in\underline{a}^*$; these inner product and norm are extented complexificaly to $\underline{a}^*_C$.

The Weyl group W of $(\underline{g},\underline{a})$ is identified to the permutation group of order 3 by setting $\sigma H_{(t_1,t_2,t_3)} = H_{(t_{\sigma(1)},t_{\sigma(2)},t_{\sigma(3)})}$ for $H_{(t_1,t_2,t_3)}\in\underline{a}$ and $\sigma\in W$. The action of $\sigma(\in W)$ on $\lambda\in\underline{a}^*$ (or $\in\underline{a}^*_C$) is given by duality, $\sigma\lambda(H) = \lambda(\sigma^{-1}H)$ (see [8]).

1.3. Let $\underline{n} = \sum\limits_{\xi\in\Delta^+}\underline{g}^\xi$. Denote by K, A, and N the analytic subgroups of G, having respectively the Lie algebra $\underline{k}$, $\underline{a}$, $\underline{n}$. Then $G = KAN$ is an Iwasawa decomposition of G. $\forall x\in G$, the Iwasawa projection $H(x)$ of x is a unique element $\in\underline{a}$ such that $x\in Ke^{H(x)}N$. The elementary spherical functions $\varphi_\lambda(x)$ are defined as[9]

$$\varphi_\lambda(x) = \int_K e^{(i\lambda-\rho)(H(xk))}dk, \quad \lambda\in\underline{a}^*_C.$$

1.4. Let $C_c^\infty(G//K)$ be the space of all bi-K-invariant smooth functions on G with a compact support. The spherical Fourier transform is defined by

$$\tilde{f}(\lambda) = \int_G f(x)\varphi_{-\lambda}(x)dx, \quad \lambda\in\underline{a}^*_C, \quad f\in C_c^\infty(G//K).$$

The Abel transform of f is given by[9] $\mathcal{A}f(H) = e^{\rho(H(x))}\int_N f(e^{H(x)}n)dn$. We know that the spherical Fourier transform ˜ is the composition of the Abel transform $\mathcal{A}$ and of the Euclidean Fourier transform $\mathcal{F}$: $\tilde{f}(\lambda) = \mathcal{F}\mathcal{A}f(\lambda), \lambda\in\underline{a}^*_C$. Conversely, if there exists the inverse Abel transform $\mathcal{A}^{-1}$, then the following inverse spherical Fourier transform holds[9]:

$$f(e^H) = \mathcal{A}^{-1}\mathcal{F}^{-1}\tilde{f}(e^H), \quad H\in\underline{a}, \tag{1.1}$$

where $\mathcal{F}^{-1}$ is the inverse Euclidean Fourier transform on $\underline{a}^*$. It is also valid that

$$f(e^H) = \int_{\underline{a}^*}\tilde{f}(\lambda)\varphi_\lambda(e^H)|\mathbf{c}(\lambda)|^{-2}d\lambda, \tag{1.2}$$

where $\mathbf{c}(\lambda)$ is the Harish-Chandra's $\mathbf{c}$-function.

1.5. The inverse Abel transform for Riemannian symmetric space $M = SL(3,\mathbb{H})/Sp(3)$

is[7] : for every regular $H\in\underline{a}$ and $f\in C_c^\infty(G//K)$,

$$f(e^H)=c\prod_{\xi\in\Delta^+}\sinh^{-2}\xi(H)\Big\{\partial^2_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}-\sum_{\alpha,\beta,\gamma}||\alpha||^2\coth\alpha(H)\partial_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}$$
$$+\sum_{\alpha,\beta,\gamma}||\alpha||^2||\beta||^2\coth\alpha(H)\coth\beta(H)\partial_{H_\alpha}\partial_{H_\beta}\partial^2_{H_\gamma}-\Big[\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)$$
$$+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big]\partial_{H_\alpha}\partial_{H_\beta}\partial_{H_\gamma}\{\mathcal{F}^{-1}\tilde{f}(H)\Big\},\qquad(1.3)$$

where $\partial_{H_\xi}(f)(H)=\lim\limits_{\varepsilon\to 0}\frac{1}{\varepsilon}[f(H+\varepsilon H_\xi)-f(H)]$, $\sum\limits_{\alpha,\beta,\gamma}$ stands for the sum for all cyclic permutations of α,β,γ, for example

$$\sum_{\alpha,\beta,\gamma}||\alpha||^2\coth\alpha(H)\partial_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}=||\alpha||^2\coth\alpha(H)\partial_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}+||\beta||^2\coth\beta(H)\partial^2_{H_\alpha}\partial_{H_\beta}\partial^2_{H_\gamma}$$
$$+||\gamma||^2\coth\gamma(h)\partial^2_{H_\alpha}\partial^2_{H_\beta}\partial_{H_\gamma}.$$

The elementary spherical functions φ_λ of the group $G=SL(3,\mathbb{H})$ are[7]

$$\varphi_\lambda(e^H)=\prod_{\xi\in\Delta^+}\Big[\frac{\langle\rho,\xi\rangle(\langle\rho,\xi\rangle+||\xi||^2)}{\langle i\lambda,\xi\rangle(\langle\lambda,\xi\rangle^2+||\xi||^4)}\times\frac{4^{-3}}{\sinh^2\xi(H)}\Big]\sum_{w\in W}\det w$$
$$\times\Big[\prod_{\xi\in\Delta^+}(\langle iw\lambda,\xi\rangle-||\xi||^2\coth\xi(H))-\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big]e^{iw\lambda(H)}.\qquad(1.4)$$

The measures on K and on $\underline{a}$ are normalized in a way such that[9]

$$\int_M f(x)dx=c\int_K dk\int_{\underline{a}}f(e^H)\prod_{\xi\in\Delta^+}\sinh^4\xi(H)dH,\quad\forall f\in C_c^\infty(G//K).\qquad(1.5)$$

We know that[8] $l=\dim\underline{a}=2$ is the rank of $M=SL(3,\mathbb{H})/Sp(3)$. If denote $m=\sum\limits_{\xi\in\Delta^+}m_\xi=12$, then it is clear that $n=\dim M=m+l=14$.

1.6. A convolutive operator on a noncompact Riemannian symmetric space G/K is an operator T on the space $L^p(G/K)$, which commutes with the left G-translations. By the spherical Plancherel theorem, for each convolutive operator T on $L^2(G/K)$, there exists a W-invariant function ψ on $\underline{a}_C^*$, such that $(Tf)^\sim(\lambda)=\psi\tilde{f}(\lambda)$. Conversely, certain functions ψ on $\underline{a}_C^*$, the so called multipliers, correspond to convolutive operators T on $L^p(G/K)$. The purpose of the present paper is to study the multipliers $m_{a,b}(\lambda)$ and their associated convolutive operators $T_{a,b}$ on Riemannian symmetric space $M=SL(3,\mathbb{H})/Sp(3)$ (see [6]): $T_{a,b}f(x)=\int_{\underline{a}^*}m_{a,b}(\lambda)f*\varphi_\lambda|\mathbf{c}(\lambda)|^{-2}d\lambda$.

