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STRONG SUBORDINATION OF SYMMETRIC DIRICHLET FORMS**

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Abstract

The author introduces a notion of subordination for symmetric Dirichlet forms and proves that the subordination is actually equivalent to the killing transformation by multiplicative functionals in the theory of symmetric Markov processes. This also gives a way to characterize bivariate smooth measures.

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§1. Introduction

Most definitions in this section will be taken from [3] and [2]. Let E be a Hausdorff topological space and $\mathcal{B}(E)$ the Borel σ-algebra on E, which is generated by all open subsets of E. Let m be a σ-finite measure on $(E, \mathcal{B}(E))$, and $L^2(m)$ the usual L^2-space on $(E, \mathcal{B}(E), m)$ with inner product denoted by $(\cdot,\cdot)$. A bilinear form $\mathcal{E}$, together with its domain $\mathcal{D} \subset L^2(m)$, is called a symmetric nonnegative definite form on $L^2(m)$ if

(1.1a) $\mathcal{D}$ is dense in $L^2(m)$;

(1.1b) $\mathcal{E}$ is symmetric, i.e., $\mathcal{E}(u,v) = \mathcal{E}(v,u)$ for $u, v \in \mathcal{D}$;

(1.1c) $\mathcal{E}$ is nonnegative definite, i.e., $\mathcal{E}(u,u) \geq 0$ for $u \in \mathcal{D}$.

For $q > 0$, let $\mathcal{E}_q(\cdot,\cdot) := \mathcal{E}(\cdot,\cdot) + q(\cdot,\cdot)$. Then $\mathcal{E}_q$ is an inner product on $\mathcal{D}$. We use the same notation for the norm $\mathcal{E}_q(u) := \mathcal{E}_q(u,u)^{\frac{1}{2}}$ for $u \in \mathcal{D}$, which is surely equivalent to $\mathcal{E}_1$. This class of equivalent norms is called $\mathcal{E}$-norm on $\mathcal{D}$. A symmetric nonnegative definite form $(\mathcal{E}, \mathcal{D})$ is called a Dirichlet form on $L^2(m)$ if, in addition, it satisfies

(1.1d) $\mathcal{D}$ is complete with respect to $\mathcal{E}$-norm;

(1.1e) $(\mathcal{E}, \mathcal{D})$ is Markovian, i.e., for any $u \in \mathcal{D}$, $0 \vee u \wedge 1 \in \mathcal{D}$ and $\mathcal{E}(0 \vee u \wedge 1, 0 \vee u \wedge 1) \leq \mathcal{E}(u,u)$.

The condition (1.1e) is equivalent to one seemingly stricter: if ϕ is a contraction on $\mathbb{R}$, namely, $|\phi(t)| \leq t$, $t \in \mathbb{R}$ and $|\phi(t) - \phi(s)| \leq |t-s|$, $t, s \in \mathbb{R}$, then for any $u \in \mathcal{D}$ we have $\phi(u) \in \mathcal{D}$ and $\mathcal{E}(\phi(u), \phi(u)) \leq \mathcal{E}(u,u)$. For example $|t|$ and $0 \vee t$ are contractions on $\mathbb{R}$. It is known that for a Dirichlet form $(\mathcal{E}, \mathcal{D})$ on $L^2(m)$ there exists a unique strongly continuous

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Markovian resolvent $(G_q)_{q>0}$ on $L^2(m)$ such that $G_q(L^2(m)) \subset \mathcal{D}$ and $\mathcal{E}_q(G_q f, u) = (f, u)$ for all $f \in L^2(m),\ u \in \mathcal{D}$ and $q > 0$.

