Chin. Ann. of Math.
19B: 1(1998),65-76.

HANKEL OPERATORS AND HANKEL ALGEBRAS**

Yan Shaozong* Chen Xiaoman* Guo Kunyu*

Abstract

The authors study the basic properties of Hankel operators and the structures of Hankel algebras relative to ordered groups, providing a new class of C^*-algebras which are very useful in general C^*-algebra theory.

Keywords Hankel operator, Hankel algebra, Commutator ideal
1991 MR Subject Classification 47B35
Chinese Library Classification O177.1

§0. Introduction

The classical version of Toeplitz theory on the unit circle $\mathbb{T}$ has been generalized in two directions in the past two decades. One direction involves replacing the open unit disc and its boundary $\mathbb{T}$ by a suitable domain in $\mathbb{C}^n$ with a nice boundary, and considering Toeplitz operators with symbols defined on this boundary. By this generalization concerning with function space theory and function theory on the domain, one obtains a series of deep results in operator theory and operator algebra (see [9, 15, 17, 18]). In another direction, its starting point is the fact that the group $\mathbb{T}$ is connected and its dual $\mathbb{Z}$ is ordered. This point of view was taken by Douglas, Murphy and Parone in [6, 7, 8, 10]. The importance of this generalization is that an interesting new class of C^*-algebras arises. This special class of C^*-algebras has a certain universal property which is very useful in general C^*-algebra theory, particular in K-theory.

It is well known in the classical version that Toeplitz operators and Hankel operators are of the same status, and present different operators classes. Halmos[1] regarded Hankel operators as an essential part of Toeplitz theory, and many authors studied Hankel operators and their related problems in [1–5].

In this paper, we consider Hankel operators and the structures of Hankel algebras relative to ordered groups, obtaining the new properties of Hankel operators, and providing a very useful class of C^*-algebras-Hankel algebras.

The paper is organized as following: In §1, we establish the basic definitions and results of Hankel operators over the connected groups. In §2, we discuss the structure of Hankel algebra N^G and show that N^G is completely determined by its related ordered group G, i.e.,

Manuscript received June 23, 1995.
*Institute of Mathematics, Fudan University, Shanghai 200433, China.
**Project supported by the National Natural Science Foundation of China and China Postdoctoral Science Foundation.

the covariant functor $G \mapsto N^G$ is full from the category of ordered groups to the category of C^*-algebras. In this section, we also establish the exact sequences of Hankel algebras and their commutator ideals.

§1. Basic Properties of Hankel Operators

We begin by recalling some definitions and results from [13]. An ordered group is a pair $(G, \preceq)$ consisting of an abelian group G and a linear order relation $\preceq$ on G which is translation-invariant, i.e., if $x \preceq y$ then $x+z \preceq y+z$ for $x, y, z \in G$. Ordered groups exist in great abundance, for example, the additive subgroups of $\mathbb{R}$ with the order inherited from $\mathbb{R}$. Let G be any abelian group. It admits an order relation making it an ordered group if and only if it is torsion-free, and if and only if its Pontryagin dual $\hat{G}$ is connected (see [13]). The fact that an ordered group has a connected dual plays an important role in our analysis.

Let G be an ordered group, and denote by G^+ its positive cone, i.e., the set of the elements $x \succeq 0$. Denote by m the normalized Harr measure of $\hat{G}$. If $x \in G$, the function

$$\epsilon_x : \hat{G} \to \mathbb{T}, \quad \epsilon_x(\gamma) = \gamma(x) = \langle x, \gamma \rangle$$

is, of course, a homomorphism, and it is well known that the family of elements $\{\epsilon_x | x \in G\}$ forms an orthogonal basis of the Hilbert space $L^2(\hat{G}, m)$. The Hilbert subspace of $L^2(\hat{G}, m)$ having orthogonal basis $\{\epsilon_x | x \in G^+\}$ is denoted by $H^2(\hat{G})$ and called the Hardy space related to G. Therefore, the classical Hardy space $H^2(\mathbb{T})$ is relative to integer group $\mathbb{Z}$. Denote by P the orthogonal projection of $L^2(\hat{G}, m)$ onto $H^2(\hat{G})$. For $\varphi \in L^\infty(\hat{G}, m)$, Toeplitz operator T_φ on $H^2(\hat{G})$ is defined by

$$T_\varphi(f) = P(\varphi f), f \in H^2(\hat{G}).$$

Let U denote the symmetric unitary operator on $L^2(\hat{G}, m)$ defined by $U\epsilon_x = \epsilon_{-x}$, $x \in G$. For $\psi \in L^2(\hat{G}, m)$, we define that $\tilde{\psi} = U\psi$. Let φ belong to $L^\infty(\hat{G}, m)$. Hankel operator H_φ on $H^2(\hat{G})$ is defined by

$$H_\varphi f = PU(\varphi f), f \in H^2(\hat{G}).$$

For Toeplitz operators relative to an ordered group, much work has been done in [6,7,8]. In this section, we consider Hankel operators relative to an ordered group. The following theorem shows that a Hankel operator is completely characterized by its algebra equation.

