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ALMOST PERIODIC SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENT***

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Abstract

The authors study the existence of almost periodic solutions to differential equations with piecewise constant arguments which found applications in certain biomedical problems.

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§1. Introduction

This note continues the investigation of differential equations with piecewise constant argument (EPCA) originated by K. L. Cooke and J. Wiener[4], and S. M. Shah and J. Wiener[7]. These equations describe hybrid dynamical systems (a combination of continuous and discrete) and therefore combine properties of both differential and difference equations. In [5], K. L. Cooke and J.Wiener gave a survey of the present status of this research. From this, we know that all of the work that has been done on the differential equations concerns the stability, the oscillation, and the existence of periodic solution. In present paper, we will investigate the existence of almost periodic solutions for differential equations with piecewise constant argument.

In what follows we denote by $|\cdot|$ the Euclidean norm and by $[\cdot]$ the greatest integer function.

We consider the delay differential equations with piecewise constant argument of the form

$$x'(t) = ax(t) + a_0x([t]) + a_1x([t-1]) + f(t), \tag{1.1}$$

where a, a_0, a_1 are constant numbers, and $f : R \to R$ is an almost periodic function, that is, for any $\epsilon > 0$, the ϵ-translation set of f

$$T(f,\epsilon) = \{\tau |\, |f(t+\tau) - f(t)| < \epsilon,\ t \in R\}$$

is a relatively dense set in R (τ is called ϵ-period for f).

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Following [1–5], we say that a function $x : R \to R$ is a solution of Equation (1.1) if the following conditions are satisfied:

(i) x is continuous on R.

(ii) The derivative $x'(t)$ of $x(t)$ exists everywhere, with possible exception of the point $[t]$, where one-sided derivatives exist.

(iii) x satisfies Equation (1.1) on each interval $[n, n+1]$, with integer n.

We also consider the differential equation alternately of retarded and advanced type

$$x'(t) = bx(t) + b_0 x(2[(t+1)/2]) + g(t), \tag{1.2}$$

where b, b_0 are constant numbers, and $g : R \to R$ is an almost periodic function. The argument deviation

$$T(t) = t - 2[(t+1)/2]$$

is negative for $2n-1 \le t < 2n$ (Equation (1.2) is of advanced type), and positive for $2n < t < 2n+1$ (Equation (1.2) is of retarded type), where n is an integer. Similarly, we can also define the solution to Equation (1.2).

§2. Main Results

Theorem 2.1. *Suppose*

$$a_1 > -\frac{[ae^a + (e^a - 1)a_0]^2}{4a(e^a - 1)}, \ a_1 \ne 0$$

and

$$a_1 \ne a_0 - a, \ a \ne 0.$$

If $f(t)$ is an almost periodic function, then Equation (1.1) *possesses an almost periodic solution. Furthermore, if $f(t)$ is ω-periodic, then the following results hold:*

(1) *If $\omega = n_0 \in Z^+$, then Equation* (1.1) *possesses an ω-periodic solution* (*called harmonic solution*).

(2) *If $\omega = \frac{n_0}{m_0}, n_0, m_0 \in Z^+$, n_0 and m_0 are mutually prime, then Equation* (1.1) *possesses an $m_0\omega$-periodic solution* (*called subharmonic solution*).

Theorem 2.2. *Suppose*

$$\begin{cases} b \ne 0, \quad b_0 \ne 0, \\ b_0 \ne -b, \\ b_0 \ne -\dfrac{b[e^b + e^{-b}]}{e^{-b} + e^b - 2}. \end{cases}$$

If $g(t)$ is an almost periodic function, then Equation (1.2) *possesses an almost periodic solution. Furthermore, if $g(t)$ is ω-periodic, then the following results hold:*

(1) *If $\omega = n_0 \in Z^+$, then Eq.(2) possesses an 2ω-periodic solution.*

(2) *If $\omega = \frac{n_0}{m_0}, n_0, m_0 \in Z^+$, n_0 and m_0 are mutually prime, then Equation* (1.2) *possesses an $m_0\omega$-periodic solution.*

§3. Proofs of Theorems

First of all, we give a definition and show some lemmas.

Definition 3.1.[6,8] *A sequence $x : Z \to R^q$ is called an almost periodic sequence if the ϵ-translation set of x*

$$T(x, \epsilon) := \{\tau \in Z | \, |x(n + \tau) - x(n)| < \epsilon \qquad \text{for all } n \in Z\}$$

is a relatively dense set in Z. τ is called the ϵ-period for x.