Assume that $(\mathcal{E}, \mathcal{D})$ is a Dirichlet form on $L^2(m)$. We now define a capacity on E. For any open subset $G \subset E$ let $\mathcal{L}_G := \{u \in \mathcal{D} :\ u \geq 1 \text{ a.e. on } G\}$. Then there exists $h_G \in \mathcal{D}$ such that h_G minimizes $\mathcal{E}_1$ on $\mathcal{L}_G$ or $\mathcal{E}_1(h_G, h_G) = \inf\{\mathcal{E}_1(u,u) :\ u \in \mathcal{L}_G\}$. For $B \subset E$ define $C_1(B) := \inf\{\mathcal{E}_1(h_G, h_G) :\ G \text{ open and } B \subset G\}$. The set function C_1 is called $\mathcal{E}_1$-capacity on E. Obviously $C_1 \geq m$ on $\mathcal{B}(E)$. A subset $B \subset E$ is said to be $(\mathcal{E}$-)exceptional if $C_1(B) = 0$. An increasing sequence $\{E_n\}$ of compact subsets of E is an $\mathcal{E}$-nest if $\bigcap\limits_n E_n^c$ is $\mathcal{E}$-exceptional. A property of points on E holds quasi-everywhere (q.e. in abbrev.) if it holds off an exceptional set. For $B \subset E$ let $\mathcal{D}_B := \{u \in \mathcal{D} :\ u = 0 \text{ q.e. on } B^c\}$. Then an increasing sequence $\{E_n\}$ of compact subsets is $\mathcal{E}$-nest if and only if $\cup_n \mathcal{D}_{E_n}$ is dense in $\mathcal{D}$ with respect to $\mathcal{E}$-norm. A function f on E is called $(\mathcal{E}$-)quasi-continuous if there exists an $\mathcal{E}$-nest $\{E_n\}$ such that $f \in C(\{E_n\})$ where $C(\{E_n\}) := \{u :\ u \text{ is continuous on all } E_n\}$. We say that f has a quasi-continuous version if there exists a quasi-continuous function $\tilde{f}$ such that $f = \tilde{f}$ a.e. m, in which case the '$\tilde{\ }$' is always used to mark the version.

Definition 1.1. *A Dirichlet form* $(\mathcal{E}, \mathcal{D})$ *on* $L^2(m)$ *is quasi-regular* (*on* E) *if*

(1.2a) *there exists an* $\mathcal{E}$*-nest* $\{E_n\}$*;*

(1.2b) *each element in* $\mathcal{D}$ *has a quasi-continuous version;*

(1.2c) *there exist* $\{u_n :\ n \in \mathbb{N}\} \subset \mathcal{D}$ *and an exceptional set* $N \subset E$ *such that* $\{\tilde{u}_n :\ n \in \mathbb{N}\}$ *separates the points of* $E \setminus N$.

Recall the Fukushima's definition of regularity: $(\mathcal{E}, \mathcal{D})$ is regular (on E) if E is a locally compact metrizable space, m is a Radon measure and $\mathcal{D} \cap C_c(E)$ is dense in $C_c(E)$ with uniform norm, in $\mathcal{D}$ with $\mathcal{E}$-norm. Fukushima showed that every Dirichlet form has a regular representation and every regular Dirichlet form associates to a symmetric Hunt process. The quasi-regularity is certainly weaker than regularity. The recent works of Albeverio, Ma, Röckner, and others showed that a Dirichlet form on $L^2(m)$ is quasi-regular on E if and only if it associates a symmetric Borel right process X with state space E, namely, if (U^q) is the resolvent of X, then $U^q f$ is a quasi-continuous version of $G_q f$ for each $f \in \mathcal{B}(E) \cap L^2(m)$ and $q > 0$.

Though a large class of Dirichlet forms is accessible to quasi-regularity but not to regularity, (e.g., when E is an infinite dimemsional space,) it is also shown by Ma, Rockner, and others that each quasi-regular Dirichlet form has a regular homeomorphism and almost all results for regularity can be immediately transferred to those for quasi-regularity through this homeomorphism. Hence we will use the applicable results in [3] and [2] without further explanation.

In this article we are going to characterize the so-called killing transform in the theory of Markov processes in terms of the theory of Dirichlet forms. This problem is closely related to the perturbation theory of Dirichlet forms on which many authors (see [2] or [3] for bibliography) have worked. The notions and main results will be given in §2 while §3 will be dedicated to a characterization for bivariate Revuz measures.

