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UNIFORM STABILITY AND ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF 2-DIMENSIONAL MAGNETOHYDRODYNAMICS EQUATIONS

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Abstract

This paper is concerned with uniform stability and asymptotic behavior for solutions of 2-dimensional Magnetohydrodynamics equations. The author establishes the corresponding temporal decay estimates when the initial data is in the following Sobolev spaces $H^2, L^1 \cap H^2$ with $\int (u_0, A_0)dx \neq 0$, or $L^1 \cap H^2$ with $\int (u_0, A_0)dx = 0$, respectively. Most of the decay rates in these estimates are optimal. Moreover, the author proves various uniform stability results, like $\sup_{t>0} \|(w, E, r)(t))\|_Y \leq C\|(w_0, E_0)\|_X$, where X and Y are Sobolev spaces. It should be pointed out that the decay estimates of the solutions for the case $(u_0, A_0) \in L^1 \cap H^2$ follow from the uniform stability estimates. The author utilizes the Fourier splitting method invented by Professor Schonbek and the new elaborate global energy estimates.

Keywords Uniform stability, Asymptotic behavior, Magnetohydrodynamics equations, Fourier transform, Energy estimates

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§1. Introduction

We study L^2 and L^∞ uniform stability and asymptotic behavior of solutions to the initial value problems for 2-dimensional Magnetohydrodynamics equations, which arises from Landau and Lifschitz[5]

$$\begin{aligned} &u_t + u \cdot \nabla u - A \cdot \nabla A - \triangle u + \nabla p = 0, \quad \nabla \cdot u = 0, \\ &A_t + u \cdot \nabla A - A \cdot \nabla u - \triangle A = 0, \quad \nabla \cdot A = 0, \\ &u(x,0) = u_0(x), \quad A(x,0) = A_0(x), \quad \nabla \cdot u_0 = \nabla \cdot A_0 = 0, \end{aligned} \tag{1.1}$$

where $x = (x_1, x_2) \in \mathbf{R}^2, \quad t > 0$ are the independent variables, $u = (u_1(x,t), u_2(x,t)),\ A = (A_1(x,t), A_2(x,t))$ are unknown vector-valued functions, $p = p(x,t)$ represents pressure.

Let the solutions of problem (1.1) satisfy

$$\lim_{|x|\to\infty} \frac{\partial^{m+n}}{\partial x_1^m \partial x_2^n}(u(x,t), A(x,t), p(x,t)) = 0,$$

for all $t > 0$, where $m \geq 0$ and $n \geq 0$ are integers with $m + n \leq 2$.

Lemma 1.1. *Let* $(u_0, A_0) \in H^2(\mathbf{R}^2)$. *Then problem* (1.1) *has a unique global solution* $(u, A, p) \in L^\infty(0,\infty; H^2) \cap W^{1,\infty}(0,\infty; L^2)$.

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It can be proved by the fixed point principle and some of the estimates displayed in Lemmas 2.6, 3.1–3.5.

Let (u, A, p) and (v, B, q) be the solutions of problem (1.1) corresponding to (u_0, A_0) and (v_0, B_0), respectively. Let $(w, E, r) = (u - v, A - B, p - q)$. Then they satisfy the equations

$$\begin{aligned} &w_t + w \cdot \nabla u + v \cdot \nabla w - E \cdot \nabla A - B \cdot \nabla E - \triangle w + \nabla r = 0, \quad \nabla \cdot w = 0, \\ &E_t + w \cdot \nabla A + v \cdot \nabla E - E \cdot \nabla u - B \cdot \nabla w - \triangle E = 0, \quad \nabla \cdot E = 0, \\ &w(x, 0) = w_0(x) = u_0(x) - v_0(x), \quad \nabla \cdot w_0 = 0, \\ &E(x, 0) = E_0(x) = A_0(x) - B_0(x), \quad \nabla \cdot E_0 = 0. \end{aligned} \tag{1.2}$$

Definition. *If there are time-independent constants $C > 0$ and $\alpha \geq 0$, such that*

$$\sup_{t>0} \Big[(1+t)^\alpha \|(w, E, r)(t)\|_Y \Big] \leq C \|(w_0, E_0)\|_X, \tag{1.3}$$

then the solutions of problem (1.1) *are uniformly stable, where X and Y are certain Sobolev spaces.*

Equations (1.1) contain the incompressible Navier-Stokes equations as an example

$$u_t + u \cdot \nabla u - \triangle u + \nabla p = 0, \quad \nabla \cdot u = 0, \quad u(x, 0) = u_0(x), \quad \nabla \cdot u_0 = 0.$$

There have been large amount of literatures on the Navier-Stokes equations (see [1, 2, 4, 6–11]).

There has been little contribution to the uniform stability and asymptotic behavior of solutions of the Magnetohydrodynamic equations. Our primary motivation is to see whether the solutions are uniformly stable. Given different initial data $(u_0(x), A_0(x))$ and $(v_0(x), B_0(x))$, the corresponding solutions (u, A, p) and (v, B, q) will be different. The least upper bounds of the L^2 and L^∞ norms of $(u - v, A - B, p - q)$ will be derived explicitly in terms of $(u_0 - v_0, A_0 - B_0)$. Whether or not the uniform stability and asymptotic behavior are true turn out to be important in understanding several aspects of the equations. Let us look at the consequences if (1.3) holds. It illustrates that if $(v_0, B_0) \to (u_0, A_0)$ in one Sobolev space, then the corresponding solutions $(v, B, q) \to (u, A, p)$ in another space, for all $t \geq 0$. The solutions depend continuously on the initial data. It is very easy to obtain the asymptotic behavior for the solutions (u, A, p) if $(u_0, A_0) \in L^1 \cap H^2$. We will establish the uniform stability and the asymptotic behavior results by various delicate integral estimates, Fourier transform, Gronwall's and Gagliardo-Nirenberg's inequalities.

It is not difficult by standard energy method to get global estimates for the solutions of equations (1.1). We employ a slightly different approach to get more elaborate estimates. However, these global estimates do not automatically lead to the uniform stability. Experiences[11−14] with similar equations show that it is important to distinguish the Sobolev spaces H^2, $L^1 \cap H^2$ when dealing with decay estimates. Therefore it is also appropriate to study the uniform stability in the Sobolev spaces H^2 and $L^1 \cap H^2$ respectively. Readers should be very careful about the different Sobolev spaces.

Let $f(x) \in L^1 \cap L^2$, define its Fourier transform by

$$F[f](\xi) = \widehat{f}(\xi) = \int_{\mathbf{R}^2} f(x) \exp(-ix \cdot \xi) dx.$$

The definition can be extended to the Hilbert space L^2 by continuity, as usual.

Denote by C any time-independent positive constant, which is also independent of the solutions (u,A,p) and (v,B,q). Moreover, denote $L^p=L^p(\mathbf{R}^2)$ and $H^m=H^m(\mathbf{R}^2)$, where $1\le p,m\le\infty$, and denote

$$\begin{aligned}
\nabla u\cdot\nabla v &= \sum_{i=1}^{2}\nabla u_i\cdot\nabla v_i=\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial u_i}{\partial x_j}\frac{\partial v_i}{\partial x_j},\\
|(u,A)|^2 &= |u|^2+|A|^2,\\
\|(u,A)(t)\|_{L^1} &= \int_{\mathbf{R}^2}(|u|^2+|A|^2)^{1/2}dx,\\
\|(u,A)(t)\|_\infty^2 &= \|u(t)\|_{L^\infty}^2+\|A(t)\|_{L^\infty}^2,\\
\|(u,A)(t)\| &= \|(u,A)(t)\|_{L^2},\quad \|(u,A)(t)\|_m=\|(u,A)(t)\|_{H^m},\\
\|(u_0,A_0)\|_{L^1\cap L^2} &= \|(u_0,A_0)\|_{L^1}+\|(u_0,A_0)\|_{L^2},\\
\|(u_0,A_0)\|_{L^1\cap H^2} &= \|(u_0,A_0)\|_{L^1}+\|(u_0,A_0)\|_{H^2}.
\end{aligned}$$

Define the weighted Sobolev space

$$\begin{aligned}
M=\Big\{(u_0,A_0)\in L^1\cap H^2\Big|\int_{\mathbf{R}^2}(u_0,A_0)dx=0,&\\
\|(u_0,A_0)\|_M=\|(u_0,A_0)\|_{H^2}+\int_{\mathbf{R}^2}(1+|x|)|(u_0,A_0)|dx<\infty\Big\}.&\qquad(1.4)
\end{aligned}$$

Define

$$\begin{aligned}
S_1(t)=&\ \|(u,A)(t)\|^2+(1+t)\|\nabla(u,A)(t)\|^2+(1+t)\|p(t)\|^2\\
&+\|\widehat{p}(t)\|_\infty^2+(1+t)^2\|\widehat{\triangle p}(t)\|_\infty^2, \qquad(1.5)\\
S_2(t)=&\ \|(u,A)(t)\|^2+(1+t)\|\nabla(u,A)(t)\|^2+(1+t)^2\|\triangle(u,A)(t)\|^2\\
&+(1+t)\|(u,A)(t)\|_\infty^2+(1+t)^2\|(u_t,A_t)(t)\|^2\\
&+(1+t)\|p(t)\|^2+(1+t)^2\|\nabla p(t)\|^2+(1+t)^3\|\triangle p(t)\|^2 \qquad(1.6)\\
&+(1+t)^2\|p(t)\|_\infty^2+\|\widehat{p}(t)\|_\infty^2+(1+t)^2\|\widehat{\triangle p}(t)\|_\infty^2+(1+t)^3\|p_t(t)\|^2,
\end{aligned}$$

$$X=\Big\{(u,A,p)|\ \|(u,A,p)\|_X^2=\sup_{t>0}\Big[\|(u,A)(t)\|^2+\|\widehat{p}(t)\|_\infty^2\Big]<\infty\Big\}, \qquad(1.7)$$

$$Y=\Big\{(u,A,p)|\ \|(u,A,p)\|_Y^2=\sup_{t>0}S_1(t)<\infty\Big\}, \qquad(1.8)$$

$$Z_k=\Big\{(u,A,p)|\ \|(u,A,p)\|_{Z_k}^2=\sup_{t>0}\Big[(1+t)^kS_2(t)\Big]<\infty\Big\}. \qquad(1.9)$$

§2. Preliminary Lemmas and Elementary Estimates

In this section we present elementary estimates concerning the solutions of (1.1) or (1.2).

Lemma 2.1. (Generalized Gronwall's inequality). *Let $g(t)\ge0$ and $h(t)\ge0$ satisfy the inequality*

$$g(t)\le C+\int_0^t g(s)h(s)ds,\quad \text{for all } 0\le t<\infty,$$

where $C\ge0$ is a constant and $h(t)\in L^1(0,\infty)$. Then we have the global estimate

$$g(t)\le C\exp\Big[\int_0^\infty h(t)dt\Big],\quad \text{for all } 0\le t<\infty. \qquad(2.1)$$

Lemma 2.2. (Gagliardo-Nirenberg's inequality). *For all $1 \le p,q,r \le \infty$ and for all natural numbers m and k with $m > k$, there exist two positive constants $\alpha \in [k/m, 1]$ and $C > 0$, such that for all $u(x) \in C_0^\infty(\mathbf{R}^n)$, we have*

$$\|D^k u\|_{L^p} \le C\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}, \tag{2.2}$$

where $n/p - k = \alpha(n/r - m) + (1-\alpha)n/q$ and

$$\|D^k u\|_{L^p}^p = \sum_{\beta_1+\cdots+\beta_k=n} \Big\| \frac{\partial^{\beta_1+\cdots+\beta_n} u}{\partial x_1^{\beta_1}\cdots\partial x_n^{\beta_n}} \Big\|_{L^p}^p.$$

The only exception is that $\alpha \ne 1$ if $m - n/r = k$ and $1 < p < \infty$.