Lemma 3.1.[6,8] *Suppose that $\{x(n)\}_{n\in Z}$ is an almost periodic sequence and $f(t)$ is an almost periodic function. Then the sets $T(f, \epsilon) \cap Z, T(x, \epsilon) \cap T(f, \epsilon)$ are relatively dense.*

Lemma 3.2.[6] *If $f(t)$ is an almost periodic function, then $\{f(n)\}$ is an almost periodic sequence.*

Lemma 3.3. *If $f(t)$ is an almost periodic function, then the sequence*

$$\{h_n\}_{n\in Z} = \left\{\int_n^{n+1} e^{a(n+1-s)} f(s)ds\right\}_{n\in Z} \tag{3.1}$$

is an almost periodic sequence.

Proof. Let $\tau \in T(f, \epsilon) \cap Z$. Then we have

$$\begin{aligned} h_{n+\tau} - h_n &= \int_{n+\tau}^{n+\tau+1} e^{a(n+\tau+1-s)} f(s)ds - \int_n^{n+1} e^{a(n+1-s)} f(s)ds \\ &= \int_n^{n+1} e^{a(n+1-s)} [f(s + \tau) - f(s)]ds. \end{aligned}$$

This implies

$$|h_{n+\tau} - h_n| \le \max\{e^a, 1\}\epsilon.$$

From definition, it follows that $\{h_n\}_{n\in Z}$ is an almost periodic sequence.

Proof of Theorem 2.1.

(1) If $x(t)$ is a soluiton of Equation (1.1) on R, then we have

$$\begin{aligned} x(t) = &\left\{e^{a(t-n)} + [e^{a(t-n)} - 1]a^{-1}a_0\right\} c_n + [e^{a(t-n)} - 1]a^{-1}a_1c_{n-1} \\ &+ \int_n^t e^{a(t-s)} f(s)ds, \ n \le t < n + 1, n \in Z, \end{aligned} \tag{3.2}$$

where $c_n = x(n), n \in Z$. Obviously, the following relations hold:

$$\begin{aligned} c_{n+1} = &[e^a + (e^a - 1)a^{-1}a_0]c_n + [e^a - 1]a^{-1}a_1c_{n-1} \\ &+ \int_n^{n+1} e^{a(n+1-s)} f(s)ds, \ n \in Z. \end{aligned} \tag{3.3}$$

Let

$$\begin{aligned} B_0 &= e^a + (e^a - 1)a^{-1}a_0, \\ B_1 &= (e^a - 1)a^{-1}a_1, \\ h_n &= \int_n^{n+1} e^{a(n+1-s)} f(s)ds. \end{aligned}$$

Then we can rewrite the inhomogeneous difference Equation (3.3) as

$$c_{n+1} = B_0c_n + B_1c_{n-1} + h_n. \tag{3.4}$$

The corresponding homogeneous difference equation is

$$c_{n+1} = B_0c_n + B_1c_{n-1}. \tag{3.5}$$

Clearly, if $x(t)$ is the almost periodic solution for Equation (1.1), then we know that $\{x(n)\}$ should be an almost periodic sequence by using Lemma 3.2. In the following, we want to show that difference equation (3.4) possesses an almost periodic sequence solution.

(2) Following [2,5], we can seek the particular solutions as $c_n = \lambda^n$ for homogeneous difference equation (3.5). Thus, λ satisfies

$$\lambda^2 - B_0\lambda - B_1 = 0. \tag{3.6}$$

Equation (3.6) has two roots

$$\lambda_{1,2} = \frac{B_0 \pm \sqrt{B_0^2 + 4B_1}}{2}.$$

At this time,

$$\{c_n\} = \{k_1\lambda_1^n + k_2\lambda_2^n\}$$

is the solution for difference Equation (3.5), where k_1, k_2 are constants. Under the conditions of Theorem 2.1 , we can know that $B_0^2 + 4B_1 > 0$ and $\lambda_{1,2} \neq \pm 1$.