Conventions. Given a class of functions $\mathcal{F}$, we use $b\mathcal{F}$ and $p\mathcal{F}$ to denote subclasses of bounded functions and nonnegative functions in $\mathcal{F}$ respectively.

§2. Subordination

In this section we fix $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ which is a symmetric Borel right process on $(E, \mathcal{B}(E))$ associated with a quasi-regular symmetric Dirichlet form $(\mathcal{E}^X, \mathcal{D}^X)$. A subset of E is exceptional if and only if it is m-polar in language of X; namely it can never be reached by (X_t) a.e. P^m. Let us now introduce the notion of subordination in Dirichlet forms.

Definition 2.1. *Let $(\mathcal{E}^1, \mathcal{D}^1)$ and $(\mathcal{E}^2, \mathcal{D}^2)$ be two Dirichlet forms on $L^2(m)$. The form $(\mathcal{E}^2, \mathcal{D}^2)$ is said to be subordinate to $(\mathcal{E}^1, \mathcal{D}^1)$, denoted by $(\mathcal{E}^2, \mathcal{D}^2) < (\mathcal{E}^1, \mathcal{D}^1)$, if $\mathcal{D}^2 \subset \mathcal{D}^1$ and $\mathcal{E}^1(u,v) \le \mathcal{E}^2(u,v)$ for $u, v \in p\mathcal{D}^2$, and strongly subordinate, denoted by $(\mathcal{E}^2, \mathcal{D}^2) \ll (\mathcal{E}^1, \mathcal{D}^1)$ if, in addition, $\mathcal{D}^2$ is dense in $\mathcal{D}^1$ with respect to $\mathcal{E}^1$-norm.*

Clearly the relation "$<$" is a partial order for Dirichlet forms on $L^2(m)$, but it is not so clear whether "$\ll$" is also a partial order. In order to give the definition of killing transform, we need to introduce multiplicative functionals for symmetric Markov processes.

Definition 2.2. *A family $M = (M_t)_{t \ge 0}$ of $[0,1]$-valued random variables on Ω is said to be a multiplicative functional of X if*

(i) *M is adapted to $(\mathcal{F}_t)$;*

(ii) *there exists a defining set $\Lambda \in \mathcal{F}$ and an exceptional (m-polar) set $N \subset E$ such that $P^x(\Lambda) = 1$ for all $x \in N^c$, $\theta_t \Lambda \subset \Lambda$ for all $t > 0$ and for each $\omega \in \Omega$, $M_{t+s}(\omega) = M_t(\omega) \cdot M_s(\theta_t \omega)$ for all $s, t > 0$.*

Let $E_M := \{x \in E : \ P^x(M_0 = 1) = 1\}$, which is the permanent set of M unique up to an m-polar set, and

$$V^q f(x) := P^x \left[\int_0^\infty e^{-qt} M_t f(X_t) dt \right], \ f \in b\mathcal{B}(E), \ q > 0, \ x \in E \setminus N. \tag{2.1}$$

Then there exists a right Markov process Y with state space E_M and resolvent $(V^q)_{q>0}$. We call Y the subprocess of X killed by M and M symmetric if (V^q) is symmetric. Let sMF(X) be the totality of the symmetric multiplicative functionals of X with E as their permanent set. A right process Y with state space $(E, \mathcal{B}(E))$ and resolvent (V^q) is said to be a subprocess of X, denoted by $Y \prec X$, if there exists $M \in$ sMF(X) with N as exceptional set such that (V^q) is given by (2.1). Define the bivariate Revuz measure of $M \in$ sMF(X) (relative to m) as

$$\nu_M(F) :=\uparrow \lim_{t \downarrow 0} \frac{1}{t} P^m \left[\int_0^t F(X_{s-}, X_s) d(-M_s) \right], \ F \in p\mathcal{B}(E \times E). \tag{2.2}$$

Then ν_M is symmetric, i.e., $\nu_M(dx \otimes dy) = \nu_M(dy \otimes dx)$. Let $\bar{\nu}_M := \nu_M(\cdot \otimes 1)$ be the marginal measure of ν_M.