Lemma 2.3. *Let $g = g(x,t)$ and $h = h(x,t) \in L^\infty(0,\infty; H^1(\mathbf{R}^2))$ satisfy the energy inequality*

$$\begin{aligned}&\frac{d}{dt}\Big[(1+t)^m\int_{\mathbf{R}^2}|\widehat{g}|^2 d\xi\Big] + \frac{3}{2}(1+t)^m\int_{\mathbf{R}^2}|\xi|^2|\widehat{g}|^2 d\xi\\ &\le M(1+t)^{m-1}\int_{\mathbf{R}^2}|\widehat{g}|^2\xi + K(1+t)^n\int_{\mathbf{R}^2}|\widehat{h}|^2 d\xi,\end{aligned}$$

where m and n are integers. Let $B(t) = \{\xi \in \mathbf{R}^2 | (1+t)|\xi|^2 \le 2M\}$. Then we have

$$\begin{aligned}&\frac{d}{dt}\Big[(1+t)^m\int_{\mathbf{R}^2}|\widehat{g}|^2 d\xi\Big] + (1+t)^m\int_{\mathbf{R}^2}|\xi|^2|\widehat{g}|^2 d\xi\\ &\le M(1+t)^{m-1}\int_{B(t)}|\widehat{g}|^2 d\xi + K(1+t)^n\int_{\mathbf{R}^2}|\widehat{h}|^2 d\xi.\end{aligned} \tag{2.3}$$

It is very easy to prove this lemma. See also [6].

Lemma 2.4. *The following preliminary estimates hold:*

$$|w\cdot\nabla u|^2 \le |w|^2|\nabla u|^2, \tag{2.4}$$

$$|\widehat{w\cdot\nabla u}|^2 \le |\xi|^2\|w(t)\|^2\|u(t)\|^2, \tag{2.5}$$

$$|\nabla\cdot(w\cdot\nabla u)|^2 \le |\nabla w|^2|\nabla u|^2, \tag{2.6}$$

$$|F[\nabla\cdot(w\cdot\nabla u)]|^2 \le \|\nabla w(t)\|^2\|\nabla u(t)\|^2, \tag{2.7}$$

$$\|\widehat{r}(t)\|_\infty \le \Big[\|(u,A)(t)\| + \|(v,B)(t)\|\Big]\|(w,E)(t)\|, \tag{2.8}$$

$$\|\widehat{\triangle r}(t)\|_\infty \le \Big[\|\nabla(u,A)(t)\| + \|\nabla(v,B)(t)\|\Big]\|\nabla(w,E)(t)\|, \tag{2.9}$$

$$|\widehat{r}|^2 \le \sum_{i=1}^2\sum_{j=1}^2\Big|\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j}\Big|^2, \tag{2.10}$$

$$\begin{aligned}\|r(t)\|^2 \le\, & C\Big[\|(u,A)(t)\|\,\|\nabla(u,A)(t)\| + \|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\Big]\\ &\times\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|,\end{aligned} \tag{2.11}$$

$$\begin{aligned}\|\nabla r(t)\|^2 \le\, & C\Big[\|(u,A)(t)\|\,\|\nabla(u,A)(t)\| + \|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\Big]\\ &\times\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\| + C\Big[\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\|\\ &+\|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\Big]\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|,\end{aligned} \tag{2.12}$$

$$\|\triangle r(t)\|^2 \le C\Big[\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\| + \|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\Big] \times \|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|, \tag{2.13}$$

$$\|(w_t,E_t)(t)\| \le 2\|(w,E)(t)\|_\infty\|\nabla(u,A)(t)\| + 2\|(v,B)(t)\|_\infty\|\nabla(w,E)(t)\| + 2\|\triangle(w,E)(t)\| + \|\nabla r(t)\|, \tag{2.14}$$

$$\|r_t(t)\| \le 2\|(w_t,E_t)(t)\|\Big[\|(u,A)(t)\|_\infty + \|(v,B)(t)\|_\infty\Big] + 2\|(w,E)(t)\|_\infty\Big[\|(u_t,A_t)(t)\| + \|(v_t,B_t)(t)\|_\infty\Big], \tag{2.15}$$

$$\int_{\mathbf{R}^2} f\cdot(g\cdot\nabla f)dx = 0, \quad \int_{\mathbf{R}^2}[f\cdot(g\cdot\nabla h) + h\cdot(g\cdot\nabla f)]dx = 0, \tag{2.16}$$

$$\Big|\int_{\mathbf{R}^2} f\cdot(g\cdot\nabla h)dx\Big| \le \int_{\mathbf{R}^2}|g|\,|h|\,|\nabla f|dx, \tag{2.17}$$

$$\|\nabla(w\cdot\nabla u)(t)\|^2 \le 2\|\nabla u(t)\|_\infty^2\|\nabla w(t)\|^2 + 2\|w(t)\|_\infty^2\|\triangle u(t)\|^2, \tag{2.18}$$

where f, g and $h \in H^1$ with $\nabla\cdot g = 0$ are arbitrary vector valued functions.

Proof. It is easy to get the following identities by using $\nabla\cdot w = 0$:

$$w\cdot\nabla u = \Big(\sum_{j=1}^{2} w_j\frac{\partial u_i}{\partial x_j}\Big)_{1\le i\le 2} = \Big(\sum_{j=1}^{2}\frac{\partial}{\partial x_j}(u_iw_j)\Big)_{1\le i\le 2},$$

$$\widehat{w\cdot\nabla u} = \sqrt{-1}\Big(\sum_{j=1}^{2}\xi_j\widehat{u_iw_j}\Big)_{1\le i\le 2}. \tag{2.19}$$

Thus we get

$$|w\cdot\nabla u|^2 = \sum_{i=1}^{2}\Big|\sum_{j=1}^{2} w_j\frac{\partial u_i}{\partial x_j}\Big|^2 \le \sum_{j=1}^{2}|w_j|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^2 = |w|^2|\nabla u|^2,$$

$$|\widehat{w\cdot\nabla u}|^2 = \sum_{i=1}^{2}\Big|\sum_{j=1}^{2}\xi_j\widehat{u_iw_j}\Big|^2 \le \sum_{i=1}^{2}\Big(\sum_{j=1}^{2}|\xi_j|^2\Big)\Big(\sum_{j=1}^{2}|\widehat{u_iw_j}|^2\Big)$$

$$\le |\xi|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\|u_i(t)\|^2\,\|w_j(t)\|^2 = |\xi|^2\|u(t)\|^2\,\|w(t)\|^2,$$

where we have applied the estimate $|\widehat{fg}| \le \|f(t)\|\,\|g(t)\|$.

Moreover

$$\nabla\cdot(w\cdot\nabla u) = \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial w_j}{\partial x_i}\frac{\partial u_i}{\partial x_j},$$

$$|\nabla\cdot(w\cdot\nabla u)|^2 \le \sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\frac{\partial w_j}{\partial x_i}\Big|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\frac{\partial u_i}{\partial x_j}\Big|^2 = |\nabla w|^2\,|\nabla u|^2,$$

$$|F[\nabla\cdot(w\cdot\nabla u)]|^2 = \Big|\sum_{i=1}^{2}\sum_{j=1}^{2}F\Big[\frac{\partial w_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\Big]\Big|^2$$

$$\le \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|\frac{\partial w_j}{\partial x_i}(t)\Big\|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|\frac{\partial u_i}{\partial x_j}(t)\Big\|^2 \le \|\nabla w(t)\|^2\|\nabla u(t)\|^2.$$

Taking the divergence of the first equation of (1.2), we get

$$\begin{aligned}\triangle r &= -\nabla\cdot(w\cdot\nabla u + v\cdot\nabla w - E\cdot\nabla A - B\cdot\nabla E)\\ &= \sum_{i=1}^{2}\sum_{j=1}^{2}\frac{\partial^2}{\partial x_i\partial x_j}(E_iA_j + B_iE_j - w_iu_j - v_iw_j).\end{aligned}\tag{2.20}$$

The Fourier transform yields

$$|\xi|^2\widehat{r} = \sum_{i=1}^{2}\sum_{j=1}^{2}\xi_i\xi_j\Big(\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j}\Big),\tag{2.21}$$

and one gets the estimates

$$\begin{aligned}|\xi|^2|\widehat{r}| &\le \sum_{i=1}^{2}\sum_{j=1}^{2}|\xi_i\xi_j|\Big[\|E_i(t)\|\,\|A_j(t)\| + \|B_i(t)\|\,\|E_j(t)\|\\ &\quad + \|w_i(t)\|\,\|u_j(t)\| + \|v_i(t)\|\,\|w_j(t)\|\Big]\\ &= \sum_{i=1}^{2}|\xi_i|\,\|E_i(t)\|\sum_{j=1}^{2}|\xi_j|\,\|A_j(t)\| + \sum_{i=1}^{2}|\xi_i|\,\|B_i(t)\|\sum_{j=1}^{2}|\xi_j|\,\|E_j(t)\|\\ &\quad + \sum_{i=1}^{2}|\xi_i|\,\|w_i(t)\|\sum_{j=1}^{2}|\xi_j|\,\|u_j(t)\| + \sum_{i=1}^{2}|\xi_i|\,\|v_i(t)\|\sum_{j=1}^{2}|\xi_j|\,\|w_j(t)\|\\ &\le |\xi|^2\Big[\|E(t)\|\,\|A(t)\| + \|B(t)\|\,\|E(t)\| + \|w(t)\|\,\|u(t)\| + \|v(t)\|\,\|w(t)\|\Big]\\ &\le |\xi|^2\|(w,E)(t)\|\,\|(u,A)(t)\| + |\xi|^2\|(w,E)(t)\|\,\|(v,B)(t)\|.\end{aligned}$$

Therefore we get (2.8).

By (2.7) and (2.20) we have

$$\begin{aligned}|\widehat{\triangle r}| &\le \Big[\|\nabla w(t)\|\,\|\nabla u(t)\| + \|\nabla v(t)\|\,\|\nabla w(t)\|\\ &\quad + \|\nabla E(t)\|\,\|\nabla A(t)\| + \|\nabla B(t)\|\,\|\nabla E(t)\|\Big]\\ &\le \Big[\|\nabla(u,A)(t)\| + \|\nabla(v,B)(t)\|\Big]\|\nabla(w,E)(t)\|.\end{aligned}$$

Moreover, by (2.21) we can easily get the estimate

$$\begin{aligned}|\xi|^4|\widehat{r}|^2 &= \Big|\sum_{i=1}^{2}\sum_{j=1}^{2}\xi_i\xi_j\Big(\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j}\Big)\Big|^2\\ &\le \sum_{i=1}^{2}\sum_{j=1}^{2}|\xi_i\xi_j|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j}\Big|^2\\ &\le |\xi|^4\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j}\Big|^2.\end{aligned}$$

Thus there holds (2.10). The Parseval's identity yields

$$\|r(t)\|^2 = \frac{1}{4\pi^2}\|\widehat{r}(t)\|^2 \le \frac{1}{4\pi^2}\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|(\widehat{E_iA_j} + \widehat{B_iE_j} - \widehat{w_iu_j} - \widehat{v_iw_j})(t)\Big\|^2$$

$$
\begin{aligned}
&\le \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|(E_iA_j+B_iE_j-w_iu_j-v_iw_j)(t)\Big\|^2\\
&\le 4\sum_{i=1}^{2}\sum_{j=1}^{2}\Big[\|(E_iA_j)(t)\|^2+\|(B_iE_j)(t)\|^2+\|(w_iu_j)(t)\|^2+\|(v_iw_j)(t)\|^2\Big]\\
&\le 4\Big[\|E(t)\|_{L^4}^2\|A(t)\|_{L^4}^2+\|B(t)\|_{L^4}^2\|E(t)\|_{L^4}^2+\|w(t)\|_{L^4}^2\|u(t)\|_{L^4}^2+\|v(t)\|_{L^4}^2\|w(t)\|_{L^4}^2\Big]\\
&\le C\Big[\|E(t)\|\,\|\nabla E(t)\|\,\|A(t)\|\,\|\nabla A(t)\|+\|B(t)\|\,\|\nabla B(t)\|\,\|E(t)\|\,\|\nabla E(t)\|\\
&\quad+\|w(t)\|\,\|\nabla w(t)\|\,\|u(t)\|\,\|\nabla u(t)\|+\|v(t)\|\,\|\nabla v(t)\|\,\|w(t)\|\,\|\nabla w(t)\|\Big]\\
&\le C\|(u,A)(t)\|\,\|\nabla(u,A)(t)\|\,\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|\\
&\quad+C\|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\,\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|\\
&=C\Big[\|(u,A)(t)\|\,\|\nabla(u,A)(t)\|+\|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\Big]\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|.
\end{aligned}
$$