(3) We define a sequence

$$c_n = \begin{cases} k_1 \sum\limits_{m\le n-1} \lambda_1^{n-(m+1)} h_m + k_2 \sum\limits_{m\le n-1} \lambda_2^{n-(m+1)} h_m, & |\lambda_1| < 1, |\lambda_2| < 1, \\ k_1 \sum\limits_{m\le n-1} \lambda_1^{n-(m+1)} h_m + k_2 \sum\limits_{m\ge n} \lambda_2^{n-(m+1)} h_m, & |\lambda_1| < 1, |\lambda_2| > 1, \\ k_1 \sum\limits_{m\ge n} \lambda_1^{n-(m+1)} h_m + k_2 \sum\limits_{m\le n-1} \lambda_2^{n-(m+1)} h_m, & |\lambda_1| > 1, |\lambda_2| < 1, \\ k_1 \sum\limits_{m\ge n} \lambda_1^{n-(m+1)} h_m + k_2 \sum\limits_{m\ge n} \lambda_2^{n-(m+1)} h_m, & |\lambda_1| > 1, |\lambda_2| > 1, \end{cases} \tag{3.7}$$

where k_1, k_2 are defined later. We prove that there exist constants k_1, k_2 such that $\{c_n\}$ is a sequence solution of difference Equation (3.4). In fact, for $|\lambda_1| < 1, |\lambda_2| < 1$, we put c_n into Equation (3.4). Then we obtain

$$\begin{cases} k_1\lambda_1 + k_2\lambda_2 = B_0, \\ k_1 + k_2 = 1. \end{cases}$$

Solving this equation, we have

$$\begin{cases} k_1 = \dfrac{\lambda_2 - B_0}{\lambda_2 - \lambda_1}, \\ k_2 = \dfrac{B_0 - \lambda_1}{\lambda_2 - \lambda_1}. \end{cases}$$

Hence, when $|\lambda_1| < 1$ and $|\lambda_2| < 1$, we obtain a sequence solution for difference Equation (3.4):

$$c_n = \frac{\lambda_2 - B_0}{\lambda_2 - \lambda_1} \sum_{m\le n-1} \lambda_1^{n-(m+1)} h_m + \frac{B_0 - \lambda_1}{\lambda_2 - \lambda_1} \sum_{m\le n-1} \lambda_2^{n-(m+1)} h_m. \tag{3.8}$$

For other cases, we can similarly write out an expression to the solution of Equation (3.4).

(4) Since f is almost periodic, it follows from Lemma 3.3 that $\{h_n\}_{n\in Z}$ is also almost periodic. Without loss of generality, we only consider the case: $|\lambda_1| < 1, |\lambda_2| < 1$. For

$\tau \in T(h,\epsilon)$, we have

$$\begin{aligned}
&|c_{n+\tau}-c_n|\\
&=\Big|\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\sum_{m\le n+\tau-1}\lambda_1^{n+\tau-(m+1)}h_m+\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\sum_{m\le n+\tau-1}\lambda_2^{n+\tau-(m+1)}h_m\\
&\quad-\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_1^{n-(m+1)}h_m-\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_2^{n-(m+1)}h_m\Big|\\
&=\Big|\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_1^{n-(m+1)}h_{m+\tau}+\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_2^{n-(m+1)}h_{m+\tau}\\
&\qquad\qquad(\text{by setting } m=m'+\tau, \text{ then replacing } m' \text{ by } m)\\
&\quad-\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_1^{n-(m+1)}h_m-\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_2^{n-(m+1)}h_m\Big|\\
&=\Big|\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_1^{n-(m+1)}(h_{m+\tau}-h_m)+\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\sum_{m\le n-1}\lambda_2^{n-(m+1)}(h_{m+\tau}-h_m)\Big|\\
&\le\Big[\Big|\frac{\lambda_2-B_0}{\lambda_2-\lambda_1}\Big|\frac{1}{1-|\lambda_1|}+\Big|\frac{B_0-\lambda_1}{\lambda_2-\lambda_1}\Big|\frac{1}{1-|\lambda_2|}\Big]\epsilon.
\end{aligned}$$

From definition, we know that $\{c_n\}$ is an almost periodic sequence.

(5) For the above mentioned almost periodic sequence $\{c_n\}_{n\in Z}$, we can obtain a solution to Equation (1.1) by using (3.2). Now, we want to show that the solution defined by (3.2) is an almost periodic solution. In fact, for $\tau\in T(c,\epsilon)\cap T(f,\epsilon)$, we have

$$\begin{aligned}
&|x(t+\tau)-x(t)|\\
&=|\{e^{a(t-n)}+[e^{a(t-n)}-1]a^{-1}a_0\}(c_{n+\tau}-c_n)+[e^{a(t-n)}-1]a^{-1}a_1(c_{n+\tau-1}-c_{n-1})\\
&\quad+\int_n^t e^{a(t-s)}[f(s+\tau)-f(s)]ds|\qquad(n\le t<n+1, n\in Z)\\
&\le[\max(e^a,1)+1](|a^{-1}a_0|+|a^{-1}a_1|+2)\epsilon.
\end{aligned}$$

It follows from definition that $x(t)$ is almost periodic.