Theorem 2.1. *If Y is a symmetric right process on $(E, \mathcal{B}(E))$ associated with its Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$ and $Y \prec X$, then $(\mathcal{E}^Y, \mathcal{D}^Y)$ is quasi-regular and strongly subordinate to $(\mathcal{E}^X, \mathcal{D}^X)$.*

Proof. It was shown in [4] that $(\mathcal{E}^Y, \mathcal{D}^Y)$ is given by

$$\begin{aligned} \mathcal{D}^Y &= \mathcal{D}^X \cap L^2(E, \bar{\nu}_M); \\ \mathcal{E}^Y(u,v) &= \mathcal{E}^X(u,v) + \nu_M(u \otimes v), \ u, v \in \mathcal{D}^Y. \end{aligned} \tag{2.3}$$

Hence $(\mathcal{E}^Y, \mathcal{D}^Y) < (\mathcal{E}^X, \mathcal{D}^X)$. We now need only to check that $(\mathcal{E}^Y, \mathcal{D}^Y)$ is quasi-regular and $\mathcal{D}^Y$ is dense in $\mathcal{D}^X$ with $\mathcal{E}^X$-norm. Firstly we claim that each $\mathcal{E}^X$-nest is an $\mathcal{E}^Y$-nest. In fact assume that $\{E_n\}$ is an $\mathcal{E}^X$-nest. Let $T_n := T_{E_n^c}$, the first exit time from E_n, $T := \lim_n T_n$ and

$$V_n^1 f := P^{\cdot}\left[\int_0^{T_n} e^{-t} M_t f(X_t)dt\right], \quad f \in p\mathcal{B}(E)\bigcap L^2(m).$$

By Lemma (IV.4.5) of [3] we have $V_n^1 f \longrightarrow V^1 f$ q.e. On the other hand $V_n^1 f$ is nothing but the orthogonal projection of $V^1 f$ on $\mathcal{D}_{E_n}^Y$ in the Hilbert space $(\mathcal{D}^Y, \mathcal{E}_1^Y)$. Hence

$$\mathcal{E}_1^Y(V^1 f - V_n^1 f, V^1 f - V_n^1 f) = \mathcal{E}_1^Y(V^1 f, V^1 f - V_n^1 f) = (f, V^1 f - V_n^1 f) \longrightarrow 0,$$

namely $\bigcup_n \mathcal{D}_{E_n}^Y$ is dense in $V^1(L^2(m))$ and therefore in $\mathcal{D}^Y$ with respect to $\mathcal{E}^Y$-norm. It follows that $(\mathcal{E}^Y, \mathcal{D}^Y)$ satisfies the quasi-regularity condition (1.2a) and (1.2b). Secondly for condition (1.2c) we choose $\{f_n : n \in \mathbb{N}\} \subset C_c(E)$ such that $\mathcal{B}(E) = \sigma(\{f_n : n \in \mathbb{N}\})$. Let N be an exceptional set of M. We set for $n \in \mathbb{N}$,

$$f_{n,m}(x) := \begin{cases} mV^m f_n(x), & x \in E \setminus N; \\ f_n(x), & x \in N. \end{cases}$$

Then $f_{n,m}(x) \longrightarrow f_n(x)$ as $m \longrightarrow \infty$ for all $x \in E$. Clearly $B_0 := \{f_{n,m} : n, m \in \mathbb{N}\}$ is a set of $\mathcal{E}^Y$-quasi-continuous functions in $\mathcal{D}^Y$ and $\sigma(B_0) = \mathcal{B}(E)$. Hence $(\mathcal{E}^Y, \mathcal{D}^Y)$ satisfies (1.2c) and is quasi-regular.