Theorem 1.1. *Let $H \in B(H^2(\hat{G}))$. Then H is a Hankel operator if and only if*

$$T^*_{\epsilon_x} H = H T_{\epsilon_x}, \quad x \in G^+. \tag{1.1}$$

Proof. Let E_0 denote the orthogonal projection from $H^2(\hat{G})$ onto $\{\epsilon_0\}$. For a Hankel operator H_φ, using the relation $PU = U(I-P) + UE_0U$, we have

$$T^*_{\epsilon_x} H_\varphi = PM_{\epsilon_{-x}}[U(I-P)+UE_0U]M_\varphi = PM_{\epsilon_{-x}}U(I-P)M_\varphi = PUM_{\varphi\epsilon_x} = H_\varphi T_{\epsilon_x}, x \in G^+.$$

In another direction, if H satisfies $T^*_{\epsilon_x} H = H T_{\epsilon_x}$ for all $x \in G^+$, then the following is obvious:

$$T_{\tilde{f}} H = H T_f, \quad \forall f \in H^\infty(\hat{G}) (\overset{\triangle}{=} L^\infty(\hat{G}, m) \cap H^2(\hat{G})).$$

Writing $H\epsilon_0 = g \in H^2(\hat{G})$, we have

$$Hf = P(\tilde{f} \cdot g), \quad \forall f \in H^\infty(\hat{G}).$$

Thus, for any $f, h \in H^2(\hat{G})$ satisfying $\overline{\tilde{f}}h \in H^2(\hat{G})$, we have

$$\langle h, Hf \rangle = \langle \overline{\tilde{f}}h, g \rangle. \tag{1.2}$$

A linear functional F on the dense subspace $H^\infty(\hat{G})$ of $H^1(\hat{G})$ is defined by

$$F(f) = \langle f, g \rangle, \quad \forall f \in H^\infty(\hat{G}).$$

Let $\epsilon > 0$, and $f \in H^\infty(\hat{G})$., By [13], there is $0 < \epsilon_f < \epsilon$ such that f can be factorized as

$$f = \alpha\beta - \epsilon_f,$$

where $\alpha, \beta \in H^2(\hat{G})$ and $\|\alpha\|^2 = \|\beta\|^2 \le \|f\| + \epsilon$. Therefore

$$\begin{aligned} |F(f)| &\le |\langle \alpha\beta, g \rangle| + \epsilon\|g\| \overset{(1.2)}{=} |\langle \beta, H\overline{\tilde{\alpha}} \rangle| + \epsilon\|g\| \\ &\le \|H\|\|\overline{\tilde{\alpha}}\|\|\beta\| + \epsilon\|H\| \le \|H\|\|f\|_1 + 2\epsilon\|H\|. \end{aligned}$$

So

$$\|F(f)\| \le \|H\|\|f\|_1,$$

i.e., F is a linear functional on $H^1(\hat{G})$ (because $H^\infty(\hat{G})$ is dense in $H^1(\hat{G})$) and $\|F\| \le \|H\|$. In fact, $\|F\| = \|H\|$. This is done by the following reason:

$$\begin{aligned} \|H\| &= \sup\{|\langle \beta, H\overline{\tilde{\alpha}} \rangle| |\alpha, \beta \in H^\infty(\hat{G}) \text{ with } \|\alpha\|, \|\beta\| \le 1\} \\ &\le \sup\{|F(\alpha\beta)| \; |\alpha, \beta \in H^\infty(\hat{G}), \text{ with } \|\alpha\|, \|\beta\| \le 1\} \text{ by } (1.2) \\ &\le \|F\|\|\alpha\beta\|_1 \le \|F\|. \end{aligned}$$

Let F' be a continuous extension of F onto $L^1(\hat{G}, m)$ and $\|F'\| = \|F\| = \|H\|$. Then there exists a $\psi \in L^\infty(\hat{G}, m)$ such that

$$\|\psi\|_\infty = \|F'\| = \|H\| \text{ and } F'(h) = \int_{\hat{G}} \psi h dm, \quad \forall h \in L^1(\hat{G}, m).$$

Thus, for $f_1, f_2 \in H^\infty(\hat{G})$,

$$\begin{aligned} \langle f_2, Hf_1 \rangle &\overset{(1.2)}{=} \langle \overline{\tilde{f}}_1 f_2, g \rangle = F(\overline{\tilde{f}}_1 f_2) = F'(\overline{\tilde{f}}_1 f_2) \\ &= \int_{\hat{G}} \psi \overline{\tilde{f}}_1 f_2 dm = \langle f_2, H_{\overline{\tilde{\psi}}} f_1 \rangle. \end{aligned}$$

This means that $H = H_{\overline{\tilde{\varphi}}}$, i.e., H is a Hankel operator and $\|H\| = \|\psi\|_\infty = \|\overline{\tilde{\psi}}\|_\infty$.

As a corollary of the above proof, we get the following

Corollary 1.1. *For $\varphi \in L^\infty(\hat{G}, m)$, we have*

$$\|H_\varphi\| = \text{dist}(\varphi, H_0^\infty(\hat{G})),$$

where $H_0^\infty(\hat{G})$ is $\{f \in H^\infty(\hat{G}) | \hat{f}(0) = \int_{\hat{G}} f dm = 0\}$.

Next we discuss the compactness of Hankel operators. The following Theorem 1.2 says that the existence of nonzero compact Hankel operators depend on the existence of a least positive element of G^+.

Denote by $\mathfrak{A}$ the set $\{\varphi \in L^\infty(\hat{G}, m) | H_\varphi \text{ is compact}\}$. A simple computation yields the following relations

$$H_{fg} = H_f T_g + T_{\tilde{f}} H_g - T_{\tilde{f}} E_0 H_g, \quad \forall f, g \in L^\infty(\hat{G}, m). \tag{1.3}$$

Immediately from the above relations, we know that $\mathfrak{A}$ is a closed subalgebra of $L^\infty(\hat{G}, m)$, containing $H^\infty(\hat{G})$.

Theorem 1.2. (1) $\mathfrak{A}$ *contains* $H^\infty(\hat{G})$ *properly if and only if* G *admits a least positive element.*

(2) *If* G *admits a least positive element* e*, denoted by* C_e *the algebra generated by the polynomials of* ϵ_e*. Then*

$$\mathfrak{A} = H^\infty(\hat{G}) + C_e.$$

Proof. (1) We begin by establishing criteria for $\mathfrak{A}$ to be equal to $H^\infty(\hat{G})$. Let $a \succcurlyeq 0$. Since

$$H^*_{\overline{\epsilon_a}} H_{\overline{\epsilon_a}} = H_{\overline{\epsilon_a}} H_{\overline{\epsilon_a}} = I - T_{\epsilon_a} T_{\overline{\epsilon_a}} + \text{compact} = P_a + \text{compact}, \tag{1.4}$$

where P_a denotes the orthogonal projection from $H^2(\hat{G})$ onto the closed span of $\{\epsilon_b | 0 \preceq b \prec a\}$, we get that the cardinal number of $[0, a)$ is infinite, where $[0, a)$ denotes the interval $\{b | 0 \preceq b \prec a\}$.