§2. Basic Facts and the Case $a>1$

In this section we set down some basic facts concerning the multipliers $m_{a,b}$ and their associated convolutive operators $T_{a,b}$. Then we treat the most trivial case $a>1$. Since $r^{-b}=\frac{1}{\Gamma(\frac{b}{a})}\int_0^\infty\sigma^{(\frac{b}{a}-1)}e^{-\sigma r^a}d\sigma, r>0$, we get

$$T_{a,b}f(x)=\frac{1}{\Gamma(\frac{b}{a})}\int_0^{+\infty}\sigma^{(\frac{b}{a}-1)}f*q_{\sigma,a}(x)d\sigma,\qquad(2.1)$$

where

$$q_{\sigma,a}(x)=\int_{\underline{a}^*}e^{(i-\sigma)(||\lambda||^2+||\rho||^2)^{\frac{a}{2}}}\varphi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}d\lambda. \tag{2.2}$$

By spherical Plancherel formula

$$||f{*}q_{\sigma,a}||_{L^2(G/K)}\leq e^{-\sigma||\rho||^a}||f||_{L^2(G/K)}, \tag{2.3}$$

thus $T_{a,b}$ extends to a bounded operator on $L^2(G/K)$ for every admissible a and b. Let $r_{a,b}$ be the radial distribution on G/K such that $\tilde{r}_{a,b}=m_{a,b}$, then $T_{a,b}f=f{*}r_{a,b}$. Let $T^*_{a,b}$ denote the adjoint operator of $T_{a,b}$. Then $T^*_{a,b}$ corresponds to the multiplier

$$m^*_{a,b}(\lambda)=(||\lambda||^2+||\rho||^2)^{-\frac{1}{2}\bar{b}}e^{-i(||\lambda||^2+||\rho||^2)^{\frac{a}{2}}},\quad \mathrm{Re}\, b\geq 0,\quad a>0.$$

Therefore, when dealing with the L^p boundedness of $T_{a,b}$, we can restrict ourselves to the case $1<p<2$. Firstly we establish an orthonormal basis of $\underline{a}$.

Lemma 2.1. *Let* $H_1=\frac{H_\alpha}{||\alpha||}, H_2=\frac{H_\alpha}{\sqrt{3}||\alpha||}+\frac{2H_\beta}{\sqrt{3}||\beta||}$. *Then* (H_1,H_2) *is an orthonormal basis of* $\underline{a}$.

Proof. For every pair $H_t=H_{(t_1,t_2,-t_1-t_2)}, H_s=H_{(s_1,s_2,-s_1-s_2)}\in\underline{a}$, it is well know that

$$\langle H_t,H_s\rangle=\sum_{\xi\in\Delta^+}2m_\xi\xi(H_t)\xi(H_s)=24(2t_1s_1+2t_2s_2+t_1s_2+t_2s_1),$$

hence $||H_t||^2=48(t_1^2+t_2^2+t_1t_2)$. Let $H_\alpha=H_(a_1,a_2,-a_1-a_2)$, with $a_1,a_2\in\mathbb{R}$. Then

$$t_1-t_2=\alpha(H_t)=\langle H_t,H_\alpha\rangle=24(2t_1a_1+2t_2a_2+t_1a_2+t_2a_1),$$

so $H_\alpha=H_{(\frac{1}{24},-\frac{1}{24},0)}$. The same argument gives

$$H_\beta=H_{(0,\frac{1}{24},-\frac{1}{24})},\quad H_\gamma=H_{(\frac{1}{24},0,-\frac{1}{24})}.$$

From these expressions we obtain easily

$$||\alpha||^2=||\beta||^2=||\gamma||^2=\frac{1}{12},\quad -\langle\alpha,\beta\rangle=\langle\beta,\gamma\rangle=\langle\gamma,\alpha\rangle=\frac{1}{24}. \tag{2.4}$$

Using these equalities we can verify that (H_1,H_2) is an orthonormal basis of $\underline{a}$.

Take $\lambda_1,\lambda_2\in\underline{a}^*$ such that $\lambda_i(H_j)=\delta_{ij}, 1\leq i,j\leq 2$. Then (λ_1,λ_2) constitutes an orthonormal basis of $\underline{a}^*$ and of $\underline{a}^*_C$. Therefore, every $\lambda\in\underline{a}^*_c$ can be written uniquely as $\lambda=\sum\limits_{j=1}^2 r_j\lambda_j$ with $r_j\in C$. By the same reason, we have $\rho=\sum\limits_{j=1}^2\rho_j\lambda_j$ with $\rho_j\in\mathbb{R}^+$. Hence

$$m_{a,b}(\lambda)=\Big[\sum_{j=1}^2(r_j^2+\rho_j^2)\Big]^{-\frac{b}{2}}\exp i\Big[\sum_{j=1}^2(r_j^2+\rho_j^2)\Big]^{\frac{a}{2}}.$$

Denote by S_ε the domain $\{\lambda\in\underline{a}^*_C|,|Im\lambda_j|<\varepsilon\rho_j\}, 0<\varepsilon<1$, then we have

Lemma 2.2. *When* $a>1$, $m_{a,b}(\lambda)$ *is not bounded in* S_1.

Proof. Writing $\lambda=\sum\limits_{j=1}^2(\sigma_j+i\tau_j)\lambda_j$, with $\sigma=\sum\sigma_j\lambda_j\in\underline{a}^*$, we have

$$|||\lambda||^2+||\rho||^2|=\Big\{\sum_{j=1}^2(\sigma_j^2+\rho_j^2-\tau_j^2)+4\Big[\sum_{j=1}^2\sigma_j\tau_j\Big]^2\Big\}^{\frac{1}{2}},$$

and

$$\arg(||\lambda||^2+||\rho||^2)=\arctan\Big\{\Big[2\sum_{j=1}^2\sigma_j\tau_j\Big]\Big[\sum_{j=1}^2(\sigma_j^2+\rho_j^2-\tau_j^2)\Big]^{-1}\Big\}.$$

It is not difficult to see that

$$|m_{a,b}(\lambda)| = |||\lambda||^2+||\rho||^2|^{-\frac{1}{2}\mathrm{Re}\, b}e^{-\frac{\mathrm{Im}\, b}{2}\arg(||\lambda||^2+||\rho||^2)-|||\lambda||^2+||\rho||^2|^{\frac{a}{2}}\sin(\frac{a}{2}\arg(||\lambda||^2+||\rho||^2))}. \quad (2.5)$$

If $a>1$, then we have

$$|||\lambda||^2+||\rho||^2|^{\frac{a}{2}}\Big|\sin\Big(\frac{a}{2}\arg(||\lambda||^2+||\rho||^2)\Big)\Big|\geq c|||\lambda||^2+||\rho||^2|^{\frac{a}{2}}\Big|\sum_{j=1}^{2}\sigma_j\tau_j\Big|. \quad (2.6)$$

If $\sigma_j(1\leq j\leq 2)$ are sufficently large,

$$\Big[\sum_{j=1}^{2}(\sigma_j^2+\rho_j^2-\tau_j^2)\Big]^2+4\Big[\sum_{j=1}^{2}\sigma_j\tau_j\Big]^2\leq 2\Big(\sum_{j=1}^{2}\sigma_j^2\Big)\Big(\sum_{j=1}^{2}\sigma_j^2+2\sum_{j=1}^{2}\tau_j^2\Big)<4\Big[\sum_{j=1}^{2}\sigma_j^2\Big]^2,$$

hence

$$\Big|\sum_{j=1}^{2}\sigma_j\tau_j||\lambda||^2+||\rho||^2\Big|^{-\frac{1}{2}}\geq\Big|\sum_{j=1}^{2}\sigma_j\tau_j\Big|\Big|2\sum_{j=1}^{2}\sigma_j^2\Big|^{-\frac{1}{2}}. \quad (2.7)$$

In view of (2.6), (2.7) and (2.5), we are ready to draw our assertion.

Proof of Part (1) of Theorem. The combination of Lemma 2.2 and Theorem 1 in [3] implies immediately this result.