Finally we will show that $\mathcal{D}^Y$ is dense in $\mathcal{D}^X$. Fix $f \in \mathcal{B}(E)$ with $0 < f \leq 1$ and $m(f) < \infty$. By the generalized Revuz formula we have

$$\bar{\nu}_M(V^1 f) = \nu_M(V^1 f \otimes 1) = \left(f, P^{\cdot}\int_0^\infty d(-M_t)\right) \leq m(f) < \infty.$$

Set $F_n := \{x : V^1 f(x) \geq \frac{1}{n}\}$, which is q.e. finely closed for each n. Since $V^1 f > 0$ q.e., F_n increases to E q.e. Now for $g \in pb\mathcal{B}(E) \cap L^2(m)$, let $T_n := T_{F_n^c}$,

$$u_n := P^{\cdot}\int_0^{T_n} e^{-t} g(X_t)dt \quad \text{and} \quad u := P^{\cdot}\int_0^{\zeta} e^{-t} g(X_t)dt.$$

Then $u_n = 0$ q.e. on F_n^c and

$$\bar{\nu}_M(u_n^2) = \bar{\nu}_M(u_n \cdot 1_{F_n}) \leq ||g||_\infty^2 \bar{\nu}_M(nV^1 f) \leq n \cdot ||g||_\infty^2 \cdot m(f) < \infty.$$

Hence $u_n \in \mathcal{D}^X \cap L^2(E, \bar{\nu}_M) = \mathcal{D}^Y$. Now since u_n is $\mathcal{E}^X$-orthogonal to $u - u_n$, $\mathcal{E}_1^X(u - u_n, u - u_n) = (g, u - u_n)$. We have $u_n \longrightarrow u$ in $\mathcal{E}^X$-norm. It follows that $\mathcal{D}^Y$ is dense in $U^1(L^2(m))$ and hence in $\mathcal{D}^X$ with respect to $\mathcal{E}^X$-norm.

This theorem tells that $Y \prec X$ implies that $(\mathcal{E}^Y, \mathcal{D}^Y) \ll (\mathcal{E}^X, \mathcal{D}^X)$. In the rest of the section we aim to show the inverse. We will first give a lemma.

Lemma 2.1. *Assume that $(\mathcal{E}^1, \mathcal{D}^1)$ and $(\mathcal{E}^2, \mathcal{D}^2)$ are Dirichlet forms on $L^2(m)$.*

(a) *If $(\mathcal{E}^2, \mathcal{D}^2) < (\mathcal{E}^1, \mathcal{D}^1)$, then $\mathcal{E}^1$-norm is dominated by $\mathcal{E}^2$-norm on $\mathcal{D}^2$.*

(b) *If $(\mathcal{E}^2, \mathcal{D}^2) \ll (\mathcal{E}^1, \mathcal{D}^1)$, then each $\mathcal{E}^2$-nest is an $\mathcal{E}^1$-nest.*

Proof. (a) let I be the inclusion operator from $(\mathcal{D}^2, \mathcal{E}^2\text{-norm})$ to $(\mathcal{D}^1, \mathcal{E}^1\text{-norm})$; i.e., $I(u) = u$ for $u \in \mathcal{D}^2$. By a basic fact in functional analysis, it suffices to show that I is continuous. Let $\{u_n\} \subset \mathcal{D}^2$ such that $\mathcal{E}_1^2(u_n, u_n) \longrightarrow 0$. Set $u_n^+ := u_n \vee 0$ and $u_n^- := -u_n \wedge 0$. Then (1.1e) and subordination imply that

$$\mathcal{E}_1^1(u_n^\pm, u_n^\pm) \leq \mathcal{E}_1^2(u_n^\pm, u_n^\pm) \leq \mathcal{E}_1^2(u_n, u_n) \longrightarrow 0.$$

Hence $\mathcal{E}_1^1(u_n, u_n) \longrightarrow 0$, i.e., I is continuous.

(b) Let $\{E_n\}$ be an $\mathcal{E}^2$-nest. Then $\bigcup_n \mathcal{D}^2_{E_n}$ is dense in $\mathcal{D}^2$ with respect to $\mathcal{E}^2$-norm, and to $\mathcal{E}^1$-norm by (a). Since $\mathcal{D}^2$ is dense in $\mathcal{D}^1$ with $\mathcal{E}^1$-norm, $\bigcup_n \mathcal{D}^2_{E_n}$ is dense in $\mathcal{D}^1$ with $\mathcal{E}^1$-norm. It follows from the fact that $\mathcal{D}^2_{E_n} \subset \mathcal{D}^1_{E_n}$ that $\bigcup_n \mathcal{D}^1_{E_n}$ is dense in $\mathcal{D}^1$ with respect to $\mathcal{E}^1$-norm; i.e., $\{E_n\}$ is an $\mathcal{E}^1$-nest.