In another direction, for any $a > 0$, if $[0, a)$'s cardinal number is infinite, then $\mathfrak{A} = H^\infty(\hat{G})$. In fact, if there exists $\varphi \in L^\infty(\hat{G}, m)$ and $\varphi \notin H^\infty(\hat{G})$ such that H_φ is compact, then we claim that there is $a > 0$ such that $H_{\overline{\epsilon_a}}$ is compact. This is done by the following process. For every $t \in \hat{G}$, an operator U_t on $L^2(\hat{G}, m)$ is defined by

$$U_t : L^2(\hat{G}, m) \to L^2(\hat{G}, m), (U_t f)(s) = f(ts), \quad \forall f \in L^2(\hat{G}, m),\ s \in \hat{G}.$$

Obviously

$$U_t^* H_\varphi U_t = H_{U_t^* \varphi} = H_{U_{t^{-1}} \varphi}. \tag{1.5}$$

Let f belong to $L^\infty(\hat{G}, m)$ and $H^{(f)}$ denote the integral of $f(t) H_{U_{t^{-1}}\varphi}$. By the compactness of $\hat{G}$, $H^{(f)}$ is compact, where

$$H^{(f)} \overset{\text{def.}}{=} \int_{\hat{G}} f(t) H_{U_{t^{-1}}\varphi} dm(t).$$

A simple computation shows that

$$H^{(f)} = H_{\varphi * f}. \tag{1.6}$$

Because φ does not belong to $H^\infty(\hat{G})$, there is $a > 0$ such that $\hat{\varphi}(-a) \neq 0$. Therefore $\varphi * \overline{\epsilon_a} = \hat{\varphi}(-a)\overline{\epsilon_a} \neq 0$. By (1.6), $H_{\overline{\epsilon_a}}$ is compact, the claim is proved. By (1.4), P_a is compact. It follows that $[0, a)$'s cardinal number is finite. This contradicts our assumption. Consequently, we obtain the following criteria for $\mathfrak{A}$ to be equal to $H^\infty(\hat{G})$.

(i) $\mathfrak{A} = H^\infty(\hat{G})$ iff the cardinal number of $[0, a)$ is infinite for any $a > 0$.

By (i), we obtain the following

(ii) $\mathfrak{A}$ contains $H^\infty(\hat{G})$ properly iff G admits a least positive element.

This completes the proof of (1).

(2) Let e be a least positive element of G and $[H^\infty(\hat{G}), \overline{\epsilon_e}]$ denote the closed subalgebra of $L^\infty(\hat{G}, m)$ generated by $H^\infty(\hat{G})$ and $\overline{\epsilon_e}$. Then we claim that the following relation is true:

$$\mathfrak{A} = [H^\infty(\hat{G}), \overline{\epsilon_e}]. \tag{1.7}$$

Clearly

$$\mathfrak{A} \supseteq [H^\infty(\hat{G}), \overline{\epsilon_e}].$$

Write H^2_e for the closed linear span of $\{\epsilon_{ne} | n \in Z^+\}$, i.e., H^2_e is a Hardy space relative to $\mathbb{Z}$, and H^2_s for $H^2(\hat{G}) \ominus H^2_e$. Let $\varphi \in \mathfrak{A}$. For $b \in G^+ - \mathbb{Z}^+ e$, the proof of (1) leads to

$\hat{\varphi}(-b)\overline{\epsilon_b} \in \mathfrak{A}$. By (1.4), we know that $\hat{\varphi}(-b) = 0$ for any $b \in G^+ - \mathbb{Z}^+ e$. Thus φ can be written as

$$\varphi = h + \sum_{n=0}^{\infty} a_n \overline{\epsilon_e^n},$$

where $h \in H^2(\hat{G})$. By the above equation, we see that $\ker H_\varphi$ contains H_s^2. For every positive integer m, there exists a function $h_m \in H^2(\hat{G})$ and $\|h_m\| = 1$ such that $\|H_{\varphi\epsilon_e^m}\| = \|H_{\varphi\epsilon_e^m} h_m\|$. Write $h_m = f_m + g_m$, where $f_m \in H_e^2$ and $g_m \in H_s^2$. Then $\epsilon_e^m f_m \xrightarrow{\mathrm{w}} 0$. Therefore we get

$$\|H_{\varphi\epsilon_e^m}\| = \|H_\varphi \epsilon_e^m h_m\| = \|H_\varphi \epsilon_e^m f_m\| \to 0.$$

By the norm formula of Hankel operator (Corollary 1.1)

$$\|H_{\varphi\epsilon_e^m}\| = \mathrm{dist}(\varphi\epsilon_e^m, H_0^\infty(\hat{G})) = \mathrm{dist}(\varphi, \overline{\epsilon_e^m} H_0^\infty(\hat{G})).$$

So $\varphi \in [H^\infty(\hat{G}), \overline{\epsilon_e}]$. In this way we have

$$\mathfrak{A} = [H^\infty(\hat{G}), \overline{\epsilon_e}],$$

completing the proof of claim (1.7).

Our next goal is to show that $[H^\infty(\hat{G}), \overline{\epsilon_e}] = H^\infty + C_e$, where C_e is the closed algebra generated by the polynomials of ϵ_e.