§3. The Case a=1

Set $q_\sigma = q_{\sigma,1}$ and let T_{q_σ} be the associated convolutive operator. Then from (2.1)

$$T_{1,b}f(e^H)=\frac{1}{\Gamma(b)}\int_0^{+\infty}\sigma^{(b-1)}T_{q_\sigma}f(e^H)d\sigma,$$

hence

$$||T_{1,b}||_{(p,p)}=\frac{1}{\Gamma(b)}\int_0^{+\infty}\sigma^{(b-1)}||T_{q_\sigma}||_{(p,p)}d\sigma,$$

where $||.||_{(p,p)}$ stands for the operator norm on the space $L^p(G/K)$.

In view of (1.1), (1.2) and (2.2), if denote $z=i-\sigma$, then

$$q_\sigma(e^H)=(e^{z(||.||^2+||\rho||^2)^{\frac{1}{2}}})\check{}(H)=\mathcal{A}^{-1}\mathcal{F}^{-1}(e^{z(||.||^2+||\rho||^2)^{\frac{1}{2}}})(H).$$

Lemma 3.1. $\mathcal{F}^{-1}\tilde{q}_\sigma(H)=cz(\sqrt{||H||^2+z^2})^{-\frac{l+1}{2}}K_{\frac{l+1}{2}}(\sqrt{||H||^2+z^2})$, *where* K_ν *is the second kind modified Bessel function* (see [10, p. 66]).

Proof. We know that $\tilde{q}_\sigma(\lambda)=e^{z\sqrt{||\lambda||^2+||\rho||^2}}$, while (see [10, p. 73])

$$e^{z\sqrt{||\lambda||^2+||\rho||^2}}=\sqrt{-\frac{2}{\pi}z}K_{-\frac{1}{2}}\big(-z\sqrt{||\lambda||^2+||\rho||^2}\big)\big(\sqrt{||\lambda||^2+||\rho||^2}\big)^{\frac{1}{2}}$$

is a radial function on $\underline{a}^*$, the Theorem 3.3 of [14] gives

$$\int_{\underline{a}^*}e^{z\sqrt{||\lambda||^2+||\rho||^2}}e^{i\lambda(H)}d\lambda=c||H||^{-\frac{l-2}{2}}\int_0^{+\infty}\sqrt{-\frac{2}{\pi}z}K_{-\frac{1}{2}}(-z\sqrt{||\lambda||^2+||\rho||^2})$$
$$\times(\sqrt{||\lambda||^2+||\rho||^2})^{\frac{1}{2}}J_{\frac{l-2}{2}}(||\lambda||||H||)||\lambda||^{\frac{l}{2}}d||\lambda||,$$

where J_ν is the Bessel function of order ν. Then from [10, p. 104] we get

$$\int_{\underline{a}^*}e^{z\sqrt{||\lambda||^2+||\rho||^2}}e^{i\lambda(H)}d\lambda=cz(\sqrt{||H||^2+z^2})^{-\frac{l+1}{2}}K_{\frac{l+1}{2}}(||\rho||\sqrt{||H||^2+z^2}).$$

In the last step we have used the property $K_{-\nu}(u)=K_\nu(u)$ (see [10, p. 67]).

Lemma 3.2. *For each* $\xi\in\Delta^+$, *we have*

$$\partial_{H_\xi}\{\mathcal{A}^{-1}\tilde{q}_\sigma(H)\} = -cz(\sqrt{||H||^2+z^2})^{-(\frac{l+1}{2}+1)}K_{(\frac{l+1}{2}+1)}(||\rho||\sqrt{||H||^2+z^2})\xi(H).$$

Proof. We know that (see [10, p. 87]) $[u^{-\nu}K_\nu(u)]' = -u^{-\nu}K_{\nu+1}(u)$, and by the definition

$$\partial_{H_\xi}(||H||^2) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\{\langle H+\varepsilon H_\xi, H+\varepsilon H_\xi\rangle - \langle H, H\rangle\} = 2\xi(H),$$

hence $\partial_{H_\xi}(||H||) = \xi(H)||H||^{-1}$. Then the chain rule gives the desired rusult.

Now, with the help of Lemma 3.2, we can draw step by step the inverse Abel transform of $\mathcal{F}^{-1}\tilde{q}_\sigma(H)$. For the sake of brevity we denote from now on

$$k_\nu(H) = (\sqrt{||H||^2+z^2})^{-\nu}K_\nu(||\rho||\sqrt{||H||^2+z^2}).$$

Hence Lemma 3.2 can be written as $\partial_{H_\xi}\{czk_{(\frac{l+1}{2})}(H)\} = -czk_{(\frac{l+1}{2}+1)}(H)\xi(H)$.

Lamma 3.3. *If denote* $g(H) = \prod_{\xi\in\Delta^+}\xi(H) = \alpha(H)\beta(H)\gamma(H)$, *then it is valid that*

$$\partial_{H_\alpha}\partial_{H_\beta}\partial_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = -czk_{(\frac{l+1}{2}+3)}(H)g(H).$$

Proof. By Lemma 3.3 we get $\partial_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = -czk_{(\frac{l+1}{2}+1)}(H)\gamma(H)$,

$$\partial_{H_\beta}\partial_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = cz\{k_{(\frac{l+1}{2}+2)}(H)\beta(H)\gamma(H) - k_{(\frac{l+1}{2}+1)}(H)\langle\beta,\gamma\rangle\},$$

$$\partial_{H_\alpha}\partial_{H_\beta}\partial_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = cz\{-k_{(\frac{l+1}{2}+3)}(H)g(H) + k_{(\frac{l+1}{2}+2)}(H)[\langle\alpha,\beta\rangle\gamma(H) + \langle\beta,\gamma\rangle\alpha(H) + \langle\gamma,\alpha\rangle\beta(H)]\},$$

but in view of (2.4) and of the fact that $\gamma(H) = \alpha(H)+\beta(H)$, the second term of the right hand side of last equation is zero, so our assertion is valid. Again by Lemma 3.2,

$$\partial_{H_\alpha}\partial_{H_\beta}\partial^2_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = cz\{k_{(\frac{l+1}{2}+4)}(H)g(H)\gamma(H) - k_{(\frac{l+1}{2}+3)}(H)(\partial_{H_\gamma}g)(H)\}$$
$$= cz\sum_{j=0}^{1}k_{(\frac{l+1}{2}+3+j)}(H)A_{\gamma,2j+2}(H),$$

where $A_{\gamma,2j+2}(H)$ is a $(2j+2)$ degree homogeneous polynomial of H (abbrivated to $(2j+2)$-deg.h.p. of H) with the coefficients depending on γ. Similarly we have

$$\partial_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = cz\{-k_{(\frac{l+1}{2}+5)}(H)g(H)\beta(H)\gamma(H)$$
$$+ k_{(\frac{l+1}{2}+4)}(H)[\beta(H)(\partial_{H_\gamma}g)(H) + \gamma(H)(\partial_{H_\beta}g)(H)] - k_{(\frac{l+1}{2}+3)}(H)(\partial_{H_\beta}\partial_{H_\gamma}g)(H)\}$$
$$= cz\sum_{j=0}^{2}k_{(\frac{l+1}{2}+3+j)}(H)B_{\beta,\gamma,2j+1}(H),$$

where $B_{\beta,\gamma,2j+1}$ is a $(2j+1)$-deg.h.p. of H with the coefficients depending on β,γ.

