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THE NEARLY KÄHLER STRUCTURE AND MINIMAL SURFACES IN S^6**

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Abstract

In terms of the almost complex connection and the unitary moving frame, a complex version on the theory of the nearly Kähler structure in S^6 is given. Under this framework, minimal surfaces in the nearly Kähler S^6 are studied. A complete classification for complete minimal surfaces in S^6 with constant Kähler angle and nonnegative curvature is given. Moreover, almost complex curves in S^6 are considered.

Keywords Nearly Kähler structure; Minimal surfaces; Kähler angle
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§1. Introduction

As is well known, there is not any almost complex structure in the $n(>2)$-dimensional, except 6-dimensional, sphere. By means of the multiplication in the Cayley number, an orthogonal almost complex structure J can be defined on the unit 6-sphere S^6. Indeed, this is a nearly Kähler structure on S^6 (see [8]). Much progress on the geometry of submanifolds in the naerly Kähler S^6 has been made recently[2,3,6,7,11]. In their research works, the nearly Kähler S^6 is considered as a real 6-manifold of constant sectional curvature with an almost complex structure satisfying the nearly Kählerian property. It seems to be uncomfortable for us to discuss some problems. A more natural idea is to introduce the almost complex affine connection on the nearly Kähler S^6 and to establish the corresponding structure equations. So far, we have not seen any statement in this direction, except in [1].

In this paper, we would like to give a complex version on the theory of the nearly Kähler structure in terms of the almost complex connection and structure equations. Under this framework, we then study minimal surfaces in the nearly Kähler S^6. The following results will be shown (see Theorem 4.1 and Theorem 5.1).

Theorem A. *Let M be a minimal surface in the nearly Kähler S^6 with constant Kähler angle α ($\neq 0,\pi$). If M is complete and has nonnegative Gauss curvature K, then either $K\equiv 1$ and M is totally geodesic; or $K\equiv 0$ and M is either totally real or superminimal.*

Theorem B. *The nearly Kähler S^6 has a Frenet frame along every almost complex curve on S^6.*

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The contents of this paper are arranged as follows. In §2 the complex version of structure equations of nearly Kähler manifolds will be given. The complex version of the theory on minimal surfaces in the nearly Kähler S^6 will be established in §3. In §4 Theorem A will be proved. Finally, in §5, we consider the almost complex curves in the nearly Kähler S^6 and the proof of Theorem B will be given.

§2. The Nearly Kähler Structure

Let N be an almost Hermitian manifold of real dimensional $2n$ with almost complex structure J and Hermitian metric $\langle\ ,\ \rangle$. We denote by ∇ the Levi-Civita connection with respect to the metric $\langle\ ,\ \rangle$ viewed as a Riemannian metric. If the almost complex structure satisfies

$$(\nabla_X J)Y + (\nabla_Y J)X = 0 \qquad \text{or} \quad \text{equivalently} \qquad (\nabla_X J)X = 0 \tag{2.1}$$

for any vector fields X and Y on N, then N is called a nearly Kähler manifold (Tachibana space; K-space). Clearly, all of the Kähler manifolds are nearly Kählerian, while any 4-dimensional nearly Kähler manifold is Kählerian[9]. Moreover, it is shown in [12] that there does not exist any dimensional, except 6-dimensional, non-Kählerian nearly Kähler manifold of constant holomorphic sectional curvature.

The most typical example of nearly Kähler manifolds is the nearly Kähler 6-dimensional unit sphere S^6. The explanation of the almost complex structure on S^6 may be found in [2] and [7].

We now introduce a (2,1)-tensor field ∇J on N defined by

$$(\nabla J)(X, Y) = (\nabla_Y J)X$$

for any vector fields $X, Y \in TN$. From (2.1) we have

$$(\nabla J)(X, Y) + (\nabla J)(Y, X) = 0, \tag{2.2}$$

$$(\nabla J)(JX, Y) + J((\nabla J)(X, Y)) = 0, \tag{2.3}$$

$$\begin{aligned} &\langle(\nabla J)(X, Y), Z\rangle + \langle(\nabla J)(X, Z), Y\rangle = 0,\\ &\langle(\nabla J)(X, Y), JZ\rangle + \langle(\nabla J)(X, Z), JY\rangle = 0. \end{aligned} \tag{2.4}$$