Clearly, $H^\infty(\hat{G}) + C_e \subseteq [H^\infty(\hat{G}), \overline{\epsilon_e}]$. Take the dense subalgebra

$$\Gamma = \Big\{ \sum_{i=0}^{k} h_i \overline{\epsilon_e^i} | h_i \in H^\infty(\hat{G}), \quad k \in Z^+ \Big\}$$

of $[H^\infty(\hat{G}), \overline{\epsilon_e}]$, then

$$\Gamma \subseteq H^\infty(\hat{G}) + C_e. \tag{1.8}$$

This is because $h_i \overline{\epsilon_e^i}$ has the form

$$h_i \overline{\epsilon_e^i} = \Big(\sum_{j=0}^{i} a_j \epsilon_e^j \Big) \overline{\epsilon}_e^i + \sum_{l \succeq ie} a_l \epsilon_{l-ie}. \tag{1.9}$$

The above fact shows immediately that

$$\mathfrak{A} = \overline{H^\infty(\hat{G}) + C_e}. \tag{1.10}$$

Therefore, what we do is to prove that $H^\infty(\hat{G}) + C_e$ is closed. It is obvious that $H^\infty(\hat{G}) + C_e$ is closed if and only if there exists a constant c, $1 < c < \infty$, such that

$$\mathrm{dist}(g, H^\infty(\hat{G}) \cap C_e) \le c\, \mathrm{dist}(g, H^\infty(\hat{G})), \quad \forall g \in C_e. \tag{1.11}$$

We write A_e for the algebra generated by analytic polynomials of ϵ_e, H_e^∞ for the w^*-closure of A_e. Then the following is true, whose proof is similar to the classical version (see [16]).

$$\mathrm{dist}(g, H^\infty(\hat{G}) \cap C_e) = \mathrm{dist}(g, A_e) = \mathrm{dist}(g, H_e^\infty). \tag{1.12}$$

Since by Corollary 1.1,

$$\mathrm{dist}(g, H_e^\infty) = \mathrm{dist}(g\epsilon_e, \epsilon_e H_e^\infty) = \|H_{g\epsilon_e}^{(e)}\|,$$

where $H^{(e)}_{g\epsilon_e}$ is a Hankel operator (relative to an ordered group $\mathbb{Z}e$), if there is not a constant c such that (1.11) is true, then we can choose $\{g_n\} \subset C_e$ such that

$$\|H^{(e)}_{g_n\epsilon_e}\| = 1, \|H_{g_n\epsilon_e}\| \to 0. \tag{1.13}$$

It follows that there exists $h^{(e)}_n \in H^2_e$ with $\|h^{(e)}_n\| = 1$ such that

$$\|H^{(e)}_{g_n\epsilon_e} h^{(e)}_n\| = 1. \tag{1.14}$$

But

$$\|H_{g_n\epsilon_e}\| \geq \|H_{g_n\epsilon_e} h^{(e)}_n\| \geq \|H^{(e)}_{g_n\epsilon_e} h^{(e)}_n\| = 1. \tag{1.15}$$

We get a contradiction by (1.13) and (1.15). Consequently, there is a positive constant c such that (1.11) is true. $H^\infty(\hat{G}) + C_e$ is then closed, and the proof of Theorem 1.2 is completed.

We say that ordered groups G_1, G_2 are ordered isomorphic if there is a group isomorphism $\varphi : G_1 \to G_2$ and $\varphi(G_1^+) \subseteq G_2^+$. From the proof of Theorem 1.2, it is obvious that $\mathfrak{A} = H^\infty(\hat{G}) + C(\hat{G})$ iff G is order-isomorphic to $\mathbb{Z}$, where $C(\hat{G})$ are all continuous functions over $\hat{G}$.

Remark 1.1. For an ordered group G, then $H^\infty(\hat{G}) + C(\hat{G})$ is a closed subspace of $L^\infty(\hat{G}, m)$, but is an algebra only when G is order-isomorphic to $\mathbb{Z}$ (see [11]).

Example 1.1. Let R_0 be an ordered subgroup of $\mathbb{R}$. Then there exist nonzero compact Hankel operators on $H^2(\hat{R}_0)$ iff $R_0 = c\mathbb{Z}$, where c is a constant.

Example 1.2. Let α be an irrational number, and take $G = \mathbb{Z} \times \mathbb{Z}$,

$$G^+ = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} | \alpha m + n \geq 0\}.$$

Then $H^2(\hat{G})$ is equal to the closed linear span of $\{e^{i(m\theta_1 + n\theta_2)} | \alpha m + n \geq 0\}$, a closed subspace of $L^2(\mathbb{T} \times \mathbb{T})$. By Theorem 1.2, there is not a nonzero compact Hankel operator on $H^2(G)$.

Take $G = \mathbb{Z} \times \mathbb{Z}$, $G^+ = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} | m > 0 \text{ or } (m = 0 \text{ and } n \geq 0)\}$. Then G has a least positive element (0,1). Consequently

$$\mathfrak{A} = H^\infty(\hat{G}) + C_{(0,1)}.$$

§2. Structure of Hankel Algebra

From [6], Toeplitz algebra associated to an ordered group has a nice structure, i.e., it has a Coburn exact sequence and its commutator ideal presents simplicity in some sense. At the time, we know that the structure of Toeplitz algebra in this context is closely related to the structure of an ordered group. In this section, our aim is to consider Hankel algebra associated to an ordered group. Finally, we find that Hankel algebra is a new class of C^*-algebras completely different from Toeplitz algebra. We first state some notations and definitions.

Let G be an ordered group. The C^*-algebra generated by Toeplitz operators with continuous symbols on Hardy space $H^2(\hat{G})$ is called the Toeplitz algebra over G and is denoted by T^G. The C^*-algebra generated by Hankel operators with continuous symbols is called the small Hankel algebra over G and is denoted by H^G. The C^*-algebra generated by Toeplitz operators and Hankel operators with continuous symbols is called the Hankel algebra and is denoted by N^G, i.e., N^G is generated by T^G and H^G. The following theorem is the main result in this section.