Finally, if consider $\langle\alpha,\beta\rangle\gamma(H) + \langle\beta,\gamma\rangle\alpha(H) + \langle\gamma,\alpha\rangle\beta(H) = 0$, we have

$$\partial^2_{H_\alpha}\partial^2_{H_\beta}\partial^2_{H_\gamma}\{\mathcal{F}^{-1}\tilde{q}_\sigma\}(H) = cz\Big\{k_{(\frac{l+1}{2}+6)}(H)[g(H)]^2 - k_{(\frac{l+1}{2}+5)}\sum_{\alpha,\beta,\gamma}\beta(H)\gamma(H)(\partial_{H_\alpha}g)(H)$$
$$+ k_{(\frac{l+1}{2}+4)}(H)\sum_{\alpha,\beta,\gamma}[\alpha(H)(\partial_{H_\beta}\partial_{H_\gamma}g)(H) + \langle\alpha,\beta\rangle(\partial_{H_\gamma}g)(H)]$$
$$- k_{(\frac{l+1}{2}+3)}(H)(\partial_{H_\alpha}\partial_{H_\beta}\partial_{H_\gamma}g)(H)\Big\}$$
$$= cz\sum_{j=0}^{3}k_{(\frac{l+1}{2}+3+j)}(H)C_{2j}(H),$$

where $C_{2j}(H)$ is a $2j$-deg.h.p. of H with the coefficients which are the symmetric functions of α, β, γ, and in particular $C_6(H) = [g(H)]^2$.

Substituting these expressions of $\partial_{H_\alpha}^{j_\alpha} \partial_{H_\beta}^{j_\beta} \partial_{H_\gamma}^{j_\gamma}$, where $1 \le j_\alpha, j_\beta, j_\gamma \le 2$, into (1.3), we obtain the explicite expression of $q_\sigma(e^H)$, i.e.,

$$\begin{aligned} q_\sigma(e^H) = cz \prod_{\xi \in \Delta^+} \sinh^{-2}\xi(H) \Big\{ & \sum_{j=0}^{3} k_{(\frac{l+1}{2}+3+j)}(H) C_{2j}(H) \\ & - \sum_{j=0}^{2} k_{(\frac{l+1}{2}+3+j)}(H) \sum_{\alpha,\beta,\gamma} ||\alpha||^2 \coth\alpha(H) B_{\beta,\gamma,2j+1}(H) \\ & + \sum_{j=0}^{1} k_{(\frac{l+1}{2}+3+j)} \sum_{\alpha,\beta,\gamma} ||\alpha||^2 ||\beta||^2 \coth\alpha(H) \coth\beta(H) A_{\gamma,2j+2}(H) \\ & - k_{(\frac{l+1}{2}+3)} \Big[\prod_{\xi \in \Delta^+} ||\xi||^2 \coth\xi(H) + \frac{1}{2} \prod_{\xi \in \Delta^+} ||\xi||^2 \sinh^{-1}\xi(H) \Big] \Big\} g(H). \end{aligned} \tag{3.1}$$

With the aid of (3.1), we can estimate the L^1-norm of q_σ. But for this purpose we yet need an auxilary fact, i.e., the following

Lemma 3.4. *If y is a fixed vector in R^n and f is a radial function in $L^1(\mathbb{R}^n), n \ge 2$, then*

$$\int_{R^n} f(x) e^{xy} dx = c||y||^{-\frac{n-2}{2}} \int_0^{+\infty} f(||x||) I_{\frac{n-2}{2}}(||y|| ||x||) ||x||^{\frac{n}{2}} d||x||,$$

where I_ν is the first kind modified Bessel function (see [10, p. 66]).

Proof. It is sufficient to replace $-ixy$ by xy in the proof of Theorem 3.3 in [14] and consider the definition of I_ν (see [10, p. 66]).

Lemma 3.5. *The following estimation is valid:*

$$||q_\sigma||_{L^1(G/K)} = ||T_{q_\sigma}||_{(1,1)} \le c \begin{cases} \sigma, & if \ \sigma \ge 1, \\ (\frac{1}{\sigma})^{\frac{n-1}{2}}, & if \ \sigma < 1. \end{cases}$$

Proof. We know that (see [10, p. 66, p. 139])

$$K_\nu(u) \sim \begin{cases} \frac{\Gamma(\nu)}{2} (\frac{u}{2})^{-\nu}, & \text{if } u \to 0, \\ \sqrt{\frac{\pi}{2u}} e^{-u}, & \text{if } u \to +\infty, \end{cases} \quad \text{and} \quad I_\nu(u) \sim \begin{cases} \frac{1}{\Gamma(\nu)} (\frac{u}{2})^\nu, & \text{if } u \to 0, \\ \frac{1}{\sqrt{2u\pi}} e^u, & \text{if } u \to +\infty. \end{cases}$$

(1) $\sigma \ge 1, ||H|| \ge 1$. From above asymptotic expansion, there exists a $c > 0$ such that

$$|k_{(\frac{l+1}{2}+j)} \le c| |\sqrt{||H||^2 + z^2}|^{-(\frac{l+2}{2}+j)} e^{-||\rho|| |\sqrt{||H||^2+z^2}|} \le c||H||^{-(\frac{l+2}{2}+j)} e^{-||\rho|| ||H||}. \tag{3.2}$$

We see that the leading term in (3.1) is $k_{(\frac{l+1}{2}+6)}(H) C_6(H)$, so by (3.2) and Lemma 3.4

$$\int_{||H|| \ge 1} |q_\sigma(e^H) \prod_{\xi \in \Delta^+} \sinh^4 \xi(H)| dH \le c\sigma \int_1^{+\infty} ||H||^{-7} e^{-||\rho|| ||H||} I_0(||\rho|| ||H||) d||H||,$$

while from the asymptotic expansion of I_ν, there exists a constant $c > 0$ such that

$$|I_0(||\rho|| ||H||)| \le c||H||^{-\frac{1}{2}} e^{||\rho|| ||H||}, \quad \forall ||H|| \ge 1, \tag{3.3}$$

hence

$$\int_{||H|| \ge 1} |q_\sigma(e^H) \prod_{\xi \in \Delta^+} \sinh^4 \xi(H)| dH \le c\sigma \int_1^{+\infty} ||H||^{-\frac{15}{2}} d||H|| < c\sigma.$$

(2) $\sigma\geq 1, ||H|| < 1$. In this case, (3.2) remains valid, while $|I_0(||\rho||||H||)|\leq c, \forall ||H|| < 1$. We observe that each term in (3.1) does not exceed $k_{(\frac{l+1}{2}+6)}(H)$, so

$$\int_{||H||<1} |q_\sigma(e^H)\prod_{\xi\in\Delta^+}\sinh^4\xi(H)|dH\leq c\sigma\int_0^1 e^{-||\rho||||H||}||H||^{\frac{l}{2}}d||H|| < c\sigma.$$

(3) $\sigma < 1, ||H|| > 1$. This time, (3.2) and (3.3) are also valid, while $|z| = |i-\sigma| < c$. The same argument as in (1) implies that $\int\limits_{||H||\geq 1} |q_\sigma(e^H)\prod\limits_{\xi\in\Delta^+}\sinh^4\xi(H)|dH < c$.