Since by (2.10)

$$
|\widehat{\nabla r}|^2\le\sum_{i=1}^{2}\sum_{j=1}^{2}|\xi|^2\Big|\widehat{E_iA_j}+\widehat{B_iE_j}-\widehat{w_iu_j}-\widehat{v_iw_j}\Big|^2,
$$

we obtain

$$
\begin{aligned}
\|\nabla r(t)\|^2&=\frac{1}{4\pi^2}\|\widehat{\nabla r}(t)\|^2\le\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|\nabla(E_iA_j+B_iE_j-w_iu_j-v_iw_j)(t)\Big\|^2\\
&=\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|(A_j\nabla E_i+E_i\nabla A_j+E_j\nabla B_i+B_i\nabla E_j\\
&\quad-u_j\nabla w_i-w_i\nabla u_j-w_j\nabla v_i-v_i\nabla w_j)(t)\Big\|^2\\
&\le 8\sum_{i=1}^{2}\sum_{j=1}^{2}\Big[\|(A_j\nabla E_i)(t)\|^2+\|(E_i\nabla A_j)(t)\|^2+\|(E_j\nabla B_i)(t)\|^2+\|(B_i\nabla E_j)(t)\|^2\\
&\quad+\|(u_j\nabla w_i)(t)\|^2+\|(w_i\nabla u_j)(t)\|^2+\|(w_j\nabla v_i)(t)\|^2+\|(v_i\nabla w_j)(t)\|^2\Big]\\
&\le C\Big[\|A(t)\|\,\|\nabla A(t)\|\,\|\nabla E(t)\|\,\|\triangle E(t)\|+\|E(t)\|\,\|\nabla E(t)\|\,\|\nabla A(t)\|\,\|\triangle A(t)\|\\
&\quad+\|E(t)\|\,\|\nabla E(t)\|\,\|\nabla B(t)\|\,\|\triangle B(t)\|+\|B(t)\|\,\|\nabla B(t)\|\,\|\nabla E(t)\|\,\|\triangle E(t)\|\\
&\quad+\|u(t)\|\,\|\nabla u(t)\|\,\|\nabla w(t)\|\,\|\triangle w(t)\|+\|w(t)\|\,\|\nabla w(t)\|\,\|\nabla u(t)\|\,\|\triangle u(t)\|\\
&\quad+\|w(t)\|\,\|\nabla w(t)\|\,\|\nabla v(t)\|\,\|\triangle v(t)\|+\|v(t)\|\,\|\nabla v(t)\|\,\|\nabla w(t)\|\,\|\triangle w(t)\|\Big]\\
&\le C\Big[\|(u,A)(t)\|\,\|\nabla(u,A)(t)\|\,\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|\\
&\quad+\|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\,\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|\\
&\quad+\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\|\,\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|\\
&\quad+\|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\,\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|\Big]\\
&\le C\Big[\|(u,A)(t)\|\,\|\nabla(u,A)(t)\|+\|(v,B)(t)\|\,\|\nabla(v,B)(t)\|\Big]
\end{aligned}
$$

$$\times \|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\| + C\Big[\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\| + \|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\Big]\|(w,E)(t)\|\,\|\nabla(w,E)(t)\|.$$

By using (2.6) and (2.20) we get the following

$$\begin{aligned}
\|\triangle r(t)\|^2 &= \|\nabla\cdot(w\cdot\nabla u + v\cdot\nabla w - E\cdot\nabla A - B\cdot\nabla E)(t)\|^2\\
&\le 4\int_{\mathbf{R}^2}|\nabla u|^2|\nabla w|^2dx + 4\int_{\mathbf{R}^2}|\nabla v|^2|\nabla w|^2dx\\
&\quad + 4\int_{\mathbf{R}^2}|\nabla A|^2|\nabla E|^2dx + 4\int_{\mathbf{R}^2}|\nabla B|^2|\nabla E|^2dx\\
&\le 4\|\nabla u(t)\|_{L^4}^2\,\|\nabla w(t)\|_{L^4}^2 + 4\|\nabla v(t)\|_{L^4}^2\,\|\nabla w(t)\|_{L^4}^2\\
&\quad + 4\|\nabla A(t)\|_{L^4}^2\,\|\nabla E(t)\|_{L^4}^2 + 4\|\nabla B(t)\|_{L^4}^2\,\|\nabla E(t)\|_{L^4}^2\\
&\le C\|\nabla u(t)\|\,\|\triangle u(t)\|\,\|\nabla w(t)\|\,\|\triangle w(t)\| + C\|\nabla v(t)\|\,\|\triangle v(t)\|\,\|\nabla w(t)\|\,\|\triangle w(t)\|\\
&\quad + C\|\nabla A(t)\|\,\|\triangle A(t)\|\,\|\nabla E(t)\|\,\|\triangle E(t)\|\\
&\quad + C\|\nabla B(t)\|\,\|\triangle B(t)\|\,\|\nabla E(t)\|\,\|\triangle E(t)\|\\
&\le C\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\|\,\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|\\
&\quad + C\|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\,\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|\\
&\le C\Big[\|\nabla(u,A)(t)\|\,\|\triangle(u,A)(t)\| + \|\nabla(v,B)(t)\|\,\|\triangle(v,B)(t)\|\Big]\\
&\quad \times\|\nabla(w,E)(t)\|\,\|\triangle(w,E)(t)\|.
\end{aligned}$$

The following estimates follow directly from equations (1.2):

$$\begin{aligned}
\|w_t(t)\| &\le \|(w\cdot\nabla u)(t)\| + \|(v\cdot\nabla w)(t)\| + \|(E\cdot\nabla A)(t)\|\\
&\quad + \|(B\cdot\nabla E)(t)\| + \|\triangle w(t)\| + \|\nabla r(t)\|\\
&\le \|w(t)\|_\infty\|\nabla u(t)\| + \|v(t)\|_\infty\|\nabla w(t)\| + \|E(t)\|_\infty\|\nabla A(t)\|\\
&\quad + \|B(t)\|_\infty\|\nabla E(t)\| + \|\triangle w(t)\| + \|\nabla r(t)\|\\
&\le \|(w,E)(t)\|_\infty\|\nabla(u,A)(t)\| + \|(v,B)(t)\|_\infty\|\nabla(w,E)(t)\| + \|\triangle w(t)\| + \|\nabla r(t)\|,\\
\|E_t(t)\| &\le \|(w\cdot\nabla A)(t)\| + \|(v\cdot\nabla E)(t)\| + \|(E\cdot\nabla u)(t)\| + \|(B\cdot\nabla w)(t)\| + \|\triangle E(t)\|\\
&\le \|w(t)\|_\infty\|\nabla A(t)\| + \|v(t)\|_\infty\|\nabla E(t)\| + \|E(t)\|_\infty\|\nabla u(t)\|\\
&\quad + \|B(t)\|_\infty\|\nabla w(t)\| + \|\triangle E(t)\|\\
&\le \|(w,E)(t)\|_\infty\|\nabla(u,A)(t)\| + \|(v,B)(t)\|_\infty\|\nabla(w,E)(t)\| + \|\triangle E(t)\|.
\end{aligned}$$

Thus we get the estimate

$$\begin{aligned}
&\|(w_t,E_t)(t)\| \le \|w_t(t)\| + \|E_t(t)\|\\
&\le 2\|(w,E)(t)\|_\infty\|\nabla(u,A)(t)\| + 2\|(v,B)(t)\|_\infty\|\nabla(w,E)(t)\| + 2\|\triangle(w,E)(t)\| + \|\nabla r(t)\|.
\end{aligned}$$

Furthermore, by (2.21) we have

$$\begin{aligned}
|\xi|^2\widehat{r}_t = \sum_{i=1}^2\sum_{j=1}^2\xi_i\xi_j\Big(&\widehat{E_{it}A_j} + \widehat{E_iA_{jt}} + \widehat{B_{it}E_j} + \widehat{B_iE_{jt}}\\
&- \widehat{w_{it}u_j} - \widehat{w_iu_{jt}} - \widehat{v_{it}w_j} - \widehat{v_iw_{jt}}\Big).
\end{aligned}$$

Thus

$$
\begin{aligned}
|\xi|^4|\widehat{r}_t|^2 \le& \sum_{i=1}^{2}\sum_{j=1}^{2}|\xi_i\xi_j|^2\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\widehat{E_{it}A_j}+\widehat{E_iA_{jt}}+\widehat{B_{it}E_j}+\widehat{B_iE_{jt}} \\
&-\widehat{w_{it}u_j}-\widehat{w_iu_{jt}}-\widehat{v_{it}w_j}-\widehat{v_iw_{jt}}\Big|^2 \\
=& |\xi|^4\sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\widehat{E_{it}A_j}+\widehat{E_iA_{jt}}+\widehat{B_{it}E_j}+\widehat{B_iE_{jt}} \\
&-\widehat{w_{it}u_j}-\widehat{w_iu_{jt}}-\widehat{v_{it}w_j}-\widehat{v_iw_{jt}}\Big|^2,
\end{aligned}
$$

or equivalently, we have

$$
|\widehat{r}_t|^2 \le \sum_{i=1}^{2}\sum_{j=1}^{2}\Big|\widehat{E_{it}A_j}+\widehat{E_iA_{jt}}+\widehat{B_{it}E_j}+\widehat{B_iE_{jt}}-\widehat{w_{it}u_j}-\widehat{w_iu_{jt}}-\widehat{v_{it}w_j}-\widehat{v_iw_{jt}}\Big|^2.
$$

Now by using this inequality, we get

$$
\begin{aligned}
\|r_t(t)\| =& \frac{1}{2\pi}\|\widehat{r}_t(t)\| \le \frac{1}{2\pi}\sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|(\widehat{E_{it}A_j}+\widehat{E_iA_{jt}}+\widehat{B_{it}E_j}+\widehat{B_iE_{jt}} \\
&-\widehat{w_{it}u_j}-\widehat{w_iu_{jt}}-\widehat{v_{it}w_j}-\widehat{v_iw_{jt}})(t)\Big\| \\
=& \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\|(E_{it}A_j+E_iA_{jt}+B_{it}E_j+B_iE_{jt} \\
&-w_{it}u_j-w_iu_{jt}-v_{it}w_j-v_iw_{jt})(t)\Big\| \\
\le& \sum_{i=1}^{2}\sum_{j=1}^{2}\Big\{\|E_{it}(t)\|\,\|A_j(t)\|_\infty+\|E_i(t)\|_\infty\|A_{jt}(t)\| \\
&+\|B_{it}(t)\|\,\|E_j(t)\|_\infty+\|B_i(t)\|_\infty\|E_{jt}(t)\|+\|w_{it}(t)\|\,\|u_j(t)\|_\infty \\
&+\|w_i(t)\|_\infty\|u_{jt}(t)\|+\|v_{it}(t)\|\,\|w_j(t)\|_\infty+\|v_i(t)\|_\infty\|w_{jt}(t)\|\Big\} \\
\le& 2\Big\{\|E_t(t)\|\,\|A(t)\|_\infty+\|E(t)\|_\infty\|A_t(t)\|+\|B_t(t)\|\,\|E(t)\|_\infty+\|B(t)\|_\infty\|E_t(t)\| \\
&+\|w_t(t)\|\,\|u(t)\|_\infty+\|w(t)\|_\infty\|u_t(t)\|+\|v_t(t)\|\,\|w(t)\|_\infty+\|v(t)\|_\infty\|w_t(t)\|\Big\} \\
\le& 2\Big\{\|(u,A)(t)\|_\infty\|(w_t,E_t)(t)\|+\|(w,E)(t)\|_\infty\|(u_t,A_t)(t)\| \\
&+\|(w,E)(t)\|_\infty\|(v_t,B_t)(t)\|+\|(v,B)(t)\|_\infty\|(w_t,E_t)(t)\|\Big\} \\
=& 2\|(w_t,E_t)(t)\|\Big[\|(u,A)(t)\|_\infty+\|(v,B)(t)\|_\infty\Big] \\
&+2\|(w,E)(t)\|_\infty\Big[\|(u_t,A_t)(t)\|+\|(v_t,B_t)(t)\|\Big].
\end{aligned}
$$