Theorem 2.1. *The commutator ideal of N^G is equal to H^G. Denote $\{T_\varphi|\varphi \in C(\hat{G})\}$ by $\mathcal{T}$, then $N^G = \mathcal{T} + H^G$ and the sum is direct. Consequently, the following sequence is exact*

$$0 \to H^G \to N^G \to C(\hat{G}) \to 0. \tag{2.1}$$

Proof. We first show that the following inequality is true:

$$\|T_\varphi\| \le \inf\{\|T_\varphi + S\| | S \in H^G\}, \quad \forall \varphi \in L^\infty(\hat{G}, m). \tag{2.2}$$

In fact, because $C(\hat{G})$ is equal to the closed linear span of $\{\epsilon_a\overline{\epsilon_b}|a, b \in G^+\}$, we only need to consider a dense subalgebra $\left\{\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m_i} H_{\epsilon_{a_{ij}}\overline{\epsilon_{b_{i,j}}}}\middle| a_{ij}, b_{ij} \in G^+,\ n, m_i \in \mathbb{Z}^+\right\}$ of H^G. Let S be equal to $\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m_i} H_{\epsilon_{a_{ij}}\overline{\epsilon_{b_{ij}}}}$, and put $u = \epsilon_{a_0}\prod\limits_{i=1}^{n}\epsilon_{b_i m_i}$, where $a_0 \succcurlyeq 0$. Then $ST_u = 0$. Thus

$$\|T_\varphi\| = \|\varphi\|_\infty = \|\varphi u\|_\infty = \|T_{\varphi u}\| = \|T_\varphi T_u\| = \|(T_\varphi + S)T_u\| \le \|T_\varphi + S\|.$$

This shows that (2.2) is true. By (2.2) we know that the sum $\mathcal{T} + H^G$ is direct.

What we next to do is the following:

(1) H^G contains all compact operators.

(2) Let $\varphi_1, \varphi_2 \in C(\hat{G})$. Then $T_{\varphi_1}H_{\varphi_2} \in H^G$.

(3) H^G contains the semicommutator ideal of T^G, i.e., the ideal of $\mathcal{T}^G$ generated by the semicommutators $\{T_{\varphi_1\varphi_2} - T_{\varphi_1}T_{\varphi_2}|\varphi_1, \varphi_2 \in C(\hat{G})\}$.

If the proofs of (1), (2) and (3) are completed, then we can prove that $N^G = \mathcal{T} \oplus H^G$.

Proof of (1). Because H^G contains compact operators (in fact, $H_{\epsilon_0} \in H^G$ and the rank of H_{ϵ_0} is equal to 1), what we shall do is to prove that H^G is irreducible. Suppose that there exists a projection P_0 such that

$$P_0A = AP_0. \text{ all } A \in H^G. \tag{2.3}$$

By the equality

$$H_fH_g = T_{\tilde{f}g} - T_{\tilde{f}}T_g + T_{\tilde{f}}E_0T_g, \tag{2.4}$$

we see that

$$H_{\epsilon_0}H_{\overline{\epsilon_a}}h = \hat{h}(a)\epsilon_0, \quad \forall a \in G^+, \quad h \in H^2(\hat{G}). \tag{2.5}$$

Write E_a for $H_{\epsilon_0}H_{\overline{\epsilon_a}}$. Then

$$P_0E_a = E_aP_0. \tag{2.6}$$

So

$$P_0E_a(\epsilon_a) = P_0\,\epsilon_0 = E_a(P_0\epsilon_a) \text{ and } E_aP_0E_b = 0, \quad a \ne b. \tag{2.7}$$

It follows that $P_0\epsilon_0$ is a constant (writing c for $P_0\epsilon_0$) and $P_0\,\epsilon_a = c\epsilon_a$ for $a \in G^+$. Since P_0 is a projection, we see that

$$P_0 = 0 \text{ or } P_0 = I.$$

This shows that H^G is irreducible, completing the proof of (1).

Proof of (2). Because the linear span of $\{\epsilon_a\overline{\epsilon_b}|a, b \in G^+\}$ is a dense subalgebra of $C(\hat{G})$, we may take $\varphi_1 = \epsilon_a\overline{\epsilon_b}$, $\varphi_2 = \epsilon_{a'}\overline{\epsilon_{b'}}$ $(a' \succeq 0)$. A simple computation shows that the following are true:

$$T_{\varphi_1}H_{\varphi_2} = T_{\varphi_1}H_{\varphi_2}H^*_{\varphi_2}H_{\varphi_2} \tag{2.8}$$

and

$$T_{\varphi_1}H_{\varphi_2}=H_{\tilde{\varphi}_1\varphi_2}-H_{\tilde{\varphi}_1}T_{\varphi_2}+H_{\tilde{\varphi}_1}E_0T_{\varphi_2}. \tag{2.9}$$

So

$$T_{\varphi_1}H_{\varphi_2}=H_{\tilde{\varphi}_1}H^*_{\varphi_2}H_{\varphi_2}-H_{\tilde{\varphi}_1}T_{\varphi_2}H^*_{\varphi_2}H_{\varphi_2}+H_{\tilde{\varphi}_1}E_0T_{\varphi_2}H^*_{\varphi_2}H_{\varphi_2}. \tag{2.10}$$

Immediately from (1) and (2.10), what is proved is that $T_{\varphi_2}H^*_{\varphi_2}$ belongs to H^G. Since

$$T_{\varphi_2}H^*_{\varphi_2}=T_{\epsilon_{a'-b'}}H_{\epsilon_{a'-b'}},$$

we see that $T_{\varphi_2}H^*_{\varphi_2}$ is compact in case $a'\succeq b'$ and by Theorem 1.1, $T_{\varphi_2}T^*_{\varphi_2}$ is equal to $H_{\epsilon_{a'-b'}}T_{\epsilon_{a'-b'}}$ in case $a'\preceq b'$. The above fact leads then to $T_{\varphi_2}H^*_{\varphi_2}$ belonging to H^G by (1). This completes the proof of (2).