(4) $\sigma < 1, ||H|| < 1$. From the asymptotic expression of K_ν, there exists a constant $c > 0$ such that

$$|k_{(\frac{l+1}{2}+j)}(H)|\leq c|\sqrt{||H||^2+z^2}|^{(l+1+2j)} = c[(||H||^2+\sigma^2-1)^2+4\sigma^2]^{\frac{1}{4}(l+1+2j)}. \quad (3.4)$$

For $||H|| < 1$, we have $\forall\xi\in\Delta^+$, $|\coth\xi(H)|\leq c||H||^{-1}$. Therefore,

$$|\coth\alpha(H)B_{\beta,\gamma,1+2j}(H)|\leq c||H||^{2j}, \quad 0\leq j\leq 2;$$

$$|\coth\alpha(H)\coth\beta(H)A_{\gamma,2+2j}(H)|\leq c||H||^{2j}, \quad 0\leq j\leq 1;$$

$$\Big[\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big]g(H)\leq c.$$

Hence we get

$$\int_{||H||<1}|q_\sigma(H)\prod_{\xi\in\Delta^+}\sinh^4\xi(H)|dH\leq c\sum_{j=0}^{3}\int_{||H||<1}k_{(\frac{l+7}{2}+j)}||H||^{2j}\prod_{\xi\in\Delta^+}\sinh^2\xi(H)dH.$$

Due to (3.4), the last integral $\leq c\int_0^1\frac{||H||}{[(||H||^2+\sigma^2-1)^2+4\sigma^2]^{\frac{l+13}{4}}}d||H||$. Change the variable $u = \frac{||H||^2+\sigma^2-1}{2\sigma}$, so

$$\int_0^1\frac{||H||}{[(||H||^2+\sigma^2-1)^2+4\sigma^2]^{\frac{n+1}{4}}}d||H|| = \Big(\frac{1}{\sigma}\Big)^{\frac{n-1}{2}}\int_{\frac{\sigma^2-1}{4\sigma}}^{\frac{\sigma}{2}}\frac{du}{(1+u^2)^{\frac{n+1}{2}}} < c\Big(\frac{1}{\sigma}\Big)^{\frac{n-1}{2}}.$$

This completes the proof of Lemma 3.5.

Proof of Part (2) of Theorem. Lemma 3.5 signifies that

$$||T_{q_\sigma}||_{(1,1)} = ||q_\sigma||_{L^1(G/K)}\leq c\begin{cases}\sigma, & \text{if } \sigma\geq 1,\\ (\frac{1}{\sigma})^{\frac{n-1}{2}}, & \text{if } \sigma < 1.\end{cases}$$

On the other hand, (2.3) means that $||q_\sigma||_{(2,2)}\leq e^{-\sigma||\rho||}$.

By the interpalation theorem, if $1 < p < 2$, there exists a constant $d_p > 0$ such that

$$||T_{q_\sigma}||_{(p,p)}\leq c\begin{cases}e^{-d_p\sigma}, & \text{if } \sigma\geq 1,\\ (\frac{1}{\sigma})^{(n-1)(\frac{1}{p}-\frac{1}{2})}, & \text{if } \sigma < 1.\end{cases}$$

Thus

$$\begin{aligned}||T_{1,b}||_{(p,p)} &\leq \frac{1}{\Gamma(b)}\int_0^{+\infty}\sigma^{(Reb-1)}||T_{q_\sigma}||_{(p,p)}d\sigma\\ &\leq \frac{1}{\Gamma(b)}\Big\{\int_0^1\sigma^{[Reb-1+(n-1)(\frac{1}{2}-\frac{1}{p})]}d\sigma+\int_1^{+\infty}\sigma^{(Reb-1)}e^{-d_p\sigma}d\sigma\Big\}.\end{aligned}$$

We see that when $1 < p < 2$ and $\frac{1}{p}-\frac{1}{2} < \frac{\mathrm{Re}\, b}{n-1}$, $||T_{1,b}||_{(p,p)} < +\infty$. For $2 < p < +\infty$, the duality deduces the same conclusion. So we obtain Part (2) of Theorem.

§4. The Case $a<1$

In this section we deal with the part (3) of Theorem; our techniques are similar to that in [3] and in [13]. Let ψ be a smooth radial function on M such that $0\le\psi\le1$, $\psi(e^H)=1$ for $||H||\le\sqrt{R_0}$ and $\psi(e^H)=0$ for $||H||\ge R_0$, where R_0 is a suitable number >1. We write $r_{a,b}=\psi r_{a,b}+(1-\psi)r_{a,b}=r^1_{a,b}+r^2_{a,b}$, and define $T^j_{a,b}$ by $T^j_{a,b}f=f*r^j_{a,b}, j=1,2$. First, we deal with $T^2_{a,b}$.

Lemma 4.1. *For $0<a<1$, $\sigma+i\tau\in S_\varepsilon$, $N=N_1+N_2$, we have*

$$|\partial^{N_1}_{\sigma_1}\partial^{N_2}_{\sigma_2}\{m_{a,b}(\sigma+i\tau)\}|\le c_\varepsilon(1+||\sigma||)^{-\mathrm{Re}\, b+N(a-1)}.$$

Proof. When $0<a<1$, it is valid that

$$|||\lambda||^2+||\rho||^2|^{\frac{a}{2}}\Big|\sin\Big(\frac{a}{2}\arg(||\lambda||^2+||\rho||^2)\Big)\Big|\le c\Big|\sum_{j=1}^{2}\sigma_j\tau_j\Big|\{|||\lambda||^2+||\rho||^2|\}^{-\frac12},$$

but $$|||\lambda||^2+||\rho||^2|^{\frac12}=\Big\{\Big[\sum_{j=1}^{2}({\sigma_j}^2+{\rho_j}^2-{\tau_j}^2)\Big]^2+4\Big(\sum_{j=1}^{2}\sigma_j\tau_j\Big)^2\Big\}^{\frac14}>\Big(\sum_{j=1}^{2}{\sigma_j}^2\Big)^{\frac12},$$

so $$|||\lambda||^2+||\rho||^2|^{\frac{a}{2}}\Big|\sin\Big(\frac{a}{2}\arg(||\lambda||^2+||\rho||^2)\Big)\Big|\le c\Big(\sum_{j=1}^{2}\tau_j\Big)^{\frac12}<c.$$

From last inequality and (2.5), it is obvious that, for $0<a<1$,

$$|m_{a,b}(\sigma+i\tau)|\le c_\varepsilon(||\lambda||^2+||\rho||^2|)^{-\frac{\mathrm{Re}\, b}{2}}\le c_\varepsilon(1+||\sigma||)^{-\mathrm{Re}\, b}.$$

The direct computation gives

$$\partial^{N_1}_{\sigma_1}\partial^{N_2}_{\sigma_2}\{m_{a,b}(\sigma+i\tau)\}=(iu)^N(||\sigma+i\tau||^2+||\rho||^2)^{(\frac{a}{2}-1)N}\times\prod_{j=1}^{2}(\sigma_j+i\tau_j)^{N_j}m_{a,b}(\sigma+i\tau)+\text{the terms of lower order.}$$

Since $|\prod_{j=1}^{2}(\sigma_j+i\tau_j)^{N_j}|\le c(1+||\sigma||)^N$, we reach immediately the desired conclusion.