Since

$$
\begin{aligned}
\int_{\mathbf{R}^2} f\cdot(g\cdot\nabla h)dx =& \int_{\mathbf{R}^2}\sum_{i=1}^{2}\sum_{j=1}^{2}f_ig_j\frac{\partial h_i}{\partial x_j}dx = \int_{\mathbf{R}^2}\sum_{i=1}^{2}\sum_{j=1}^{2}f_i\frac{\partial}{\partial x_j}(g_jh_i)dx \\
=& -\int_{\mathbf{R}^2}\sum_{i=1}^{2}\sum_{j=1}^{2}g_jh_i\frac{\partial f_i}{\partial x_j}dx = -\int_{\mathbf{R}^2}h\cdot(g\cdot\nabla f)dx,
\end{aligned}
$$

we get

$$\int_{\mathbf{R}^2} f\cdot(g\cdot\nabla f)dx=0,\quad \int_{\mathbf{R}^2}[f\cdot(g\cdot\nabla h)+h\cdot(g\cdot\nabla f)]dx=0.$$

In addition, since

$$\Big|\sum_{i=1}^2\sum_{j=1}^2 g_jh_i\frac{\partial f_i}{\partial x_j}\Big|\le\Big(\sum_{i=1}^2\sum_{j=1}^2|g_jh_i|^2\Big)^{1/2}\Big(\sum_{i=1}^2\sum_{j=1}^2\Big|\frac{\partial f_i}{\partial x_j}\Big|^2\Big)^{1/2}=|g|\,|h|\,|\nabla f|,$$

we have

$$\Big|\int_{\mathbf{R}^2} f\cdot(g\cdot\nabla h)dx\Big|=\Big|\int_{\mathbf{R}^2}\sum_{i=1}^2\sum_{j=1}^2 g_jh_i\frac{\partial f_i}{\partial x_j}dx\Big|\le\int_{\mathbf{R}^2}|g|\,|h|\,|\nabla f|dx.$$

Finally we have

$$\begin{aligned}|\nabla(w\cdot\nabla u)|^2&=\sum_{i=1}^2\sum_{j=1}^2\Big|\frac{\partial}{\partial x_j}\Big(\sum_{k=1}^2 w_k\frac{\partial u_i}{\partial x_k}\Big)\Big|^2\\&=\sum_{i=1}^2\sum_{j=1}^2\Big|\sum_{k=1}^2\frac{\partial w_k}{\partial x_j}\frac{\partial u_i}{\partial x_k}+\sum_{k=1}^2 w_k\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\\&\le 2\sum_{i=1}^2\sum_{j=1}^2\Big[\Big|\sum_{k=1}^2\frac{\partial w_k}{\partial x_j}\frac{\partial u_i}{\partial x_k}\Big|^2+\Big|\sum_{k=1}^2 w_k\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\Big]\\&\le 2\sum_{i=1}^2\sum_{j=1}^2\Big[\sum_{k=1}^2\Big|\frac{\partial w_k}{\partial x_j}\Big|^2\sum_{k=1}^2\Big|\frac{\partial u_i}{\partial x_k}\Big|^2+\sum_{k=1}^2|w_k|^2\sum_{k=1}^2\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2\Big]\\&=2|\nabla u|^2|\nabla w|^2+2|w|^2\sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2,\end{aligned}$$

$$\int_{R^2}\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2dx=\int_{R^2}\frac{\partial^2 u_i}{\partial x_j^2}\frac{\partial^2 u_i}{\partial x_k^2}dx,\quad\int_{R^2}\sum_{i=1}^2\sum_{j=1}^2\sum_{k=1}^2\Big|\frac{\partial^2 u_i}{\partial x_j\partial x_k}\Big|^2dx=\|\triangle u(t)\|^2,$$

one obtains

$$\|\nabla(w\cdot\nabla u)(t)\|^2\le 2\|\nabla u(t)\|_\infty^2\|\nabla w(t)\|^2+2\|w(t)\|_\infty^2\|\triangle u(t)\|^2.$$

Lemma 2.5. *Let* (u_0,A_0) *and* $(v_0,B_0)\in L^1\cap H^2$. *We have*

$$|(\widehat{w},\widehat{E})|\le\|(w_0,E_0)\|_{L^1}+3|\xi|\int_0^t\|(w,E)(s)\|\Big[\|(u,A)(s)\|+\|(v,B)(s)\|\Big]ds.\tag{2.22}$$

Proof. Applying the Fourier transform to the equations (1.2) yields

$$\widehat{w}_t+|\xi|^2\widehat{w}+F[w\cdot\nabla u+v\cdot\nabla w-E\cdot\nabla A-B\cdot\nabla E+\nabla r]=0.$$

It follows easily that

$$[\widehat{w}e^{|\xi|^2t}]_t+F[w\cdot\nabla u+v\cdot\nabla w-E\cdot\nabla A-B\cdot\nabla E+\nabla r]e^{|\xi|t}=0.$$

Integrating in time gives

$$\widehat{w}=e^{-|\xi|^2t}\Big[\widehat{w_0}-\int_0^t F[w\cdot\nabla u+v\cdot\nabla w-E\cdot\nabla A-B\cdot\nabla E+\nabla r]e^{|\xi|^2s}ds\Big].$$

Similarly, we have

$$\widehat{E}=e^{-|\xi|^2t}\Big[\widehat{E_0}-\int_0^t F[w\cdot\nabla A+v\cdot\nabla E-E\cdot\nabla u-B\cdot\nabla w]e^{|\xi|^2s}ds\Big].$$

Thus one obtains

$$(\widehat{w},\widehat{E})=e^{-|\xi|^2t}\Big[(\widehat{w_0},\widehat{E_0})-\int_0^t(\widehat{K},\widehat{L})e^{|\xi|^2s}ds\Big],$$

where

$$K=w\cdot\nabla u+v\cdot\nabla w-E\cdot\nabla A-B\cdot\nabla E+\nabla r,$$
$$L=w\cdot\nabla A+v\cdot\nabla E-E\cdot\nabla u-B\cdot\nabla w.$$

Therefore by using the first two estimates in Lemma 2.4, we obtain

$$|(\widehat{w},\widehat{E})|\le|(\widehat{w_0},\widehat{E_0})|+\int_0^t|(\widehat{K},\widehat{L})|ds,$$

and

$$\begin{aligned}|\widehat{K}|\le{}&|\xi|\Big[\|u(s)\|\,\|w(s)\|+\|v(s)\|\,\|w(s)\|+\|A(s)\|\,\|E(s)\|+\|B(s)\|\,\|E(s)\|\Big]\\&+|\xi|\,\|(w,E)(s)\|\Big[\|(u,A)(s)\|+\|(v,B)(s)\|\Big]\\&\le2|\xi|\,\|(w,E)(s)\|\Big[\|(u,A)(s)\|+\|(v,B)(s)\|\Big],\\|\widehat{L}|\le{}&|\xi|\Big[\|A(s)\|\,\|w(s)\|+\|v(s)\|\,\|E(s)\|+\|u(s)\|\,\|E(s)\|+\|B(s)\|\,\|w(s)\|\Big]\\&\le|\xi|\,\|(w,E)(s)\|\Big[\|(u,A)(s)\|+\|(v,B)(s)\|\Big].\end{aligned}$$

Thus

$$|(\widehat{w},\widehat{E})|\le\|(w_0,E_0)\|_{L^1}+3|\xi|\int_0^t\|(w,E)(s)\|\Big[\|(u,A)(s)\|+\|(v,B)(s)\|\Big]ds.$$

Lemma 2.6. *Let (u,A,p) be the solutions of problem* (1.1) *corresponding to $(u_0,A_0)\in H^2$. Then*

$$\begin{aligned}\|\widehat{p}(t)\|_\infty&\le\|(u,A)(t)\|^2,\quad\|\widehat{\triangle p}(t)\|_\infty\le\|\nabla(u,A)(t)\|^2,\\\|p(t)\|^2&\le C\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2,\\\|\nabla p(t)\|^2&\le C\|(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2,\\\|\triangle p(t)\|^2&\le C\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2,\\\|(u_t,A_t)(t)\|&\le2\|(u,A)(t)\|_\infty\,\|\nabla(u,A)(t)\|+2\|\triangle(u,A)(t)\|+\|\nabla p(t)\|,\\\|p_t(t)\|&\le4\|(u,A)(t)\|_\infty\,\|(u_t,A_t)(t)\|.\end{aligned}\tag{2.23}$$

Moreover, if $(u_0,A_0)\in L^1\cap H^2$, then

$$|(\widehat{u},\widehat{A})|\le\|(u_0,A_0)\|_{L^1}+3|\xi|\int_0^t\|(u,A)(s)\|^2ds,\tag{2.24}$$

where C is independent of (u_0,A_0).

Proof. Let $(u_0,B_0)=(0,0)$. Then $(v,B,q)=(0,0,0)$. By Lemmas 2.4 and 2.5, we get the desired estimates.

§3. Decay Estimates for Problem (1.1) with $(u_0,A_0)\in H^2$

Let us establish some decay estimates for the solutions of problem (1.1) for the case $(u_0,A_0)\in H^2$.

Lemma 3.1. *Let (u, A, p) be the solutions of problem* (1.1) *corresponding to* $(u_0, A_0) \in L^2$. *Then* $(u, A, p) \in X$ *and*

$$\sup_{t>0} \|(u,A)(t)\|^2, \quad \sup_{t>0} \|\widehat{p}(t)\|_\infty, \quad 2\int_0^\infty \|\nabla(u,A)(t)\|^2 dt \le \|(u_0, A_0)\|^2. \tag{3.1}$$

Proof. It is easy to get the identity from the equations (1.1)

$$\frac{d}{dt}\|(u,A)(t)\|^2 + 2\|\nabla(u,A)(t)\|^2 = 0,$$

where

$$\int_{\mathbf{R}^2} u\cdot(u\cdot\nabla u)dx = 0, \quad \int_{\mathbf{R}^2} A\cdot(u\cdot\nabla A)dx = 0,$$
$$\int_{\mathbf{R}^2} [u\cdot(A\cdot\nabla A) + A\cdot(A\cdot\nabla u)]dx = 0,$$
$$\int_{\mathbf{R}^2} u\cdot\nabla p dx = 0.$$

It is very easy to get the desired estimates now.