Proof of (3). Note the relations

$$T_{fg}-T_fT_g=H_{\tilde{f}}H_g-T_fE_0T_g \tag{2.11}$$

and (1) and (2). We see that H^G contains the semicommutator ideal of T^G, completing the proof of (3).

By (1), (2) and (3) and [6], we obtain the following

$$N^G=\mathcal{T}\oplus H^G. \tag{2.12}$$

We next prove that $H^G=\text{comm}N^G$. Obviously, $\text{comm}N^G$ contains all compact operators. By the relations

$$H^*_\varphi H_\varphi=T_{\tilde{\varphi}\varphi}-T_{\tilde{\varphi}}T_\varphi+T_{\tilde{\varphi}}E_0T_\varphi \tag{2.13}$$

and [6], we see that $\hat{H}^*_\varphi\hat{H}_\varphi=0$, where $\hat{H}_\varphi$ is the image of H_φ in $N^G/\text{comm}N^G$. This shows that $\text{comm}N^G=H^G$. Consequently, from (2.12) we have the following exact sequence

$$0\to H^G\to N^G\to C(\hat{G})\to 0.$$

This completes the proof of Theorem 2.1.

Finally, we give a corollary of Theorem 2.1, which says that Hankel algebra determines the ordered group.

Theorem 2.2 *If G_1,G_2 are two ordered groups, then N^{G_1} and N^{G_2} are isomorphic C^*-algebras iff G_1 and G_2 are order-isomorphic, i.e., the covariant functor $G\mapsto N^G$ is full from the category of ordered groups to the category of C^*-algebras.*

Proof. Sufficiency. A unitary operator $V:L^2(\hat{G}_1,m_1)\to L^2(\hat{G}_2,m_2)$ is defined by $V\epsilon^{(1)}_a=\epsilon^{(2)}_{\varphi(a)}$ for $a\in G_1$, where φ is the order isomorphism of G_1 to G_2. $\{\epsilon^{(i)}_t\}$ is the canonical orthogonal bases of $L^2(G_i,m_i)$'s. Write P_i to be the orthogonal projection of $L^2(\hat{G}_i,m_i)$ onto $H^2(\hat{G}_i)$ $(i=1,2)$ and U_i for the symmetric unitary operator on $L^2(G_i,m_i)$ $(i=1,2)$. Then the following equations are obvious:

$$VP_1=P_2V,\quad VM^{(1)}_{f_1}V^*=M^{(2)}_{Vf},\quad VU_1=U_2V. \tag{2.14}$$

So

$$VT^{(1)}_fV^*=T^{(2)}_{Vf},\quad VH^{(1)}_fV^*=H^{(2)}_{Vf}, \tag{2.15}$$

where $T^{(i)}_*$ and $H^{(i)}_*$ denote Toeplitz operators and Hankel operators, respectively on $H^2(\hat{G}_i)$ $(i=1,2)$. The equations (2.15) say that N^{G_1} and N^{G_2} are isomorphic C^*-algebras.

Necessity. Let π be an isomorphism from N^{G_1} to N^{G_2}. We have thus that π induces an isomorphism between $N^{G_1}/\mathrm{comm}N^{G_1}$ and $N^{G_2}/\mathrm{comm}N^{G_2}$. It follows that $C(\hat{G}_1)$ is isomorphic to $C(\hat{G}_2)$ by Theorem 2.1.

A result of van Kampen (see [14]) implies that each component of the groups of invertible elements in $C(\hat{G}_1)$ and $C(\hat{G}_2)$ contains respectively a characteristic function and hence we obtain an isomorphism π_* between G_1 and G_2. Moreover, since $a \in G_1^+$ iff some element in the coset $T_{\epsilon_a} + H^{G_1}$ has a left inverse, and the latter property is preserved by π, it follows that π_* is the order isomorphism of G_1 to G_2. The proof is completed.

Example 2.1. Let α, β be irrational numbers, and take the ordered subgroups R_α, R_β of $\mathbb{R}$ as the following

$$R_\alpha = \{m + n\alpha | m, n \in \mathbb{Z}\}, \quad R_\beta = \{m + n\beta | m, n \in \mathbb{Z}\}.$$

Then N^{R_α} is isomorphic to N^{R_β} iff there are integers i, j, k and l such that

$$\begin{pmatrix} i & j \\ k & l \end{pmatrix} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \beta \end{pmatrix} \quad \text{and} \quad il - jk = 1.$$

By the above fact, we see that $N^{R_{\sqrt{2}}}$ and $N^{R_{\sqrt{5}}}$ are not isomorphic.

Remark 2.1. From the process of Theorem 2.2's proof and [6], we can prove that T^{G_1} is isomorphic to T^{G_2} iff G_1 is order-isomorphic to G_2 and they are order-isomorphic to $\mathbb{Z}$. Thus Toeplitz algebra is completely different from Hankel algebra.

We are now ready to study the structure of H^G, i.e., the commutator ideal of Hankel algebra N^G. From [6], we know that the commutator ideal of Toeplitz algebra $\mathcal{T}^G$ has a nice structure. However, $H^G \supseteq \mathrm{comm}T^G$, thus our technique is that $\mathrm{comm}T^G$ is separated from H^G by direct sum. What we use is the odd-even decomposition of a C^*-algebra associated with its generators (see [3]). The following concepts are needed. Let H be a Hilbert space. For $\mathfrak{F} \subset B(H)$, $C^*(\mathfrak{F})$ denotes the C^*-algebra generated by $\mathfrak{F}$, and $C_e^*(\mathfrak{F})$ (resp. $C_0^*(\mathfrak{F})$) denotes the closed linear span of operators of the form $F_1F_2\cdots F_n$, where $F_i \in \mathfrak{F} \cup \mathfrak{F}^*(i = 1, 2, \cdots, n)$ and n is even (resp. odd). Then $C_e^*(\mathfrak{F}) + C_o^*(\mathfrak{F})$ is closed and so equal to $C^*(\mathfrak{F})$ (see [3]). In our context, take

$$\mathfrak{F} = \{H_f | f \in C(\hat{G})\}.$$

Then $\mathfrak{F}$ is a self-adjoint operator space. By the above state, we have

$$H^G = C_e^*(\mathfrak{F}) + C_o^*(\mathfrak{F}). \tag{2.18}$$

It is natural to consider what $C_e^*(\mathfrak{F})$ and $C_o^*(\mathfrak{F})$ are, respectively.