Lemma 4.2. *We have*

$$r^2_{a,b}(e^H)=(1-\psi)(e^H)\prod_{\xi\in\Delta^+}\sinh^{-2}\xi(H)\int_{\underline{a}^*}m_{a,b}(\lambda)\sum_{j=0}^{3}c_jP_{(3+j)}(\lambda,H)e^{i\lambda(H)}dH,$$

where
$$P_6(\lambda,H)=\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2,$$
$$P_5(\lambda,H)=\sum_{\alpha,\beta,\gamma}\langle\lambda,\alpha\rangle^2\langle\lambda,\beta\rangle^2\langle\lambda,\gamma\rangle||\gamma||^2\coth\gamma(H),$$
$$P_4(\lambda,H)=\sum_{\alpha,\beta,\gamma}\langle\lambda,\alpha\rangle^2\langle\lambda,\beta\rangle\langle\lambda,\gamma\rangle||\beta||^2||\gamma||^2\coth\beta(H)\coth\gamma(H),$$
$$P_3(\lambda,H)=\Big\{\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)+\frac12\prod_{\xi\in\Delta^+}\sinh^{-1}\xi(H)\Big\}\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle.$$

Proof. From the definition of $r^2_{a,b}$ and the formula (1.2),

$$r^2_{a,b}(e^H)=(1-\psi)(e^H)\int_{\underline{a}^*}m_{a,b}(\lambda)\varphi_\lambda(e^H)\frac{d\lambda}{|\mathbf{c}(\lambda)|^2}. \tag{4.1}$$

We know that[3] $|\mathbf{c}(\lambda)|^2=\mathbf{c}(\lambda)\mathbf{c}(-\lambda)$, and for $\underline{g}=sl(3,\mathbb{H})$ (see [7]),

$$\mathbf{c}(\lambda)=\prod_{\xi\in\Delta^+}\frac{\langle\rho,\xi\rangle(\langle\rho,\xi\rangle+||\xi||^2)}{\langle i\lambda,\xi\rangle(\langle i\lambda,\xi\rangle+||\xi||^2)}.$$

Substituting this expression of $\mathbf{c}(\lambda)$ and the expression (1.4) of φ_λ into (4.1), we have

$$\begin{aligned}r^2_{a,b}(e^H)=&(1-\psi)(e^H)\prod_{\xi\in\Delta^+}\sinh^{-2}\xi(H)\int_{\underline{a}^*}m_{a,b}\prod_{\xi\in\Delta^+}\langle -i\lambda,\xi\rangle\sum_{w\in W}\det w\\ &\times\Big\{\prod_{\xi\in\Delta^+}\langle iw\lambda,\xi\rangle-\sum_{\alpha,\beta,\gamma}\langle iw\lambda,\alpha\rangle\langle iw\lambda,\beta\rangle||\gamma||^2\coth\gamma(H)\\ &+\sum_{\alpha,\beta,\gamma}\langle iw\lambda,\alpha\rangle||\beta||^2||\gamma||^2\coth\beta(H)\coth\gamma(H)\\ &-\Big[\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-2}\xi(H)\Big]\Big\}e^{iw\lambda(H)}d\lambda.\end{aligned}$$

Change the variable $\lambda'=w^{-1}\lambda$. Since $\prod_{\xi\in\Delta^+}\langle -i\lambda',\xi\rangle=\det w\prod_{\xi\in\Delta^+}\langle -i\lambda,\xi\rangle$, as in [12], if denote by $|W|$ the cardinality of the Weyl group W, we obtain

$$\begin{aligned}r^2_{a,b}(e^H)=&(1-\psi)(e^H)|W|\prod_{\xi\in\Delta^+}\sinh^{-2}\xi(H)\int_{\underline{a}^*}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle\\ &\times\Big\{\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle+i\sum_{\alpha,\beta,\gamma}\langle\lambda,\alpha\rangle\langle\lambda,\beta\rangle||\gamma||^2\coth\gamma(H)\\ &-\sum_{\alpha,\beta,\gamma}\langle\lambda,\alpha\rangle||\beta||^2||\gamma||^2\coth\beta(H)\coth\gamma(H)\\ &-i\Big[\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big]\Big\}e^{i\lambda(H)}d\lambda,\end{aligned}$$

which is the desired result.

By the same argument as in Lemma 4 in [3], we have for every $N\in\mathbb{N}$,

$$r^2_{a,b}(e^H)=(1-\psi)(e^H)\frac{\rho(H)^{-N}}{\prod\limits_{\xi\in\Delta^+}\sinh^2\xi(H)}\int_{\underline{a}^*}D^N_\rho\Big[m_{a,b}(\lambda)\sum_{j=0}^{3}P_{(3+j)}(\lambda,H)\Big]e^{i\lambda(H)}d\lambda,$$

where D_ρ is the directional derivative along ρ.

Since the function $D^N_\rho\big[m_{a,b}(.)\sum\limits_{j=0}^{3}P_{(3+j)}(.,H)\big](\lambda)$ is holomorphic for $\lambda\in S_1$, we may change the contour of integral to $\underline{a}^*+i(1-\varepsilon)\rho$, so

$$r^2_{a,b}(e^H)=c(1-\psi)^{\frac{1}{2}}(e^H)\frac{\rho(H)^{-N}}{\prod\limits_{\xi\in\Delta^+}\sinh^2\xi(H)}e^{-(1-\varepsilon)\rho(H)}U_\varepsilon(H),$$

where

$$U_\varepsilon(H)=(1-\psi)^{\frac{1}{2}}(h)\int_{\underline{a}^*}D^N_\rho\Big[m_{a,b}(\cdot)\sum_{j=0}^{3}P_{(3+j)}(\cdot,H)\Big](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma.$$

Lemma 4.3. *We have* $\int\limits_{\underline{a}}|U_\varepsilon(H)|^2dH<c_\varepsilon$.

Proof. Denote $U_\varepsilon(H)=\sum_{j=0}^{3}U_{3+j,\varepsilon}(H)$, where

$$U_{6,\varepsilon}(H)=(1-\psi)^{\frac{1}{2}}(e^H)\int_{\underline{a}^*}D_\rho^N\Big[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2\Big](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma,$$

$$U_{5,\varepsilon}(H)=(1-\psi)^{\frac{1}{2}}(H)\sum_{\alpha,\beta,\gamma}||\gamma||^2\coth\gamma(H)$$
$$\times\int_{\underline{a}^*}D_\rho^N[m_{a,b}(\cdot)\langle\cdot,\alpha\rangle^2\langle\cdot,\beta\rangle^2\langle.,\gamma\rangle](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma,$$

$$U_{4,\varepsilon}(H)=(1-\psi)^{\frac{1}{2}}(H)\sum_{\alpha,\beta,\gamma}||\beta||^2||\gamma||^2\coth\beta(H)\coth\gamma(H)$$
$$\times\int_{\underline{a}^*}D_\rho^N[m_{a,b}(\cdot)\langle\cdot,\alpha\rangle^2\langle\cdot,\beta\rangle\langle\cdot,\gamma\rangle](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma,$$

$$U_{3,\varepsilon}(H)=(1-\psi)^{\frac{1}{2}}(H)\Big[\prod_{\xi\in\Delta^+}||\xi||^2\coth\xi(H)+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big]$$
$$\times\int_{\underline{a}^*}D_\rho^N\Big[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2\Big](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma.$$

By Euclidean Plancherel formula,

$$\int_{\underline{a}}|U_{6,\varepsilon}(H)|^2dH\le\int_{\underline{a}}\Big\{\int_{\underline{a}^*}D_\rho^N\Big[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2\Big](\sigma+i(1-\varepsilon)\rho)e^{i\sigma(H)}d\sigma\Big\}^2dH$$
$$=\int_{\underline{a}^*}\Big|D_\rho^N\Big[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2\Big](\sigma+i(1-\varepsilon)\rho)\Big|^2d\sigma.$$

When $||H||\ge\sqrt{R_0}$, $(1-\psi)^{\frac{1}{2}}(e^H)=0$, and $\forall\xi\in\Delta^+$, $\coth\xi(H)$ is bounded, so

$$\int_{\underline{a}}|U_{5,\varepsilon}(H)|^2dH\le\sum_{\alpha,\beta,\gamma}\int_{\underline{a}^*}|D_\rho^N[m_{a,b}(\cdot)\langle\cdot,\alpha\rangle^2\langle\cdot,\beta\rangle^2\langle\cdot,\gamma\rangle](\sigma+i(1-\varepsilon)\rho)|^2d\sigma.$$