Lemma 3.2. *Let (u, A, p) be the solutions of problem* (1.1) *corresponding to* $(u_0, A_0) \in L^1 \cap L^2$. *Then we have the elementary estimate*

$$\|(u,A)(t)\| \le 2\Big[\|(u_0,A_0)\|_{L^1} + \|(u_0,A_0)\| + \|(u_0,A_0)\|^2\Big]/\ln(e+t). \tag{3.2}$$

Proof. The starting point is the energy equation

$$\frac{d}{dt}\|(u,A)(t)\|^2 + 2\|\nabla(u,A)(t)\|^2 = 0.$$

Applying Parseval's identity to the last equation yields

$$\frac{d}{dt}\int_{\mathbf{R}^2} |(\widehat{u},\widehat{A})|^2 d\xi + 2\int_{\mathbf{R}^2} |\xi|^2 (\widehat{u},\widehat{A})|^2 d\xi = 0.$$

Thus we get

$$\frac{d}{dt}\Big[(\ln(e+t))^3 \int_{\mathbf{R}^2} |(\widehat{u},\widehat{A})|^2 d\xi\Big] + 2[\ln(e+t)]^3 \int_{\mathbf{R}^2} |\xi|^2 |(\widehat{u},\widehat{A})|^2 d\xi$$
$$= \frac{3[\ln(e+t)]^2}{e+t}\int_{\mathbf{R}^2} |(\widehat{u},\widehat{A})|^2 d\xi.$$

Let

$$B(t) = \{\xi \in \mathbf{R}^2 | 2(e+t)\ln(e+t)|\xi|^2 \le 3\}.$$

Since

$$2[\ln(e+t)]^3 \int_{\mathbf{R}^2} |\xi|^2 |(\widehat{u},\widehat{A})|^2 d\xi$$
$$= 2[\ln(e+t)]^3 \int_{B(t)} |\xi|^2 |(\widehat{u},\widehat{A})|^2 d\xi + 2[\ln(e+t)]^3 \int_{B(t)^c} |\xi|^2 |(\widehat{u},\widehat{A})|^2 d\xi$$
$$\ge \frac{3[\ln(e+t)]^2}{e+t}\int_{B(t)^c} |(\widehat{u},\widehat{A})|^2 d\xi$$
$$= \frac{3[\ln(e+t)]^2}{e+t}\int_{\mathbf{R}^2} |(\widehat{u},\widehat{A})|^2 d\xi - \frac{3[\ln(e+t)]^2}{e+t}\int_{B(t)} |(\widehat{u},\widehat{A})|^2 d\xi,$$

we get

$$
\begin{aligned}
&\frac{d}{dt}\Big[(\ln(e+t))^3\int_{\mathbf{R}^2}|(\widehat{u},\widehat{A})|^2d\xi\Big]\\
&\le \frac{3[\ln(e+t)]^2}{e+t}\int_{B(t)}|(\widehat{u},\widehat{A})|^2d\xi\\
&\le \frac{3[\ln(e+t)]^2}{e+t}\int_{B(t)}\Big[\|(u_0,A_0)\|_{L^1}+3|\xi|\int_0^t\|(u,A)(s)\|^2ds\Big]^2d\xi\\
&\le \frac{3[\ln(e+t)]^2}{e+t}\int_0^{2\pi}\int_0^{D}\Big[\|(u_0,A_0)\|_{L^1}+3\|(u_0,A_0)\|^2rt\Big]^2rdrd\theta\\
&\le 9\pi\|(u_0,A_0)\|_{L^1}^2/2(e+t)+9\sqrt{6}\pi\|(u_0,A_0)\|_{L^1}\|(u_0,A_0)\|^2/(e+t)\\
&\quad+243\pi\|(u_0,A_0)\|^4/8(e+t)\\
&\le 35\pi\|(u_0,A_0)\|_{L^1}^2/(e+t)+35\pi\|(u_0,A_0)\|^4/(e+t)\\
&\le 35\pi\Big[\|(u_0,A_0)\|_{L^1}^2+\|(u_0,A_0)\|^4\Big]/(e+t),
\end{aligned}
$$

where $D>0$ is the solution of $2(e+t)\ln(e+t)D^2=3$. We have utilized the estimates

$$
\begin{aligned}
&\|(u,A)(t)\|\le\|(u_0,A_0)\|,\\
&\quad|(\widehat{u},\widehat{A})|\le\|(u_0,A_0)(t)\|_{L^1}+3|\xi|\int_0^t\|(u,A)(s)\|^2ds.
\end{aligned}
$$

Integrating in time to get

$$
[\ln(e+t)]^3\int_{\mathbf{R}^2}|(\widehat{u},\widehat{A})|^2d\xi\le\int_{\mathbf{R}^2}|(\widehat{u_0},\widehat{A_0})|^2d\xi+35\pi\Big[\|(u_0,A_0)\|_{L^1}^2+\|(u_0,A_0)\|^4\Big]\ln(e+t).
$$

Therefore one obtains

$$
\begin{aligned}
&\int_{\mathbf{R}^2}|(\widehat{u},\widehat{A})|^2d\xi\\
&\le\int_{\mathbf{R}^2}|(\widehat{u_0},\widehat{A_0})|^2d\xi/[\ln(e+t)]^2+35\pi\Big[\|(u_0,A_0)\|_{L^1}^2+\|(u_0,A_0)\|^4\Big]/[\ln(e+t)]^2\\
&\le 35\pi\Big[\|(u_0,A_0)\|_{L^1}^2+\|(u_0,A_0)\|^2+\|(u_0,A_0)\|^4\Big]/[\ln(e+t)]^2,
\end{aligned}
$$

thus

$$
\|(u,A)(t)\|\le 2\Big[\|(u_0,A_0)\|_{L^1}+\|(u_0,A_0)\|+\|(u_0,A_0)\|^2\Big]/\ln(e+t).
$$

Lemma 3.3. *Let* (u,A,p) *be the solutions of problem* (1.1) *corresponding to* $(u_0,A_0)\in L^2$. *Then we have*

$$
\lim_{t\to\infty}\|(u,A)(t)\|=\lim_{t\to\infty}\|\widehat{p}(t)\|_\infty=0. \tag{3.3}
$$

Proof. Since

$$
\frac{d}{dt}\|(u,A)(t)\|^2=-2\|\nabla(u,A)(t)\|^2\le0,
$$

the limit $\lim\limits_{t\to\infty}\|(u,A)(t)\|=\alpha\ge0$ exists. Assume that $\alpha>0$. Let

$$
(v_{0\varepsilon},B_{0\varepsilon})=(u_0,A_0)/(1+\varepsilon|x|^2),
$$

where $\varepsilon>0$ is a constant. Then $(v_{0\varepsilon},B_{0\varepsilon})\in L^1\cap L^2$, and

$$
\|(v_{0\varepsilon},B_{0\varepsilon})\|\le\|(u_0,A_0)\|,\quad \|(v_{0\varepsilon},B_{0\varepsilon})\|_{L^1}\le\sqrt{\pi/\varepsilon}\|(u_0,A_0)\|,
$$

and $(v_{0\varepsilon}, B_{0\varepsilon}) \to (u_0, A_0)$ in L^2, as $\varepsilon \to 0$, by Lebesgue's dominant convergence theorem. Moreover, (because of the Corollary of Theorem 4.1)

$$\|(u - v_\varepsilon, A - B_\varepsilon)(t)\| \le C\|(u_0 - v_{0\varepsilon}, A_0 - B_{0\varepsilon})\|, \quad \text{for all } 0 \le t < \infty,$$

where C depends only on $\|(u_0, A_0)\|$. Choose ε small enough, such that

$$\|(u - v_\varepsilon, A - B_\varepsilon)(t)\| \le \alpha/4, \quad \text{for all } 0 \le t < \infty.$$

Thus

$$\|(u, A)(t)\| \le \|(v_\varepsilon, B_\varepsilon)(t)\| + \alpha/4, \quad \text{for all } 0 \le t < \infty.$$

The corresponding solutions $(v_\varepsilon, B_\varepsilon)$ of problem (1.1) with $(v_{0\varepsilon}, B_{0\varepsilon}) \in L^1 \cap L^2$ satisfy

$$\begin{aligned}\|(v_\varepsilon, B_\varepsilon)(t)\| &\le 2\Big[\|(v_{0\varepsilon}, B_{0\varepsilon})\|_{L^1} + \|(v_{0\varepsilon}, B_{0\varepsilon})\| + \|(v_{0\varepsilon}, B_{0\varepsilon})\|^2\Big]/\ln(e+t)\\ &\le 2\Big[\sqrt{\pi/\varepsilon} + 1 + \|(u_0, A_0)\|\Big]\|(u_0, A_0)\|/\ln(e+t).\end{aligned}$$

Choose t_0 large enough, such that

$$\|(v_\varepsilon, B_\varepsilon)(t_0)\| \le 2\Big[\sqrt{\pi/\varepsilon} + 1 + \|(u_0, A_0)\|\Big]\|(u_0, A_0)\|/\ln(e+t_0) \le \alpha/4.$$

Thus we obtain

$$\|(u, A)(t)\| \le \|(v_\varepsilon, B_\varepsilon)(t)\| + \alpha/4 \le \alpha/2, \quad \text{for all } t_0 \le t < \infty.$$

This leads to a contradiction, i.e.,

$$\alpha/2 \ge \lim_{t\to\infty} \|(u, A)(t)\| = \alpha > 0.$$

This implies that α must be zero. Moreover,

$$\|\widehat{p}(t)\|_\infty \le \|(u, A)(t)\|^2,$$

the lemma is proved.

Lemma 3.4. *Let (u, A, p) be the solutions of problem* (1.1) *corresponding to* $(u_0, A_0) \in H^1$. *Then $(u, A, p) \in Y$ and*

$$\begin{aligned}&\sup_{t>0}\Big[(1+t)\|\nabla(u, A)(t)\|^2\Big] \le \|(u_0, A_0)\|_1^2 \exp\Big[4\|(u_0, A_0)\|^4\Big],\\ &\int_0^\infty (1+t)\|\triangle(u, A)(t)\|^2 dt \le \|(u_0, A_0)\|_1^2 \exp\Big[4\|(u_0, A_0)\|^4\Big],\\ &\sup_{t>0} S_1(t) \le C\Big[\|(u_0, A_0)\|_1^2 + \|(u_0, A_0)\|^4\Big]\exp\Big[8\|(u_0, A_0)\|^4\Big],\\ &\lim_{t\to\infty}\Big[(1+t)\|\nabla(u, A)(t)\|^2\Big] = \lim_{t\to\infty} S_1(t) = 0.\end{aligned} \tag{3.4}$$

Proof. Making the scalar products of equations (1.1) and $\triangle(u, A)$ and integrating, we get

$$\begin{aligned}&\frac{d}{dt}\|\nabla u(t)\|^2 + 2\|\triangle u(t)\|^2 = 2\int_{\mathbf{R}^2} \triangle u \cdot (u \cdot \nabla u - A \cdot \nabla A)dx,\\ &\frac{d}{dt}\|\nabla A(t)\|^2 + 2\|\triangle A(t)\|^2 = 2\int_{\mathbf{R}^2} \triangle A \cdot (u \cdot \nabla A - A \cdot \nabla u)dx.\end{aligned}$$

Combining these equations together, we get

$$\begin{aligned}
&\frac{d}{dt}\|\nabla(u,A)(t)\|^2 + 2\|\triangle(u,A)(t)\|^2\\
&= 2\int_{\mathbf{R}^2}[\triangle u\cdot(u\cdot\nabla u - A\cdot\nabla A) + \triangle A\cdot(u\cdot\nabla A - A\cdot\nabla u)]dx\\
&\le \frac{1}{2}\|\triangle(u,A)(t)\|^2 + 4\|(u\cdot\nabla u)(t)\|^2 + 4\|(A\cdot\nabla A)(t)\|^2\\
&\quad + 4\|(u\cdot\nabla A)(t)\|^2 + 4\|(A\cdot\nabla u)(t)\|^2\\
&\le \frac{1}{2}\|\triangle(u,A)(t)\|^2 + 4\|u(t)\|_\infty^2\|\nabla u(t)\|^2 + 4\|A(t)\|_\infty^2\|\nabla A(t)\|^2\\
&\quad + 4\|u(t)\|_\infty^2\|\nabla A(t)\|^2 + 4\|A(t)\|_\infty^2\|\nabla u(t)\|^2\\
&\le \frac{1}{2}\|\triangle(u,A)(t)\|^2 + 4\|(u,A)(t)\|_\infty^2\|\nabla(u,A)(t)\|^2\\
&\le \frac{1}{2}\|\triangle(u,A)(t)\|^2 + 4\|(u,A)(t)\|\,\|\triangle(u,A)(t)\|\,\|\nabla(u,A)(t)\|^2\\
&\le \|\triangle(u,A)(t)\|^2 + 8\|(u,A)(t)\|^2\|\nabla(u,A)(t)\|^4,
\end{aligned}$$

or we get

$$\frac{d}{dt}\|\nabla(u,A)(t)\|^2 + \|\triangle(u,A)(t)\|^2 \le 8\|(u,A)(t)\|^2\|\nabla(u,A)(t)\|^4, \tag{3.5}$$

$$\begin{aligned}
&\frac{d}{dt}\Big[(1+t)\|\nabla(u,A)(t)\|^2\Big] + (1+t)\|\triangle(u,A)(t)\|^2\\
&\le \|\nabla(u,A)(t)\|^2 + 8(1+t)\|(u,A)(t)\|^2\|\nabla(u,A)(t)\|^4.
\end{aligned}$$