Lemma 2.1. *Let K be the compact ideal of $B(H^2(\hat{G}))$. Then*

$$C_e^*(\mathfrak{F}) + K = \mathrm{comm}T^G + K.$$

Proof. Repeating the proofs of (2) and (3) of Theorem 2.1, we have $C_e^*(\mathfrak{F}) + K =$ the semicommutator ideal of $T^G + K$. By [6], the semicommutator ideal of $T^G = \mathrm{comm}T^G$, and thus

$$C_e^*(\mathfrak{F}) + K = \mathrm{comm}T^G + K.$$

Lemma 2.2. $K \subset C_o^*(\mathfrak{F})$.

Proof. It is obvious that $C_e^*(\mathfrak{F}) \cap C_o^*(\mathfrak{F})$ is an ideal of C^*-algebra $C_e^*(\mathfrak{F})$ and

$$H_1 = H_1^2 = E_0 \in C_e^*(\mathfrak{F}) \cap C_o^*(\mathfrak{F}).$$

The remainder is an exact analogue of (3) of Theorem 2.1.

Denote by S^G the self-adjoint operator space of $C_0^*(\mathfrak{F})$ (i.e., the closed linear span of all operators of the form $H_{f_1}H_{f_2}\cdots H_{f_n}$ where n is odd), and call it the singular part of small Hankel algebra H^G. By Lemmas 2.1, 2.2 and (2.18), we get

$$H^G = \text{comm}T^G + S^G \tag{2.19}$$

and $S^G \cdot \text{comm}T^G \subset S^G$, $\text{comm}T^G \cdot S^G \subset S^G$, i.e., S^G is a C^*-bimodule over comm T^G.

What we pay attention to is when sum (2.19) is direct, i.e, when $\text{comm}T^G$ is separated from H^G.

Our methods are closely related to the study of the Weyl commutation relations and Hankel commutation relations associated with group G. For each $\xi \in \hat{G}$, we consider the unitary operator

$$U_\xi = S \circ T - \lim_{b\in G^+} \sum_{a\preceq b} \langle \xi, a\rangle P_a,$$

where G^+ is regarded as a net. P_a is the orthogonal projection of $H^2(\hat{G})$ onto $\{\epsilon_a\}$.

A simple computation yields the following Weyl commutation's relations and Hankel commutation relations:

$$T_{\epsilon_a}U_\xi = \overline{\langle \xi, a\rangle} U_\xi T_{\epsilon_a}, \quad a \in G, \quad \xi \in \hat{G}, \tag{2.20}$$

and

$$H_{\overline{\epsilon_a}}U_\xi = \langle \xi, a\rangle U_\xi^* H_{\overline{\epsilon_a}}, \quad a \in G, \quad \xi \in \hat{G}. \tag{2.21}$$

Let $\xi \in \hat{G}$. The operator α_ξ is defined from H^G to $B(H^2(\hat{G}))$ by $\alpha_\xi(S) = U_\xi^* S U_\xi$ for $S \in H^G$. We consider the bounded mean of ρ defined on H^G by the following

$$\rho(s) = \int_{\hat{G}} \alpha_\xi(S) dm(S).$$

Then ρ has the following properties:

(1) If $S > 0$, then $\rho(S) > 0$.

(2) $\rho(\text{comm}T^G) \subset \text{comm}T^G$.

(3) $\rho(S^G) \subset K$.

Proof of (1). Obviously, $\rho(S) \geq 0$. If $\rho(S) = 0$, then

$$\begin{aligned}\langle \rho(S)\epsilon_a, \epsilon_a\rangle &= \int_{\hat{G}} \langle \alpha_\xi(S)\epsilon_a,\, \epsilon_a\rangle dm(S) = \int_{\hat{G}} |\langle \xi, a\rangle|^2 \langle S\epsilon_a, \epsilon_a\rangle dm(S)\\ &= \int_{\hat{G}} \|\sqrt{S}\epsilon_a\|^2 dm(S) = \|\sqrt{S}\epsilon_a\|^2\end{aligned}$$

for $a \in G^+$. Thus we have $S = 0$, completing the proof of (1).

Proof of (2). Note that the linear span of the finite products of the operators of the form $T_{\epsilon_{a_3}}(T_{\epsilon_{a_1+a_2}} - T_{\epsilon_{a_1}}T_{\epsilon_{a_2}})T_{\epsilon_{a_4}}$ is dense in $\text{comm}T^G$ and

$$\alpha_\xi(T_{\epsilon_{a_3}}(T_{\epsilon_{a_1+a_2}} - T_{\epsilon_{a_1}}T_{\epsilon_{a_2}})T_{\epsilon_{a_4}}) = \overline{\langle \xi, a_1 + a_2 + a_3 + a_4\rangle} T_{\epsilon_{a_3}}(T_{\epsilon_{a_1+a_2}} - T_{\epsilon_{a_1}}T_{\epsilon_{a_2}})T_{\epsilon_{a_4}},$$

where a_i's are in G.

Since ρ is continuous, we see that (2) is true.

