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THE TOPOLOGY OF JULIA SETS FOR GEOMETRICALLY FINITE POLYNOMIALS**

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Abstract

By means of the Branner-Hubbard puzzle, the author studies the topology of filled-in Julia sets for geometrically finite polynomials, and proves a conjecture of C. McMullen and a conjecture of B. Branner and J. H. Hubbard partially.

Keywords Filled-in Julia set, Geometrically finite polynomial, Topology
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§1. Introduction

Let $R:\overline{\mathbb{C}}\to\overline{\mathbb{C}}$ be a rational map with degree $d\geq 2$. Denote by R^n the n-th iteration of R. The point z is called stable if there exists a neighborhood U of z such that $\{R^n|_U\}$ is a normal family. The stable set $F(R)$ is the set of stable points of R. Its complement, $J(R)$, is called the Julia set of R. The Julia set $J(R)$ is perfect, completely invariant and never empty[4].

The postcritical set P_R is the union of forward orbits of critical points of R. A rational map is called hyperbolic if $\overline{P}_R\cap J(R)=\emptyset$ and R is called geometrically finite if $\overline{P}_R\cap J(R)$ is a finite set.

C. McMullen conjectured that each component of the Julia set for a geometrically finite rational map is locally connected[2]. In [5], Tan Tei and Yin Yongcheng proved that each eventually periodic component of $J(R)$ for a geometrically finite rational map is locally connected. It remains to show that each wandering component is locally connected.

For a polynomial $f(z)$, the set $K_f=\{z\in\mathbb{C}\mid \text{the sequence } \{f^n(z)\} \text{ is bounded}\}$ is called the filled-in Julia set. Its complement $\mathbb{C}\setminus K_f=A_f(\infty)$ is the attracting basin of infinity. Then $\partial K_f=\partial A_f(\infty)=J(f)$. The component of K_f containing critical points is called critical component.

In [1], B. Branner and J. H. Hubbard conjectured that $J(f)$ is a Cantor set if and only if each critical component of K_f is not periodic. They proved this conjecture in degree 3.

By means of Branner-Hubbard puzzle, we prove the following statements.

Theorem A. *For a geometrically finite polynomial $f(z)$, each component of $J(f)$ is locally connected.*

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Theorem B. *For a geometrically finite polynomial $f(z)$, $J(f)$ is a Cantor set if and only if each critical component of K_f is not periodic.*

§2. Branner-Hubbard Puzzle and the Main Result

Let $f : \mathbb{C} \to \mathbb{C}$ be a geometrically finite polynomial of degree $d \geq 2$. It follows from the Bötkher theorem that there exists a conformal map $\Phi(z)$ which is defined throughout some neighborhood of infinity, with $\Phi(\infty) = \infty$, and which satisfies $\Phi \circ f \circ \Phi^{-1}(z) = z^d$.

We set $G(z) = \lim\limits_{n\to\infty} \log^+ | f^n(z) |/d^n$, where $\log^+(x) = \log(x)$ for $x \geq 1$ and $\log^+(x) = 0$ for $0 \leq x \leq 1$. The function $G(z)$ is continuous in $\mathbb{C}$ and satisfies that

(1) $G(f(z)) = d \cdot G(z)$, (2) $K_f = \{ z | G(z) = 0\}$,

(3) The critical points of G in $A_f(\infty) = \mathbb{C} \setminus K_f$ are the preimages of the critical points of f in $A_f(\infty)$.

Let $f(z)$ be a geometrically finite polynomial. Each critical component of K_f is eventually periodic and its forward orbit is finite. The Branner-Hubbard puzzle of f is constructed as follows. Choose a small number $r_0 > 0$ which is not a critical value of G, so that the region $G^{-1}(0, r_0]$ contains no critical value of f, each component of $G^{-1}[0, r_0]$ contains at most one critical component of K_f and any two components of K_f which belong to two disjoint forward orbits of critical components of K_f are contained in different components of $G^{-1}[0, r_0]$. Each locus $G^{-1}[0, r_0/d^k] = \{z \in \mathbb{C} \,|\, G(z) \leq r_0/d^k\}$ is the disjoint union of a finite number of closed topological disks and each of these closed disks will be called a puzzle piece P_k of depth k. Thus each point $x \in K_f$ determines a nested sequence $P_0(x) \supset P_1(x) \supset \cdots$. We look at the intermediate annuli $A_k(x) = \text{int } (P_k(x)) \setminus P_{k-1}(x)$.

All of the annuli $A_k(x)$ are non-degenerate with strictly positive modulus and the intersection $\bigcap\limits_k P_k(x) = K_f(x)$ is the component of K_f containing x. If $P_k(x)$ contains no critical points of f, then $f : A_k(x) \to A_{k-1}(f(x))$ is conformal and $\bmod A_k(x) = \bmod A_{k-1}(f(x))$; $\bmod A_k(x) \geq \frac{1}{2} \bmod A_{k-1}(f(x))$ otherwise. From a proposition in [1], $K_f(x) = \{x\}$ if and only if $\sum\limits_{k=0}^{\infty} \bmod A_k(x) = \infty$.