Integrate in time to get

$$\begin{aligned}
&(1+t)\|\nabla(u,A)(t)\|^2 + \int_0^t (1+s)\|\triangle(u,A)(s)\|^2 ds\\
&\le \|\nabla(u_0,A_0)\|^2 + \int_0^\infty \|\nabla(u,A)(t)\|^2 dt + 8\int_0^t (1+s)\|(u,A)(s)\|^2\|\nabla(u,A)(s)\|^4 ds.
\end{aligned}$$

By using Lemmas 2.1 and 3.1, we get the estimate

$$\begin{aligned}
&(1+t)\|\nabla(u,A)(t)\|^2 + \int_0^t (1+s)\|\triangle(u,A)(s)\|^2 ds\\
&\le \Big[\|\nabla(u_0,A_0)\|^2 + \int_0^\infty \|\nabla(u,A)(t)\|dt\Big]\exp\Big[8\|(u_0,A_0)\|^2\int_0^\infty \|\nabla(u,A)(t)\|^2 dt\Big]\\
&\le \|(u_0,A_0)\|_1^2\exp\Big[4\|(u_0,A_0)\|^4\Big].
\end{aligned}$$

Moreover, by (3.5)

$$\frac{d}{dt}\|\nabla(u,A)(t)\|^2 + \|\triangle(u,A)(t)\|^2 \le 8\|(u,A)(t)\|^4\|\triangle(u,A)(t)\|^2. \tag{3.6}$$

Since $\lim_{t\to\infty}\|(u,A)(t)\|^2 = 0$, there is a $t_0 > 0$, such that for all $t \ge t_0$, $2\|(u,A)(t)\| \le 1$.

Thus one obtains

$$\frac{d}{dt}\|\nabla(u,A)(t)\|^2 \le 0, \quad \text{for all } t \ge t_0.$$

Now we have

$$\int_s^\infty \|\nabla(u,A)(\tau)\|^2 d\tau \ge \int_s^t \|\nabla(u,A)(\tau)\|^2 d\tau \ge (t-s)\|\nabla(u,A)(t)\|^2,$$

for all $t > s \geq t_0$. Letting $t \to \infty$, we get

$$\int_s^\infty \|\nabla(u,A)(\tau)\|^2 d\tau \geq \limsup_{t\to\infty}\Big[t\|\nabla(u,A)(t)\|^2\Big].$$

Letting $s \to \infty$, we get

$$\lim_{t\to\infty}\Big[t\|\nabla(u,A)(t)\|^2\Big] = 0. \tag{3.7}$$

By Lemma 2.6 we have estimates

$$\begin{aligned}
\sup_{t>0}\Big[(1+t)\|p(t)\|^2\Big] &\leq C\sup_{t>0}\Big[(1+t)\|(u,A)(t)\|^2\|\nabla(u,A)(t)\|^2\Big]\\
&\leq C\|(u_0,A_0)\|^2\,\|(u_0,A_0)\|_1^2\exp\Big[4\|(u_0,A_0)\|^4\Big],\\
\sup_{t>0}\|\widehat{p}(t)\|_\infty &\leq \sup_{t>0}\|(u,A)(t)\|^2 \leq \|(u_0,A_0)\|^2,\\
\sup_{t>0}\Big[(1+t)\|\widehat{\triangle p}(t)\|_\infty\Big] &\leq \sup_{t>0}\Big[(1+t)\|\nabla(u,A)(t)\|^2\Big]\\
&\leq \|(u_0,A_0)\|_1^2\exp[4\|(u_0,A_0)\|^4].
\end{aligned}$$

It is obviously true that $\lim\limits_{t\to\infty} S_1(t) = 0$.

Lemma 3.5. *Let (u,A,p) be the solutions of problem (1.1) corresponding to $(u_0,A_0) \in H^2$. Then $(u,A,p) \in Z_0$ and*

$$\begin{aligned}
&\sup_{t>0}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big] \leq \|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big],\\
&\int_0^\infty (1+t)^2\|\nabla\triangle(u,A)(t)\|^2 dt \leq \|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big],\\
&\sup_{t>0} S_2(t) \leq C\Big[\|(u_0,A_0)\|_2^2 + \|(u_0,A_0)\|_2^4\Big]\exp\Big[C\|(u_0,A_0)\|^4\Big],\\
&\lim_{t\to\infty}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big] = \lim_{t\to\infty} S_2(t) = 0.
\end{aligned} \tag{3.8}$$

Proof. If we make the scalar product of equations (1.1) and $2\triangle^2(u,A)$, integrate in the sapce $\mathbf{R}^2$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\triangle u(t)\|^2 + 2\|\nabla\triangle u(t)\|^2 = 2\int_{\mathbf{R}^2}\nabla\triangle u\cdot\nabla(u\cdot\nabla u - A\cdot\nabla A)dx,\\
&\frac{d}{dt}\|\triangle A(t)\|^2 + 2\|\nabla\triangle A(t)\|^2 = 2\int_{\mathbf{R}^2}\nabla\triangle A\cdot\nabla(u\cdot\nabla A - A\cdot\nabla u)dx.
\end{aligned}$$

We have the following bounds

$$\begin{aligned}
&2\Big|\int_{\mathbf{R}^2}\nabla\triangle u\cdot\nabla(u\cdot\nabla u)dx\Big|\\
&\leq \frac{1}{8}\|\nabla\triangle u(t)\|^2 + 8\int_{\mathbf{R}^2}|\nabla(u\cdot\nabla u)|^2 dx\\
&\leq \frac{1}{8}\|\nabla\triangle u(t)\|^2 + C\|\nabla u(t)\|_\infty^2\|\nabla u(t)\|^2 + C\|u(t)\|_\infty^2\|\triangle u(t)\|^2\\
&\leq \frac{1}{8}\|\nabla\triangle u(t)\|^2 + C\|u(t)\|\,\|\triangle u(t)\|\,\|\nabla u(t)\|\,\|\nabla\triangle u(t)\|\\
&\leq \frac{1}{4}\|\nabla\triangle u(t)\|^2 + C\|u(t)\|^2\,\|\nabla u(t)\|^2\,\|\triangle u(t)\|^2,
\end{aligned}$$

$$
\begin{aligned}
&2\Big|\int_{\mathbf{R}^2}\nabla\triangle u\cdot\nabla(A\cdot\nabla A)dx\Big|\\
&\le\frac18\|\nabla\triangle u(t)\|^2+\frac18\|\nabla\triangle A(t)\|^2+C\|A(t)\|^2\,\|\nabla A(t)\|^2\,\|\triangle A(t)\|^2,\\
&2\Big|\int_{\mathbf{R}^2}\nabla\triangle A\cdot\nabla(u\cdot\nabla A)dx\Big|\\
&\le\frac14\|\nabla\triangle A(t)\|^2+C\|u(t)\|^2\,\|\nabla A(t)\|^2\,\|\triangle u(t)\|^2,\\
&2\Big|\int_{\mathbf{R}^2}\nabla\triangle A\cdot\nabla(A\cdot\nabla u)dx\Big|\\
&\le\frac18\|\nabla\triangle A(t)\|^2+\frac18\|\nabla\triangle u(t)\|^2+C\|A(t)\|^2\,\|\nabla u(t)\|^2\,\|\triangle A(t)\|^2.
\end{aligned}
$$

Coupling together the above estimates, we get

$$
\frac{d}{dt}\|\triangle(u,A)(t)\|^2+\frac32\|\nabla\triangle(u,A)(t)\|^2\le C\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2,
$$

or

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big]+\frac32(1+t)^2\|\nabla\triangle(u,A)(t)\|^2\\
&\le 2(1+t)\|\triangle(u,A)(t)\|^2+C(1+t)^2\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2\\
&\le 2(1+t)\|\nabla(u,A)(t)\|\,\|\nabla\triangle(u,A)(t)\|^2\\
&\quad+C(1+t)^2\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2\\
&\le 2\|\nabla(u,A)(t)\|^2+\frac12(1+t)^2\|\nabla\triangle(u,A)(t)\|^2\\
&\quad+C(1+t)^2\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big]+(1+t)^2\|\nabla\triangle(u,A)(t)\|^2\\
&\le 2\|\nabla(u,A)(t)\|^2+C(1+t)^2\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2. \qquad (3.9)
\end{aligned}
$$

Integrate in time to get the following inequality

$$
\begin{aligned}
&(1+t)^2\|\triangle(u,A)(t)\|^2+\int_0^t(1+s)^2\|\nabla\triangle(u,A)(s)\|^2ds\\
&\le\|\triangle(u_0,A_0)\|^2+2\int_0^\infty\|\nabla(u,A)(t)\|^2\\
&\quad+C\int_0^t(1+s)^2\|(u,A)(s)\|^2\,\|\nabla(u,A)(s)\|^2\,\|\triangle(u,A)(s)\|^2ds\\
&\le\|(u_0,A_0)\|_2^2+C\|(u_0,A_0)\|^2\int_0^t(1+s)^2\|\nabla(u,A)(s)\|^2\,\|\triangle(u,A)(s)\|^2ds.
\end{aligned}
$$

By means of Lemma 2.1, we obtain the global estimates

$$
\begin{aligned}
&\sup_{t>0}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big]\le\|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big],\\
&\int_0^\infty(1+t)^2\|\nabla\triangle(u,A)(t)\|^2dt\le\|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big].
\end{aligned}
$$

Moreover, by these estimates and (3.9) we observe

$$
\frac{d}{dt}\Big[(1+t)^2\|\triangle(u,A)(t)\|^2\Big]\in L^1(0,\infty). \qquad (3.10)
$$

Thus $\lim_{t\to\infty}[(1+t)^2\|\triangle(u,A)(t)\|^2]$ exists. A similar argument to Lemma 3.3 shows that the limit is zero. By using Lemma 2.2, it is very easy to get the estimate

$$\begin{aligned}\sup_{t>0}\Big[(1+t)\|(u,A)(t)\|_\infty^2\Big] &\le \sup_{t>0}\Big[(1+t)\|(u,A)(t)\|\,\|\triangle(u,A)(t)\|\Big]\\ &\le \|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big].\end{aligned}$$

By Lemma 2.6, we can now obtain other global estimates.

$$\begin{aligned}\sup_{t>0}\Big[(1+t)^2\|\nabla p(t)\|^2\Big] &\le C\sup_{t>0}\Big[(1+t)^2\|(u,A)(t)\|^2\|\triangle(u,A)(t)\|^2\Big]\\ &\le C\|(u_0,A_0)\|_2^4\exp\Big[C\|(u_0,A_0)\|^4\Big],\\ \sup_{t>0}\Big[(1+t)^3\|\triangle p(t)\|^2\Big] &\le C\sup_{t>0}\Big[(1+t)^3\|\nabla(u,A)(t)\|^2\,\|\triangle(u,A)(t)\|^2\Big]\\ &\le C\|(u_0,A_0)\|_2^4\exp\Big[C\|(u_0,A_0)\|^4\Big],\end{aligned}$$

$$\begin{aligned}\sup_{t>0}\Big[(1+t)\|(u_t,A_t)(t)\|\Big] &\le 2\sup_{t>0}\Big[(1+t)\|(u,A)(t)\|_\infty\|\nabla(u,A)(t)\|\Big]\\ &\quad+2\sup_{t>0}\Big[(1+t)\|\triangle(u,A)(t)\|\Big]+\sup_{t>0}\Big[(1+t)\|\nabla p(t)\|\Big]\\ &\le C\|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big],\\ \sup_{t>0}\Big[(1+t)^{3/2}\|p_t(t)\|\Big] &\le 4\sup_{t>0}\Big[(1+t)^{3/2}\|(u,A)(t)\|_\infty\|(u_t,A_t)(t)\|\Big]\\ &\le C\|(u_0,A_0)\|_2^2\exp\Big[C\|(u_0,A_0)\|^4\Big].\end{aligned}$$

Since

$$\lim_{t\to\infty}\|(u,A)(t)\|^2=0,\quad \lim_{t\to\infty}[(1+t)\|\nabla(u,A)(t)\|^2]=0,$$
$$\lim_{t\to\infty}[(1+t)^2\|\triangle(u,A)(t)\|^2]=0,$$

it is obviously true that $\lim_{t\to\infty}S_2(t)=0$.