Proof of (3). By induction, for the case that n is odd, $\alpha_\xi(H_{\overline{\epsilon}_{a_1}} H_{\overline{\epsilon}_{a_2}} \cdots H_{\overline{\epsilon}_{a_n}})$ has the form $\langle\xi, b_n\rangle U_\xi^{*2} H_{\overline{\epsilon}_{a_1}} H_{\overline{\epsilon}_{a_2}} \cdots H_{\overline{\epsilon}_{a_n}}$ where $a_1, a_2, \cdots, a_n$ and b_n are in G. Since

$$\int_{\hat{G}} \langle\xi, b_n\rangle U_\xi^{*2} H_{\overline{\epsilon}_{a_1}} H_{\overline{\epsilon}_{a_2}} \cdots H_{\overline{\epsilon}_{a_n}} dm(s)$$
$$= \begin{cases} P_a H_{\overline{\epsilon}_0} \cdots H_{\overline{\epsilon}_{a_n}}, & \text{if there exists } a \in G^+ \text{ such that } 2a = b_n, \\ 0, & \text{otherwise,} \end{cases}$$

and P is continuous, we see that (3) is true.

Lemma 2.3. *Let J be a nontrivial ideal of T^G. Then there exists an ideal I of G such that the ideal F_I generated by $\{I - T_{\epsilon_x} T_{\overline{\epsilon}_x} | x \in I^+\}$ is contained in J.*

Proof. By [6], we have the canonical homomorphism $\beta : T^G \to T^G/J$ by $\beta(T_{\epsilon_x}) = T_{\epsilon_x} + J$. Since $J \neq \{0\}$, β is not injective. Hence the set $I^+ = \{x \in G^+ | \beta(T_{\epsilon_x}) \text{ is unitary}\}$ is nontrivial. Denote by I the ideal of G, generated by I^+. Then it is obvious that $F_I \subset J$.

Remark 2.1. $K \subset T^G$ iff G has a least positive element by Lemma 2.3.

Theorem 2.3. $H^G = \text{comm}T^G + S^G$. *The sum is direct iff G has not a least positive element.*

Proof. Necessary is obvious by Lemma 2.2 and the remark of Lemma 2.3.

Sufficiency. Assume that $\text{comm}T^G \cap S^G$ is nontrivial. Then $\text{comm}T^G \cap S^G$ is a nontrivial ideal of $\text{comm}T^G$. Taking a positive element S of $\text{comm}T^G \cap S^G$, we have $\rho(S) > 0$, $\rho(S) \in \text{comm}T^G$ and $\rho(S) \in K$ by the properties (1), (2) and (3) of ρ. It follows that $\text{comm}T^G$ contains K since $\text{comm}T^G$ is irreducible. So G has a least positive element by the remark of Lemma 2.3. Thus, if G has not a least positive element, then the sum is direct. The proof is completed.

At the same time, in the case that G has not a least positive element, by Theorems 2.1 and 2.3 and the remark of Lemma 2.3, we have the following

Theorem 2.4. *If G has not a least positive element, then $N^G = T^G \oplus S^G$. In other words, if the above sum is direct, then G has not a least positive element.*

Remark 2.2. Note that N^G, T^G and S^G are C^*-bimodules over $\text{comm}T^G$. By Theorems 2.3 and 2.4, when G has not a least positive element, we have the following C^*-bimodules over $\text{comm}T^G$ exact sequences

$$0 \to \text{comm}T^G \to H^G \to S^G \to 0,$$
$$0 \to T^G \to N^G \to S^G \to 0.$$

References

[1] Barria, J. & Halmos, P. R., Asymptotic Toeplitz operators, *Trans. Amer. Math. Soc.,* **273** (1982), 621–630.

[2] Power, S. C., Hankel operator on Hilbert space, Pitman Publishing Inc., Boston, London, Melbourne, 1982.

[3] Power, S. C., C^*-modules and an odd-even decomposition for C^*-algebra, *Bull. London Math. Soc.,* **8** (1976), 268–272.

[4] Power, S. C., C^*-algebras generated by Hankel operators and Toeplitz operators, *J. Funct. Analy.,* **31** (1979), 52–68.

[5] Chen Xiaoman & Chen Feng, Hankel operators in the set of essential Toeplitz operators, *Acta Math. Sinica, New Series,* **6** (1990), 354–363.

[6] Murphy, G. J., Ordered groups and Toeplitz algebras, *J. Operator Theory,* **18** (1987), 303–326.

[7] Murphy, G. J., Toeplitz operators and algebra, *Math. Zeit,* **208** (1991), 143–152.

[8] Parone, M., Toeplitz operators on discrete groups, *J. Operator Theory,* **27** (1992), 359–384.

[9] Douglas, R. G. & Howe, R., On the C^*-algebra of Toepltiz operators on the quarter-plane, *Trans, Amer. Math. Soc.,* **158** (1971), 203–217.

[10] Coburn, L. A., Douglas, R. G., Schaeffer, D. & Singer, I. M., C^*-algebra of operators on half-space II. Index theory, *Inst. Hautes Etudes Sci. Publ. Math.,* **40** (1971), 69–79.

[11] Rudin, W., Space of type $H^\infty + C$, *Ann. Inst. Fourier Grenoble,* **25** (1975), 99–125.

[12] Berger, C. A. & Coburn, L. A., Toeplitz operators on the Segal-Bergman space, *Trans. Amer. Math. Soc.,* **301** (1987), 813–827.

[13] Rudin, W., Fourier analysis on groups, Interscience Publishers, New York, 1962.

[14] van Kampen, E., On almost periodic functions of constant absolute value, *J. London Math. Soc.,* **12** (1937), 3–6.

[15] Davie, A. M. & Jewell, N. P., Toeplitz operators in several complex variables, *J. Funct. Analy.,* **26** (1977), 356-368.

[16] Garnett, J. B., Bounded analytic functions, Academic Press Inc., New York, 1981.

[17] Xu Shengzhi, Chen Xiaoman & Yan Shaozong, Toeplitz C^*-algebra over the L-shaped domain in C^n, *Chin. Ann. of Math.,* **17B:**3(1996), 325–334.

[18] Yan Shaozong, Chen Xiaoman & Xu Shengzhi,, The K-theory and the classification of the irrational rotation groupoid C^*-algebras, *Chin. Ann. of Math.,* **18B:**2(1997), 257–264.