The same argument gives

$$\int_{\underline{a}}|U_{4,\varepsilon}(H)|^2dH\le\sum_{\alpha,\beta,\gamma}\int_{\underline{a}^*}|D_\rho^N m_{a,b}(\cdot)\langle\cdot,\alpha\rangle^2\langle.,\beta\rangle\langle\cdot,\gamma\rangle](\sigma+i(1-\varepsilon)\rho)|^2d\sigma,$$

$$\int_{\underline{a}}|U_{3,\varepsilon}(H)|^2dH\le\sum_{\alpha,\beta,\gamma}\int_{\underline{a}^*}|D_\rho^N[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle](\sigma+i(1-\varepsilon)\rho)|^2d\sigma.$$

It is clear that D_ρ is a linear combination of ∂_α and ∂_β, hence a linear combination of ∂_1 and ∂_2. So in view of Lemma 4.1, we have

$$\Big|D_\rho^N\Big[m_{a,b}(\cdot)\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2\Big](\sigma+i(1-\varepsilon)\rho)\Big|$$
$$\le\sum_{k=0}^{N}\Big|c_k\Big[(D_\rho^{N-k}(\cdot))(D_\rho^k\prod_{\xi\in\Delta^+}\langle\cdot,\xi\rangle^2)\Big](\sigma+i(1-\varepsilon)\rho)\Big|$$
$$\le c_\varepsilon(1+||\sigma||)^{6-k+(a-1)(N-k)}\le c_\varepsilon(1+||\sigma||)^{(6+(a-1)N)}.$$

Taking $N=[\frac{n}{2(1-a)}]+1$, we have

$$\int_{\underline{a}}|U_{6,\varepsilon}(H)|^2dH\le c_\varepsilon\int_0^{+\infty}(1+||\sigma||)^{(12+2(a-1)N+l-1)}d||\sigma||<+\infty.$$

Since $$|D_\rho^N[m_{a,b}(\cdot)\langle\cdot,\alpha\rangle^2\langle\cdot,\beta\rangle^2\langle\cdot,\gamma\rangle](\sigma+i(1-\varepsilon)\rho)|\le c_\varepsilon\sum_{k=0}^N(1+||\sigma||)^{(5-k+(a-1)(N-k))},$$

we see that $\int_{\underline{a}}|U_{5,\varepsilon}(H)|^2dH\le\int_{\underline{a}}|U_{6,\varepsilon}(H)|^2dH$, the same argument gives analogous inequalites for $U_{3+j,\varepsilon}(H)j=0,1$. Hence we obtain $\int_{\underline{a}}|U_\varepsilon(H)|^2dH\le c\int_{\underline{a}}|U_{6,\varepsilon}(H)|^2dH\le+\infty$.

Lemma 4.4. *For $1<p<2$, $r_{a,b}^2$ is a convolutive operator on $L^p(M)$, more precisely $||T_{a,b}^2||_{(p,p)}=||r_{a,b}^2||_{L^p(M)}<+\infty$.*

Proof. In view of (2.5),

$$\begin{aligned}||r_{a,b}^2||_{L^p(M)}^p&\le\int_{||H||\ge\sqrt{R_0}}\Big|\frac{||H||^{-N}}{\prod\limits_{\xi\in\Delta^+}\sinh^2\xi(H)}\Big|^pe^{-(1-\varepsilon)p\rho(H)}U_\varepsilon^p(H)\prod_{\xi\in\Delta^+}\sinh^4\xi(H)dH\\&\le\Big\{\int_{||H||\ge\sqrt{R_0}}|U_\varepsilon^p(H)|^{\frac{2}{p}}dH\Big\}\\&\quad\times\Big\{\int_{||H||\ge\sqrt{R_0}}|||H||^{-pN}\prod_{\xi\in\Delta^+}\sinh^{2(2-p)}\xi(H)e^{-(1-\varepsilon)p\rho(H)}|^{\frac{2}{2-p}}dH\Big\}^{\frac{2-p}{2}}.\end{aligned}$$

When $\varepsilon<2(1-\frac1p)$, the second factor of the right-hand side of above inequality is finite, the first factor is also finite by Lemma 4.3. Hence we reach the desired conclusion.

In order to deal with the part near the origin of $r_{a,b}$, we have to use the following statement of Coifman and Weiss[4]:

Let r be a compactly supported bi-K-invariant function on G/K. If the convolution with $\prod\limits_{\xi\in\Delta^+}\sinh^{m_\xi}\xi(H)r(e^H)$ is a bounded operator on $L^p(\underline{a})$, then the convolution with r is a bounded operator on $L^p(G/K)$. Therefore, it is sufficient to prove that $\prod\limits_{\xi\in\Delta^+}\sinh^4\xi(H)r_{a,b}^1$ $\cdot(e^H)$ is a convolutive operator on $L^p(\underline{a})$. For this purpose, we only need to check that the so-called Mikhlin condition[2] is satisfied:

Let r be a tempered distribution on $\mathbb{R}^n$ and $m=\mathcal{F}r$ its Fourier transform. If m satisfies certain symbol estimates, e.g., $\sup\limits_{\lambda\in\mathbb{R}^n}||\lambda||^i||\bigtriangledown^im(\lambda)||<+\infty, 0\le i\le[\frac n2]+1$, then $Tf=f*r$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $1<p<+\infty$.

Lemma 4.5. *If $\mathrm{Re}\,b\ge-\frac n2a(=-7a)$, there exists functions r_0,ε_0, such that*

$$\prod_{\xi\in\Delta^+}\sinh^4\xi(H)r_{a,b}^1(e^H)=r_0(H)+\varepsilon_0(H),$$

where $\varepsilon_0(H)$ is in $L^1(\underline{a})$ and $r_0(H)$ satisfies $\Big|D_y^j\int_{\underline{a}}e^{-i\mu(H)}r_0(H)dH\Big|\le c_j(1+||\mu||)^{-j}, 0\le j\le2$, in other words, r_0 satisfies the Mikhlin condition.

Proof. Similar to the proof of Lemma 4.3, we have

$$\begin{aligned}&\prod_{\xi\in\Delta^+}\sinh^4\xi(H)r_{a,b}^1(e^H)=\psi(e^H)\prod_{\xi\in\Delta^+}\sinh^2\xi(H)\int_{\underline{a}^*}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle-i\lambda,\xi\rangle\\&\times\sum_{w\in W}\det w\Big\{\prod_{\xi\in\Delta^+}(\langle iw\lambda,\xi\rangle-||\xi||^2\coth\xi(H))-\frac12\prod_{\xi\in\Delta^+}||\xi||^2\sinh^{-1}\xi(H)\Big\}e^{iw\lambda(H)}d\lambda\\&=\sum_{j=0}^3c_jQ_{(3+j)}(H),\end{aligned}$$

where
$$Q_6(H)=\psi(e^H)\prod_{\xi\in\Delta^+}\sinh^2\xi(H)\int_{\underline{a}^*}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2 e^{i\lambda(H)}d\lambda,$$
$$Q_5(H)=\psi(e^H)\sum_{\alpha,\beta,\gamma}\sinh^2\alpha(H)\sinh^2\beta(H)\sinh2\gamma(H)$$
$$\times\int_{\underline{a}^*}m_{a,b}(\lambda)\langle\lambda,\alpha\rangle^2\langle\lambda,\beta\rangle^2\langle\lambda,\gamma\rangle e^{i\lambda(H)}d\lambda,$$
$$Q_4(H)=\psi(e^H)\sum_{\alpha,\beta,\gamma}\sinh^2\alpha(H)\sinh2\beta(H)\sinh2\gamma(H)$$
$$\times\int_{\underline{a}^*}m_{a,b}(\lambda)\langle\lambda,\alpha\rangle^2\langle\lambda,\beta\rangle\langle\lambda,\gamma\rangle e^{i\lambda(H)}d\lambda,$$
$$Q_3(H)=\psi(e^H)\Big[\prod_{\xi\in\Delta^+}||\xi||^2\sinh2\xi(H)+\frac{1}{2}\prod_{\xi\in\Delta^+}||\xi||^2\sinh\xi(H)\Big]$$
$$\times\int_{\underline{a}^*}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle e^{i\lambda(H)}d\lambda.$$