(6) If $f(t)$ is ω-periodic and $\omega=n_0\in Z^+$, then we can see that the sequence $\{h_n\}_{n\in Z}$ defined by (3.1) is an ω-periodic sequence, that is, $h_{n+\omega}=h_n$, for all $n\in Z$. At this time, the sequence $\{c_n\}$ defined by (3.8) is also an ω-periodic sequence. Hence, the solution $x(t)$ defined by (3.2) is an ω-periodic solution.

(7) If $f(t)$ is ω-periodic and $\omega=\frac{n_0}{m_0}, n_0, m_0\in Z^+$, then the sequence $\{h_n\}_{n\in Z}$ defined by (3.1) is an $m_0\omega$-periodic sequence. At this time, the sequence $\{c_n\}$ defined by (3.8) is also an $m_0\omega$-periodic sequence. Hence, the solution $x(t)$ defined by (3.2) is an $m_0\omega$-periodic solution. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Assuming that $x_n(t)$ is a solution of Equation (1.2) on the interval $2n-1\le t<2n+1$, with the condition $x_n(2n)=c_{2n}$, we have

$$x_n(t)=[e^{b(t-2n)}+b^{-1}b_0(e^{b(t-2n)}-1)]c_{2n}+\int_{2n}^t e^{b(t-s)}g(s)ds,\qquad 2n-1\le t<2n+1.$$

Let

$$\mu(t)=e^{bt}+b^{-1}b_0(e^{bt}-1),\quad \mu_1=\mu(1),\quad \mu_{-1}=\mu(-1).$$

For $t = 2n - 1$, we have

$$x_n(2n-1) = c_{2n-1} = \mu_{-1}c_{2n} + h_n^{(1)},$$

and for $t = 2n + 1$,

$$x_n(2n+1) = c_{2n+1} = \mu_1 c_{2n} + h_n^{(2)},$$

where

$$h_n^{(1)} = -\int_{2n-1}^{2n} e^{b(2n-1-s)}g(s)ds,$$
$$h_n^{(2)} = \int_{2n}^{2n+1} e^{b(2n+1-s)}g(s)ds.$$

This implies

$$c_{2n+2} = \frac{\mu_1}{\mu_{-1}}c_{2n} + \frac{1}{\mu_{-1}}[h_n^{(2)} - h_{n+1}^{(1)}]. \tag{3.9}$$

From the conditions of Theorem 1.2, it follows that $\left|\frac{\mu_1}{\mu_{-1}}\right| \neq 1$. We define a sequence $\{c_n\}$ as follows:

$$c_{2n} = \begin{cases} \sum\limits_{m\le n-1} \left(\frac{\mu_1}{\mu_{-1}}\right)^{n-(m+1)} \frac{1}{\mu_{-1}}(h_m^{(2)} - h_{m+1}^{(1)}), & \left|\frac{\mu_1}{\mu_{-1}}\right| < 1, \\ \sum\limits_{m\ge n} \left(\frac{\mu_1}{\mu_{-1}}\right)^{n-(m+1)} \frac{1}{\mu_{-1}}(h_m^{(2)} - h_{m+1}^{(1)}), & \left|\frac{\mu_1}{\mu_{-1}}\right| > 1. \end{cases}$$

Using the same argument as in the proof of Theorem 1.1, we know that $\{c_{2n}\}$ is an almost periodic sequence solution for Equation (3.8). At this time, we then imply that $x(t) = x_n(t), 2n-1 \le t < 2n+1$, is an almost periodic solution to Equation (1.2).

When $g(t)$ is an ω-periodic function, we know that Equation (1.2) possesses a 2ω (or $2m_0\omega$)-periodic solution if $\omega = n_0 \in Z^+$ (or $\omega = \frac{n_0}{m_0}, n_0, m_0 \in Z^+$), by using the same argument as in (3.4) and (3.5). This completes the proof of Theorem 2.2.

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