In order to estimate $\sum\limits_{k=0}^{\infty} \bmod A_k(x)$ for $x \in K_f$, we define its tableau $T(x)$; that is the two dimensional array of $P_{k,l} = f^l(P_k(x))$. The position (k, l) is called critical if $P_{k,l}(x)$ contains critical points of f. We call a tableau $T(x)$ of $x \in K_f$ recurrent if to the right of every position there is a critical position.

For a geometrically finite polynomial $f(z)$, we have the following tableau properties.

Lemma 2.1. (a) *If (k, l) is a critical position, so is (j, l) for all $0 \leq j \leq k$.*

(b) *If (k, l) and $(k - m, l + m)$ are critical, $(k + 1, l)$ and $(k - j, l + j)$ are not critical for $0 < j < m$, then $(k - m + 1, l + m)$ is not critical.*

(c) *A critical component $K_f(\omega)$ is a preimage of some periodic critical component or its tableau $T(\omega)$ is not recurrent.*

Proof. (a) and (c) are trivial.

(b) Let U be the critical puzzle piece of depth $k+1$ contained in $P_{k,l}(x)$. Then $f^m(U)$ is the critical puzzle piece of depth $k-m+1$ contained in $P_{k-m,l+m}$. If U contains n critical points (with multiplity) of f, from the Riemann-Hurewicz formula, $f^m : P_{k,l}(x) \to P_{k-m,l+m}(x)$

and $f^m : U \to f^m(U)$ are proper mappings with the same degree $n+1$. This implies that $f^m(U)$ and $P_{k-m+1,l+m}(x)$ are different puzzle pieces of depth $k-m+1$, that is, $(k-m+1, l+m)$ is not critical.

Property (a) Property (b)

Proposition 2.1. *Let $K_f(x)$ be a component which is not a preimage of some periodic critical component. Then $K_f(x) = \{x\}$.*

Proof. There exists an integer $n \geq 0$ such that $f^k(K_f(x))$ is not critical for all $k \geq n$. Suppose $f^k(K_f(x))$ is not critical for all $k \geq 0$. Otherwise, we replace $K_f(x)$ by $f^n(K_f(x))$.

If the tableau $T(x)$ is not recurrent, then there is a k_0 such that (k, l) is not critical for all $k \geq k_0$, $l \geq 0$. The mapping $f^{k-k_0} : A_k(x) \to A_{k_0}(f^{k-k_0}(x_0))$ is conformal and $\mod A_k(x) = \mod A_{k_0}(f^{k-k_0}(x)) \geq \alpha_{k_0}$, where $\alpha_{k_0} = \min\{\mod A_{k_0}\}$. Since there are only finitely many annuli A_{k_0} at level k_0, α_{k_0} is a positive number. Hence $\sum\limits_{k=0}^{\infty} \mod A_k(x) = \infty$ and $K_f(x) = \{x\}$.

If the tableau $T(x)$ is recurrent, then there is a first critical position (k, l_k) at level k. Since $K_f(x)$ is not a preimage of some critical component of K_f, there is an integer $m_k \geq k$ such that (m_k, l_k) is critical but $(m_k + 1, l_k)$ is not critical. From Lemma 2.1, the diagonal $\{(i,j) \,|\, i + j = m_k + l_k + 1 = n_k\}$ contains no critical position and $\{n_k\}$ is an increasing sequence. The mapping $f^{n_k} : A_{n_k}(x) \to A_0(f^{n_k}(x))$ is conformal, and $\mod A_{n_k}(x) = \mod A_0(f^{n_k}(x)) \geq \alpha_0 > 0$. Hence $\sum\limits_{k=0}^{\infty} \mod A_k(x) \geq \sum\limits_{k=0}^{\infty} \mod A_{n_k}(x) = \infty$. Hence $K_f(x) = \{x\}$.

We end the proof of this proposition.

or

$T(w)$ $T(w)$

Property (c)

The Recurrent Tableau $T(x)$

The following is the main result in this section.

Theorem 2.1. *For a geometrically finite polynomial $f(z)$ and $x \in K_f$, $K_f(x) = \{x\}$ if and only if $K_f(x)$ is not a preimage of some periodic critical component of K_f.*

Proof. For a periodic critical component $K_f(\omega)$, $g = f^k : P_k(\omega) \to P_0(\omega)$ is a polynomial-like mapping with degree $\deg(g|_{P_k(\omega)}) > 1$ in the sence of [3], where k is the period of $K_f(\omega)$. By the straightening theorem, g is hybrid equivalent to a polynomial of the same degree with connected Julia set[3]. The component $K_f(\omega)$ which is homeomorphic to K_g is not a point, and each preimage of $K_f(\omega)$ is not a point. From Proposition 2.2, $K_f(x) = \{x\}$ for $K_f(x)$ being not a preimage of some critical component.

The proof of the theorem is completed.

Now we can prove Theorems A and B.

Proof of Theorem A. By the theorem in [5], each eventually periodic component of $J(f)$ is locally connected. From Theorem 2.1, each wandering component of $J(f)$ is a point. Therefore, each component of $J(f)$ for a geometrically finite polynomial $f(z)$ is locally connected.

Proof of Theorem B. If $J(f)$ is a Cantor set, then $J(f) = K_f$ and each critical component of K_f is strictly preperiodic. On the other hand, if each critical component of K_f is not periodic, Theorem 2.1 says that $K_f(x) = \{x\}$ for all $x \in K_f$, that is, $J(f) = K_f$ is a Cantor set.

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