Since $$\sinh^2x=\sum_{n=0}^{\infty}\Big[\frac{x^{2n+1}}{(2n+1)!!}\Big]^2+2\sum_{m,n=0,m\neq n}^{\infty}\frac{x^{2m+2n+2}}{(2m+1)!!(2n+!)!!}=\sum_{n=1}^{\infty}\frac{x^{2n}}{a_{2n}},\quad a_{2n}\in\mathbb{R},$$
we have
$$\prod_{\xi\in\Delta^+}\sinh^2\xi(H)=\Big\{\sum_{6\le2(m+n+k)\le N}+\sum_{2(m+n+k)\ge(N+1)}\Big\}\frac{\alpha^{2m}(H)\beta^{2n}(H)\gamma^{2k}(H)}{a_{2m}a_{2n}a_{2k}},$$
where, $N=[\frac{n}{2(1-a)}]+1$.

According to this decomposition of $\prod_{\xi\in\Delta^+}\sinh^2\xi(H)$, $Q_6(H)$ is also decomposed into two parts: $Q_6(H)=X(H)+Y(H)$, where
$$X(H)=\sum_{6\le2(m+n+k)\le N}\psi(e^H)\int_{\underline{a}^*}\frac{\alpha^{2m}(H)\beta^{2n}(H)\gamma^{2k}(H)}{a_{2m}a_{2n}a_{2k}}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2 e^{i\lambda(H)}d\lambda,$$
$$Y(H)=\psi(e^H)\sum_{2(m+n+k)>N}\int_{\underline{a}^*}\frac{\alpha^{2m}(H)\beta^{2n}(H)\gamma^{2k}(H)}{a_{2m}a_{2n}a_{2k}}m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2 e^{i\lambda(H)}d\lambda$$
$$=\sum_{2(m+n+k)>N}\psi(e^H)\frac{2}{a_{2m}a_{2n}a_{2k}}F_{(m,n,k)}(H).$$
For $2(m+n+k)>N$, the same reason as in the proof of Lemma 4.3 shows that
$$\int_{\underline{a}^*}|\partial_\alpha^{2m}\partial_\beta^{2n}\partial_\gamma^{2k}[m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2]|^2d\lambda<c<+\infty,$$
where the constant c does not depend on $\{m,n,k\}$. Then, again due to Euclidean Planchere Formula, we get
$$\int_{\underline{a}}|Y(H)|dH\le\sum_{2(m+n+k)>N}\frac{1}{a_{2m}a_{2n}a_{2n}}\Big(\int_{\underline{a}^*}\Big|\partial_\alpha^{2m}\partial_\beta^{2n}\partial_\gamma^{2k}\Big[m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2\Big]\Big|^2d\lambda\Big)^{\frac{1}{2}}$$
$$\le c\sum_{2(m+n+k)>N}\frac{1}{a_{2m}a_{2n}a_{2k}}<+\infty.$$

As in [13], in order to prove that $\mathcal{F}X$ satisfies the Mikhlin condition, it is sufficient to prove that, for every $\{m,n,k\}$ such that $6\leq(m+n+k)\leq N$,

$$E_{(m,n,k)}(\lambda)=\partial_\alpha^{2m}\partial_\beta^{2n}\partial_\gamma^{2k}\Big[m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2\Big]$$

also satisfies this condition. Lemma 4.1 implies that, when $l\geq7$, $\partial^l(\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2)=0$. Then, if $m'+n'+k'+\dot m+\dot n+\dot k=2(m+n+k)$,

$$\Big|\partial_\alpha^{2m}\partial_\beta^{2n}\partial_\gamma^{2k}\Big[m_{a,b}(\lambda)\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2\Big]\Big|\leq\sum_{l=0}^{6}\Big|\partial_\alpha^{m'}\partial_\beta^{n'}\partial_\gamma^{k'}m_{a,b}(\lambda)\times\partial_\alpha^{\dot m}\partial_\beta^{\dot n}\partial_\gamma^{\dot k}\Big(\prod_{\xi\in\Delta^+}\langle\lambda,\xi\rangle^2\Big)\Big|$$

$$\leq c(1+||\lambda||)^{[6-l-7a+(a-1)(2(m+n+k)-l)]}<c(1+||\lambda||)^{(6-l+(a-1)(6-l)-7a)}<+\infty.$$

The same argument showes that for $6\leq2(m+n+k)\leq N$, and $j_1+j_2+j_3=1,2$, $\partial_\alpha^{j_1}\partial_\beta^{j_2}\partial_\gamma^{j_3}E_{(m,n,k)}(\lambda)$ is also bounded. We can use the analogous methods to treat the functions $Q_{(3+j)}(H)$ with $0\leq j\leq2$. This completes the proof of Lemma 4.5.

Proof of Part (3) of Theorem. Lemmas 4.4 and 4.5 deduce that for $1<p<2$, $||T_{a,-\frac{n}{2}a+iw}||_{(p,p)}=||r_{a,-\frac{n}{2}a+iw}||_{L^p(M)}\leq c(w)$, while (2.1) and (2.3) give $||T_{a,iw'}||_{(2,2)}\leq c(w')$. If $b=b_1+ib_2$ with $0<b_1<\frac{n}{2}a$, then there exists a $t\in(0,1)$ such that $b=t(\frac{n}{2}a+iw)+i(b_2-wt)$.

The Stein complex interpolation theorem gives us $||T_{a,b}||_{(p',p')}\leq c(t,w)$, where $\frac{1}{p'}=\frac{1-t}{2}+\frac{1}{p}$, hence $\frac{1}{p'}-\frac{1}{2}=(\frac{2}{p}-1)\frac{t}{2}=(\frac{2}{p}-1)\frac{\mathrm{Re}\,b}{na}$. Since $0<(\frac{2}{p}-1)<1$, we have $0<(\frac{1}{p'}-\frac{1}{2})<\frac{\mathrm{Re}\,b}{na}$.

If $\mathrm{Re}\,b=b_1>\frac{n}{2}a$, then $m_{a,b}(\lambda)=m_{a,\frac{n}{2}a+iImb}(\lambda)m_{(b_1-\frac{n}{2}a)}(\lambda)$, where $m_{(b_1-\frac{n}{2}a)}(\lambda)=(||\lambda^2+||\rho||^2)^{-\frac{1}{2}(b_1-\frac{n}{2}a)}$. Note that $m_\gamma(\lambda)$ with $\mathrm{Re}\gamma>0$ is a bounded multiplier on $L^P(G/K)$ for $1<p<+\infty$, which is assured by Theorem1 of [1], so the full results follow from this remark.

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