It follows from this lemma that each $\mathcal{E}^2$-exceptional set is $\mathcal{E}^1$-exceptional and the quasi-regularity of $(\mathcal{E}^2, \mathcal{D}^2)$ implies that of $(\mathcal{E}^1, \mathcal{D}^1)$. It is also easily seen from this lemma that "$\ll$" and "$\prec$" are partial orders in their respective defining classes.

Theorem 2.2. *Let Y be a symmetric Borel right process on state space $(E, \mathcal{B}(E))$ which is associated with a quasi-regular Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(m)$. Then $(\mathcal{E}^Y, \mathcal{D}^Y) \ll (\mathcal{E}^X, \mathcal{D}^X)$ implies that $Y \prec X$.*

Proof. Let (V^q) be the resolvent of Y. Since $\mathcal{D}^Y$ is dense in $\mathcal{D}^X$ with respect to $\mathcal{E}^X$-norm, there exists, for $f, g \in p\mathcal{B}(E) \cap L^2(m)$ and $q > 0$, $\{u_n\} \subset \mathcal{D}^Y$ such that $u_n \longrightarrow U^q f$ in $\mathcal{E}^X$-norm and also in L^2-norm. Then $u_n^+ \longrightarrow U^q f$ and

$$(u_n, g) = \mathcal{E}^Y_q(u_n^+, V^q g) \geq \mathcal{E}^X_q(u_n^+, V^q g).$$

Bringing n to infitity, we have

$$(U^q f, g) \geq \mathcal{E}^X_q(U^q f, V^q g) = (f, V^q g) = (V^q f, g).$$

It follows that $U^q f \geq V^q f$ a.e. m for $q > 0$ and $f \in p\mathcal{B}(E) \cap L^2(m)$.

Since $(\mathcal{E}^Y, \mathcal{D}^Y)$ is quasi-regular, there exists an $\mathcal{E}^Y$-nest $\{E_n\}$, which is an $\mathcal{E}^X$-nest by Lemma 2.1. Set $F := \cup_n E_n$. Then F is separable and both X and Y actually live on F. Let $\{q_i\}$ be a dense set in $]0, \infty[$ and f_j a dense set in $pC_c(F)$. By Lemma 2.1 the $\mathcal{E}^Y$-quasi-continuous function $V^{q_i} f_j$, $i, j \in \mathbb{N}$, is also $\mathcal{E}^X$-quasi-continuous. Then there exists an $\mathcal{E}^X$-nest $\{E'_n\}$ such that $\{U^{q_i} f_j, V^{q_i} f_j :\ i, j \in \mathbb{N}\} \subset C(\{E'_n\})$. Hence $U^{q_i} f_j(x) \geq V^{q_i} f_j(x)$ for all $x \in \cup_n E'_n$ and $i, j \in \mathbb{N}$. Let $N := (\bigcap_n E^c_n) \cup (\bigcap_n (E'_n)^c)$, which is $\mathcal{E}^X$-exceptional. By continuity of $U^{\cdot} f(x)$ and $V^{\cdot} f(x)$ for $f \in C_c(E)$ and $x \in E$, we find that $U^q f_j(x) \geq V^q f_j(x)$ for $q > 0$, $j \in \mathbb{N}$ and $x \in E \setminus N$. A similar limiting reasonning gives

$$U^q f(x) \geq V^q f(x), \quad \text{for } x \in E \setminus N, \quad f \in p\mathcal{B}(E), \quad q > 0.$$

By the results in Chapter III of [1], there exists $M \in \mathrm{sMF}(X)$ with N as its exceptional set such that (V^q) is given by (2.1).

Combining the theorems above, we show that the killing transform in the theory of Markov processes is essentially equivalent to strong subordination in the theory of Dirichlet forms. But we may ask what specific can be said about subordination.