§4. Main Results

Let us first derive two important inequalities. Then we prove our main results.

Lemma 4.1. *We have the following elementary estimates*

$$\frac{d}{dt}\Big[(1+t)^3\|(w,E)(t)\|^2\Big]+(1+t)^3\|\nabla(w,E)(t)\|^2\le C(1+t)^2\int_{B(t)}|(\widehat{w},\widehat{E})|^2d\xi, \tag{4.1}$$

$$\frac{d}{dt}\Big[(1+t)^5\|\triangle(w,E)(t)\|^2\Big]+(1+t)^5\|\nabla\triangle(w,E)(t)\|^2\le C(1+t)^2\|(w,E)(t)\|^2, \tag{4.2}$$

where $B(t)=\{\xi\in\mathbf{R}^2|(1+t)|\xi|^2\le C\}$.

Proof. It is easy to get the following identities from equations (1.2):

$$\frac{d}{dt}\|w(t)\|^2+2\|\nabla w(t)\|^2=2\int_{\mathbf{R}^2}w\cdot(E\cdot\nabla A+B\cdot\nabla E-w\cdot\nabla u-v\cdot\nabla w)dx,$$
$$\frac{d}{dt}\|E(t)\|^2+2\|\nabla E(t)\|^2=2\int_{\mathbf{R}^2}E\cdot(E\cdot\nabla u+B\cdot\nabla w-w\cdot\nabla A-v\cdot\nabla E)dx.$$

These equations are combined together to reach

$$\begin{aligned}&\frac{d}{dt}\|(w,E)(t)\|^2+2\|\nabla(w,E)(t)\|^2\\&=2\int_{\mathbf{R}^2}[w\cdot(E\cdot\nabla A-w\cdot\nabla u)+E\cdot(E\cdot\nabla u-w\cdot\nabla A)]dx,\end{aligned}$$

where

$$\begin{aligned}&\int_{\mathbf{R}^2} w\cdot(v\cdot\nabla w)dx=0,\quad \int_{\mathbf{R}^2} E\cdot(v\cdot\nabla E)dx=0,\\&\int_{\mathbf{R}^2} w\cdot\nabla r dx=0,\quad \int_{\mathbf{R}^2}[w\cdot(B\cdot\nabla E)+E\cdot(B\cdot\nabla w)]dx=0.\end{aligned}$$

Taking (2.17) into account, we have the following simplifications

$$\begin{aligned}&2\Big|\int_{\mathbf{R}^2}[w\cdot(E\cdot\nabla A-w\cdot\nabla u)+E\cdot(E\cdot\nabla u-w\cdot\nabla A)]dx\Big|\\&\le 2\int_{\mathbf{R}^2}\Big[|A|\,|E|\,|\nabla w|+|u|\,|w|\,|\nabla w|+|u|\,|E|\,|\nabla E|+|A|\,|w|\,|\nabla E|\Big]dx\\&\le 2\|u(t)\|_\infty\|w(t)\|\,\|\nabla w(t)\|+2\|u(t)\|_\infty\|E(t)\|\,\|\nabla E(t)\|\\&\quad+2\|A(t)\|_\infty\|E(t)\|\,\|\nabla w(t)\|+2\|A(t)\|_\infty\|w(t)\|\,\|\nabla E(t)\|\\&\le 4\|(u,A)(t)\|_\infty^2\|(w,E)(t)\|^2+\frac{1}{2}\|\nabla(w,E)(t)\|^2.\end{aligned}$$

Therefore we get

$$\frac{d}{dt}\|(w,E)(t)\|^2+\frac{3}{2}\|\nabla(w,E)(t)\|^2\le 4\|(u,A)(t)\|_\infty^2\|(w,E)(t)\|^2.$$

By Lemma 3.5, we have the inequality

$$\frac{d}{dt}\|(w,E)(t)\|^2+\frac{3}{2}\|\nabla(w,E)(t)\|^2\le C(1+t)^{-1}\|(w,E)(t)\|^2.$$

Obviously the following relations hold:

$$\begin{aligned}&\frac{d}{dt}\Big[(1+t)^3\|(w,E)(t)\|^2\Big]+\frac{3}{2}(1+t)^3\|\nabla(w,E)(t)\|^2\le(3+C)(1+t)^2\|(w,E)(t)\|^2,\\&\frac{d}{dt}\Big[(1+t)^3\int_{\mathbf{R}^2}|(\widehat{w},\widehat{E})|^2d\xi\Big]+\frac{3}{2}(1+t)^3\int_{\mathbf{R}^2}|(\widehat{\nabla w},\widehat{\nabla E})|^2d\xi\\&\le(3+C)(1+t)^2\int_{\mathbf{R}^2}|(\widehat{w},\widehat{E})|^2d\xi.\end{aligned}$$

By virtue of Lemma 2.3, we get estimate

$$\frac{d}{dt}\Big[(1+t)^3\int_{\mathbf{R}^2}|(\widehat{w},\widehat{E})|^2d\xi\Big]+(1+t)^3\int_{\mathbf{R}^2}|(\widehat{\nabla w},\widehat{\nabla E})|^2d\xi\le C(1+t)^2\int_{B(t)}|(\widehat{w},\widehat{E})|^2d\xi.$$

Applying the Parseval's identity, we can now prove (4.1).

If we make the scalar product of equations (1.2) and $2\triangle^2(w,E)$, integrate in the space $\mathbf{R}^2$, and combine them together, we get

$$\begin{aligned}&\frac{d}{dt}\|\triangle(w,E)(t)\|^2+2\|\nabla\triangle(w,E)(t)\|^2\\&=2\int_{\mathbf{R}^2}\nabla\triangle w\cdot\nabla(w\cdot\nabla u+v\cdot\nabla w-E\cdot\nabla A-B\cdot\nabla E)dx\\&\quad+2\int_{\mathbf{R}^2}\nabla\triangle E\cdot\nabla(w\cdot\nabla A+v\cdot\nabla E-E\cdot\nabla u-B\cdot\nabla w)dx.\end{aligned}$$

We have the following simplifications

$$
\begin{aligned}
&2\Big|\int_{\mathbf{R}^2}\nabla\triangle w\cdot\nabla(w\cdot\nabla u)dx\Big|\\
\le{}&\frac18\|\nabla\triangle w(t)\|^2+8\|\nabla(w\cdot\nabla u)(t)\|^2\\
\le{}&\frac18\|\nabla\triangle w(t)\|^2+C\|w(t)\|_\infty^2\|\triangle u(t)\|^2+C\|\nabla u(t)\|_{L^4}^2\|\nabla w(t)\|_{L^4}^2\\
\le{}&\frac18\|\nabla\triangle w(t)\|^2+C\|w(t)\|\,\|\triangle w(t)\|\,\|\triangle u(t)\|^2\\
&+C\|u(t)\|^{1/2}\|\triangle u(t)\|^{3/2}\|w(t)\|^{1/2}\|\triangle w(t)\|^{3/2}\\
\le{}&\frac18\|\nabla\triangle w(t)\|^2+C\|\triangle u(t)\|\,\|\triangle w(t)\|^2+C\|\triangle u(t)\|^3\|w(t)\|^2\\
&+C\|\triangle u(t)\|\,\|\triangle w(t)\|^2+C\|u(t)\|^2\|\triangle u(t)\|^3\|w(t)\|^2\\
\le{}&\frac18\|\nabla\triangle w(t)\|^2+C(1+t)^{-1}\|\triangle w(t)\|^2+C(1+t)^{-3}\|w(t)\|^2.
\end{aligned}
$$

Other simplifications are similar to these. Therefore we obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\triangle(w,E)(t)\|^2+\frac32\|\nabla\triangle(w,E)(t)\|^2\\
&\le C(1+t)^{-1}\|\triangle(w,E)(t)\|^2+C(1+t)^{-3}\|(w,E)(t)\|^2.
\end{aligned}
$$

It is easy to obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^5\|\triangle(w,E)(t)\|^2\Big]+\frac32(1+t)^5\|\nabla\triangle(w,E)(t)\|^2\\
&\le(5+C)(1+t)^4\|\triangle(w,E)(t)\|^2+C(1+t)^2\|(w,E)(t)\|^2.
\end{aligned}
$$

Applying the Parseval's identity to the inequality yields

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^5\int_{\mathbf{R}^2}|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi\Big]+\frac32(1+t)^5\int_{\mathbf{R}^2}|\xi|^2|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi\\
\le{}&(5+C)(1+t)^4\int_{\mathbf{R}^2}|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi+C(1+t)^2\int_{\mathbf{R}^2}|(\widehat w,\widehat E)|^2d\xi.
\end{aligned}
$$

By using Lemma 2.3, we now obtain

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^5\int_{\mathbf{R}^2}|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi\Big]+(1+t)^5\int_{\mathbf{R}^2}|\xi|^2|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi\\
\le{}&C(1+t)^4\int_{B(t)}|(\widehat{\triangle w},\widehat{\triangle E})|^2d\xi+C(1+t)^2\int_{\mathbf{R}^2}|(\widehat w,\widehat E)|^2d\xi\\
\le{}&C(1+t)^2\int_{B(t)}|(\widehat w,\widehat E)|^2d\xi+C(1+t)^2\int_{\mathbf{R}^2}|(\widehat w,\widehat E)|^2d\xi\\
\le{}&C(1+t)^2\int_{\mathbf{R}^2}|(\widehat w,\widehat E)|^2d\xi.
\end{aligned}
$$

Applying the Parseval's identity to the inequality gives (4.2).

Theorem 4.1. *We have the uniform stability if* (u_0,A_0) *and* $(v_0,B_0)\in H^2$,

$$
\sup_{t>0}\|(w,E,r)(t)\|_{Z_0}\le C\|(w_0,E_0)\|_2. \tag{4.3}
$$

Proof. As in Lemma 4.1, we have

$$
\begin{aligned}
&\frac{d}{dt}\|(w,E)(t)\|^2 + 2\|\nabla(w,E)(t)\|^2 \\
&\le 2\int_{\mathbf{R}^2}\Big[|A|\,|E|\,|\nabla w| + |u|\,|w|\,|\nabla w| + |u|\,|E|\,|\nabla E| + |A|\,|w|\,|\nabla E|\Big]dx \\
&\le 2\|A(t)\|_{L^4}\,\|E(t)\|_{L^4}\,\|\nabla w(t)\| + 2\|u(t)\|_{L^4}\,\|w(t)\|_{L^4}\,\|\nabla w(t)\| \\
&\quad + 2\|u(t)\|_{L^4}\,\|E(t)\|_{L^4}\,\|\nabla E(t)\| + 2\|A(t)\|_{L^4}\,\|w(t)\|_{L^4}\,\|\nabla E(t)\| \\
&\le C\|A(t)\|^{1/2}\|\nabla A(t)\|^{1/2}\,\|E(t)\|^{1/2}\|\nabla E(t)\|^{1/2}\,\|\nabla w(t)\| \\
&\quad + C\|u(t)\|^{1/2}\|\nabla u(t)\|^{1/2}\,\|w(t)\|^{1/2}\,\|\nabla w(t)\|^{3/2} \\
&\quad + C\|u(t)\|^{1/2}\,\|\nabla u(t)\|^{1/2}\,\|E(t)\|^{1/2}\,\|\nabla E(t)\|^{3/2} \\
&\quad + C\|A(t)\|^{1/2}\,\|\nabla A(t)\|^{1/2}\,\|w(t)\|^{1/2}\,\|\nabla w(t)\|^{1/2}\,\|\nabla E(t)\| \\
&\le \|\nabla(w,E)(t)\|^2 + C\|A(t)\|^2\,\|\nabla A(t)\|^2\|E(t)\|^2 \\
&\quad + C\|A(t)\|^2\,\|\nabla A(t)\|^2\|w(t)\|^2 + C\|u(t)\|^2\,\|\nabla u(t)\|^2\|w(t)\|^2 \\
&\quad + C\|u(t)\|^2\,\|\nabla u(t)\|^2\|E(t)\|^2 \\
&\le \|\nabla(w,E)(t)\|^2 + C\|(u,A)(t)\|^2\,\|\nabla(u,A)(t)\|^2\,\|(w,E)(t)\|^2.
\end{aligned}
$$

Integrating the above inequality in time to reach

$$
\begin{aligned}
&\|(w,E)(t)\|^2 + \int_0^t \|\nabla(w,E)(s)\|^2 ds \\
&\le \|(w_0,E_0)\|^2 + C\int_0^t \|(u,A)(s)^2\|\nabla(u,A)(s)\|^2\|(w,E)(s)\|^2 ds.
\end{aligned}
$$

By using Lemma 2.1 and estimate (3.1), we obtain

$$
\begin{aligned}
&\|(w,E)(t)\|^2 + \int_0^t \|\nabla(w,E)(s)\|^2 ds \\
&\le \|(w_0,E_0)\|^2 \exp\Big[C\int_0^\infty \|(u,A)(t)\|^2\|\nabla(u,A)(t)\|^2 dt\Big] \\
&\le \|(w_0,E_0)\|^2 \exp\Big[C\|(u_0,A_0)\|^4\Big].
\end{aligned}
$$

Thus

$$
\sup_{t>0}\|(w,E)(t)\|^2 \le C\|(w_0,E_0)\|^2, \quad \int_0^\infty \|\nabla(w,E)(t)\|^2 dt \le C\|(w_0,E_0)\|^2. \tag{4.4}
$$

The constants are independent of (v_0,B_0).

Now the estimates (4.2) and (4.4) yield

$$
\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^5\|\triangle(w,E)(t)\|^2\Big] + (1+t)^5\|\nabla\triangle(w,E)(t)\|^2 \\
&\le C(1+t)^2\|(w,E)(t)\|^2 \le C(1+t)^2\|(w_0,E_0)\|^2.
\end{aligned}
$$

Integrating in time, we get

$$
(1+t)^5\|\triangle(w,E)(t)\|^2 \le \|\triangle(w_0,E_0)\|^2 + C(1+t)^3\|(w_0,E_0)\|^2.
$$

Thus one obtains

$$
\sup_{t>0}\Big[(1+t)^2\|\triangle(w,E)(t)\|^2\Big] \le C\|(w_0,E_0)\|_2^2.
$$

By Lemmas 2.2 and 2.4, we can obtain the uniform estimates for others.

Corollary 4.1. *We have the uniform stability if* (u_0, A_0) *and* $(v_0, B_0) \in L^2$,

$$\sup_{t>0} \|(w,E)(t)\| \le C\|(w_0,E_0)\|. \tag{4.5}$$

Proof. Follows from the proof of Theorem 4.1.

The results illustrate that if $(u_0, A_0) \in L^2$ or H^2, which guarantees the existence of global solutions of problem (1.1), the solutions are uniformly stable. We do not necessarily require that the initial data decay more rapidly as $|x| \to \infty$, i.e. $(u_0, A_0) \in L^r \cap H^2$, for some $1 \le r < 2$.

Now let us consider the case the initial data in $L^1 \cap H^2$.

Theorem 4.2. *Let* (u_0, A_0) *and* $(v_0, B_0) \in L^1 \cap H^2$. *Then we have the uniform stability*

$$\sup_{t>0} \|(w,E,r)(t)\|_{Z_1} \le C\|(w_0,E_0)\|_{L^1\cap H^2}. \tag{4.6}$$

Proof. Using Lemmas 2.5 and 4.1, we have

$$\begin{aligned}
&\frac{d}{dt}\Big[(1+t)^3\|(w,E)(t)\|^2\Big] + (1+t)^3\|\nabla(w,E)(t)\|^2 \\
&\le C(1+t)^2\int_{B(t)} |(\widehat{w},\widehat{E})(t)|^2 d\xi \\
&\le C(1+t)^2\int_{B(t)} \Big\{\|(w_0,E_0)\|_{L^1} + 3|\xi|\int_0^t \|(w,E)(s)\|[\|(u,A)(s)\| + \|(v,B)(s)\|]ds\Big\}^2 d\xi \\
&\le C(1+t)\|(w_0,E_0)\|_{L^1}^2 + C(1+t)\int_0^t \|(w,E)(s)\|^2\Big[\|(u,A)(s)\|^2 + \|(v,B)(s)\|^2\Big]ds.
\end{aligned}$$

Integrating in time, we have

$$\begin{aligned}
(1+t)^3\|(w,E)(t)\|^2 &\le \|(w_0,E_0)\|^2 + C(1+t)^2\|(w_0,E_0)\|_{L^1}^2 \\
&\quad + C(1+t)^2\int_0^t \|(w,E)(s)\|^2\Big[\|(u,A)(s)\|^2 + \|(v,B)(s)\|^2\Big]ds,
\end{aligned}$$

or we get

$$\begin{aligned}
(1+t)\|(w,E)(t)\|^2 &\le \|(w_0,E_0)\|^2 + C\|(w_0,E_0)\|_{L^1}^2 \\
&\quad + C\int_0^t \|(w,E)(s)\|^2\Big[\|(u,A)(s)\|^2 + \|(v,B)(s)\|^2\Big]ds \\
&\le C\|(w_0,E_0)\|_{L^1\cap L^2}^2 + C\int_0^t \|(w,E)(s)\|^2[\ln(e+s)]^{-2}ds,
\end{aligned}$$

where (3.2) has been applied.

Set

$$g(t) = (1+t)\|(w,E)(t)\|^2, \quad h(t) = C(1+t)^{-1}[\ln(e+t)]^{-2}.$$

By Lemma 2.1, one obtains

$$(1+t)\|(w,E)(t)\|^2 \le C\|(w_0,E_0)\|_{L^1\cap L^2}^2 \exp\Big[C\int_0^\infty (1+t)^{-1}[\ln(e+t)]^{-2}dt\Big],$$

or

$$\sup_{t>0}\Big[(1+t)\|(w,E)(t)\|^2\Big] \le C\|(w_0,E_0)\|_{L^1\cap L^2}^2.$$

By (4.2), we have the estimate

$$\frac{d}{dt}\Big[(1+t)^5\|\triangle(w,E)(t)\|^2\Big] \le C(1+t)^2\|(w,E)(t)\|^2 \le C(1+t)\|(w_0,E_0)\|_{L^1\cap L^2}^2.$$

Integrating in time, we obtain

$$(1+t)^5\|\triangle(w,E)(t)\|^2 \le \|\triangle(w_0,E_0)\|^2 + C\|(w_0,E_0)\|^2_{L^1\cap L^2}(1+t)^2.$$

Therefore

$$\sup_{t>0}\Big[(1+t)^3\|\triangle(w,E)(t)\|^2\Big] \le C\|(w_0,E_0)\|^2_{L^1\cap H^2}.$$

As before, using Lemmas 2.2 and 2.4, we get other estimates.

Corollary 4.2. *Let* (u_0,A_0) *and* $(v_0,B_0)\in L^1\cap L^2$. *Then*

$$\sup_{t>0}\Big[(1+t)\|(w,E)(t)\|^2\Big] \le C\|(w_0,E_0)\|^2_{L^1\cap L^2}. \tag{4.7}$$

Proof. Follows from the proof of Theorem 4.2.

If we let $(u_0,A_0)\in L^1\cap L^2$ and $(v_0,B_0)=0$, the latter implies that $(v,B)=(0,0)$ for all $t\ge 0$, then we have

$$\sup_{t>0}\Big[(1+t)\|(u,A)(t)\|^2\Big] \le C.$$

Theorem 4.3. *Let* (u_0,A_0) *and* $(v_0,B_0)\in M$. *We have the uniform stability*

$$\sup_{t>0}\|(w,E,r)(t)\|_{Z_2} \le C\|(w_0,E_0)\|_M. \tag{4.8}$$

Proof. By the assumption there is a constant $\delta>0$, such that

$$|(\widehat{w_0},\widehat{E_0})| \le C|\xi|\int_{\mathbf{R}^2}|x|\,|(w_0,E_0)|dx \le C|\xi|\,\|(w_0,E_0)\|_M, \quad \text{for all} \quad |\xi|\le\delta. \tag{4.9}$$

Using estimates (4.1) and (4.9), we have

$$\frac{d}{dt}\Big[(1+t)^3\|(w,E)(t)\|^2\Big] \le C(1+t)^2\int_{B(t)}|(\widehat{w},\widehat{E})(t)|^2d\xi$$

$$\le C(1+t)^2\int_{B(t)}\Big\{|(\widehat{w_0},\widehat{E_0})| + 3|\xi|\int_0^t\|(w,E)(s)\|[\|(u,A)(s)\|+\|(v,B)(s)\|]ds\Big\}^2d\xi$$

$$\le C\|(w_0,E_0)\|^2_M + C\Big\{\int_0^t\|(w,E)(t)\|[\|(u,A)(s)\|+\|(v,B)(s)\|]ds\Big\}^2$$

$$\le C\|(w_0,E_0)\|^2_M[\ln(1+t)]^2.$$

Integrating in time gives

$$(1+t)^3\|(w,E)(t)\|^2 \le \|(w_0,E_0))\|^2 + C\|(w_0,E_0)\|^2_M t[\ln(1+t)]^2,$$

i.e.,

$$(1+t)^2\|(w,E)(t)\|^2 \le C\|(w_0,E_0)\|^2_M[\ln(1+t)]^2.$$

Iterating once more, we obtain

$$(1+t)^2\|(w,E)(t)\|^2 \le C\|(w_0,E_0)\|^2_M.$$

Now (4.2) yields

$$\frac{d}{dt}\Big[(1+t)^5\|\triangle(w,E)(t)\|^2\Big] \le C(1+t)^2\|(w,E)(t)\|^2 \le C\|(w_0,E_0)\|^2_M.$$

Integrating in time gives

$$(1+t)^5\|\triangle(w,E)(t)\|^2 \le \|\triangle(w_0,E_0)\|^2 + C\|(w_0,E_0)\|^2_M t.$$

Therefore we have

$$(1+t)^4\|\triangle(w,E)(t)\|^2 \le C\|(w_0,E_0)\|_M^2.$$

Other estimates follow lines which are by now familiar.

Corollary 4.3. *Let (u_0,A_0) and $(v_0,B_0)\in M$. We have the uniform stability*

$$\sup_{t>0}\Big[(1+t)^2\|(w,E)(t)\|^2\Big] \le C\|(w_0,E_0)\|_M^2. \tag{4.10}$$

Theorem 4.4. *Let (u,A,p) be the solutions of problem (1.1) corresponding to $(u_0,A_0)\in H^2$. Then we have the decay estimates*

$$\lim_{t\to\infty}\Big[\|(u,A)(t)\|^2+(1+t)\|\nabla(u,A)(t)\|^2+(1+t)^2\|\triangle(u,A)(t)\|^2\Big]=0,$$
$$\lim_{t\to\infty}\Big[(1+t)\|p(t)\|^2+(1+t^2)\|\nabla p(t)\|^2+(1+t)^3\|\triangle p(t)\|^2\Big]=0,$$
$$\sup_{t>0}\Big[(1+t)^kS_2(t)\Big]\le C,\quad \lim_{t\to\infty}S_i(t)=0,\quad i=1,2,$$

where $k=0$. Moreover, if $(u_0,A_0)\in L^1\cap H^2$, then $k=1$. If $(u_0,A_0)\in M$, then $k=2$.

Proof. Let $(v_0,B_0)=(0,0)$. Then $(v,B,q)=(0,0,0)$. By Lemmas 3.1–3.5 and Theorem 4.1–4.3, we can justify these results.

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