Corollary 2.1. *Let Y be a symmetric Borel right process on $(E, \mathcal{B}(E))$ which is associated with a quasi-regular Dirichlet form $(\mathcal{E}^Y, \mathcal{D}^Y)$ on $L^2(m)$. If $(\mathcal{E}^Y, \mathcal{D}^Y) < (\mathcal{E}^X, \mathcal{D}^X)$, then there exists a measure σ on $E \times E$ such that $\mathcal{E}^Y(u, v) = \mathcal{E}^X(u, v) + \sigma(u \otimes v)$, $u, v \in \mathcal{D}^Y$.*

Proof. Let $\mathcal{D}$ be the closure of $\mathcal{D}^Y$ in $\mathcal{D}^X$ with respect to $\mathcal{E}^X$-norm. It is clear that $(\mathcal{E}^X, \mathcal{D})$ is also a Dirichlet form on $L^2(m)$ and quasi-regular by Lemma 2.1. Then $(\mathcal{E}^Y, \mathcal{D}^Y) \ll (\mathcal{E}^X, \mathcal{D})$. Using Theorem 2.2 and the Feynman-Kac formula (3.14) in [4] we have our conclusion proven.

An interesting question is under what condition subordination is equivalent to strong subordination.

§3. Characterization

In this section we assume X and $(\mathcal{E}^X, \mathcal{D}^X)$ as in §2. Given a σ-finite measure μ charging no exceptional sets, let

$$\begin{aligned} \mathcal{D}^{(\mu)} &:= \mathcal{D}^X \cap L^2(E, \mu), \\ \mathcal{E}^{(\mu)}(u,v) &:= \mathcal{E}^X(u,v) + \mu(\tilde{u}\tilde{v}),\ u,v \in \mathcal{D}^{(\mu)}. \end{aligned} \tag{3.1}$$

The pair $(\mathcal{E}^{(\mu)}, \mathcal{D}^{(\mu)})$ is called μ-perturbation (of $(\mathcal{E}^X, \mathcal{D}^X)$). The measure μ is said to be ($\mathcal{E}^X$-)smooth on E if it does not charge exceptional sets and there exists an $\mathcal{E}^X$-nest $\{E_n\}$ such that $\mu(E_n) < \infty$ for all n. It follows from (IV.4.c) of [3] that the measure μ is smooth if and only if the μ-perturbation is quasi-regular and strongly subordinate to $(\mathcal{E}^X, \mathcal{D}^X)$.

Let J be the canonical measure of X (or jumping measure of $(\mathcal{E}^X, \mathcal{D}^X)$. Given now a positive symmetric bivariate measure ν, denote by $\bar{\nu}$ the marginal measure of ν. We call ν smooth if $\bar{\nu}$ is smooth and $\nu|_{D^c} \leq J$, where $D := \{(x,x) :\ x \in E\}$. Let

$$\begin{aligned} \mathcal{D}^{(\nu)} &:= \mathcal{D}^X \cap L^2(E, \bar{\nu}); \\ \mathcal{E}^{(\nu)}(u,v) &:= \mathcal{E}^X(u,v) + \nu(\tilde{u} \otimes \tilde{v}),\ u,v \in \mathcal{D}^{(\nu)}. \end{aligned} \tag{3.2}$$

The pair $(\mathcal{E}^{(\nu)}, \mathcal{D}^{(\nu)})$ is called ν-perturbation (of $(\mathcal{E}^X, \mathcal{D}^X)$). Two Dirichlet forms on $L^2(m)$ are equivalent if they have a common domain and their corresponding norms are equivalent.

Lemma 3.1. *Let ν be a smooth symmetric bivariate measure. Then the ν-perturbation is equivalent to $\bar{\nu}$-perturbation. Consequently ν-perturbation is quasi-regular and strongly subordinate to $(\mathcal{E}^X, \mathcal{D}^X)$.*

Proof. The equivalence follows from the well-known Beurling-Deny's decomposition. The other results are easy to check.

Having all previous results at hands, the following characterization theorem is almost trivial.

Theorem 3.1. *A positive symmetric measure on $E \times E$ is a bivariate Revuz measure of some $M \in sMF(X)$ if and only if ν is smooth.*

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