In general, the connection ∇ is not almost complex with respect to the almost complex structure J, namely, $\nabla J \neq 0$. However, from the connection ∇ we can construct an almost complex affine connection $\widetilde{\nabla}$ which is defined by[10]

$$\widetilde{\nabla}_X Y = \nabla_X Y - \tfrac{1}{4}(\nabla_{JY} J)X - \tfrac{1}{4}J((\nabla_X J)Y) = \nabla_X Y - \tfrac{1}{2}J((\nabla_X J)Y), \tag{2.5}$$

so that $\widetilde{\nabla} J = 0$. Moreover, the connection $\widetilde{\nabla}$ is compatible with the metric $\langle\ ,\ \rangle$ according to (2.2) and (2.4), i.e., $Z\langle X, Y\rangle = \langle\widetilde{\nabla}_Z X, Y\rangle + \langle X, \widetilde{\nabla}_Z Y\rangle$ for any $X, Y \in TN$.

We now choose a local field of orthonormal frames $\{e_A\}$ in N such that $Je_\alpha = e_{\alpha^*}$, where the convention of the range of indices is as follows:

$$\alpha, \beta, \gamma, \cdots = 1, \cdots, n; \qquad A, B, C, \cdots = 1, \cdots, n, 1^*, \cdots, n^*.$$

Let $\{\theta^A\}$ be the coframes dual to $\{e_A\}$. Then the almost complex structure can be written as $J = J^A_B \theta^B e_A$, where

$$(J^A_B) = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}, \qquad J^A_B + J^B_A = 0. \tag{2.6}$$

Let

$$(\nabla J)(e_A, e_B) = J^C_{AB} e_C, \tag{2.7}$$

where, and from now on, the Einstein sum convention is used. We then have

$$J^A_{BC}\theta^C = dJ^A_B - J^A_C\theta^C_B + J^C_B\theta^A_C, \tag{2.8}$$

$$d\theta^A = -\theta^A_B \wedge \theta^B, \quad \theta^A_B + \theta^B_A = 0,$$
$$d\theta^A_B = -\theta^A_C \wedge \theta^C_B + \Phi^A_B, \tag{2.9}$$

which are the structure equations related to ∇. From (2.7), (2.2)–(2.4) are reduced to

$$J^A_{BC} + J^A_{CB} = 0, \quad J^A_{BC} + J^B_{AC} = 0, \tag{2.10}$$

$$J^A_{CD}J^C_B + J^A_C J^C_{BD} = 0. \tag{2.11}$$

Let $\{\widetilde{\theta}^A_B\}$ be the connection 1-forms related to the almost complex affine connection $\widetilde{\nabla}$ described as in (2.5). Then, by the definition that $\widetilde{\nabla}_X e_B = \tilde{\theta}^A_B(X)e_A$, it follows from (2.5) and (2.7) that

$$\tilde{\theta}^A_B = \theta^A_B - \frac{1}{2}J^A_C J^C_{BD}\theta^D, \tag{2.12}$$

which together with $(2.9)_1$ yields

$$d\theta^A = -\tilde{\theta}^A_B \wedge \theta^B + \Psi^A, \quad \Psi^A = \frac{1}{2}J^A_D J^D_{BC}\theta^B \wedge \theta^C, \tag{2.13}$$

$$d\tilde{\theta}^A_B = -\tilde{\theta}^A_C \wedge \tilde{\theta}^C_B + \widetilde{\Phi}^A_B, \tag{2.14}$$

$$\widetilde{\Phi}^A_B = \frac{1}{2}(\Phi^A_B - J^A_C\Phi^C_D J^D_B) - \frac{1}{8}(J^A_{EC}J^E_{BD} - J^A_{ED}J^E_{BC})\theta^C \wedge \theta^D. \tag{2.15}$$

On putting

$$\begin{aligned} \omega^\alpha &= \theta^\alpha + \sqrt{-1}\theta^{\alpha^*}, & \Theta^\alpha &= \Psi^\alpha + \sqrt{-1}\Psi^{\alpha^*}, \\ \omega^\alpha_{\bar\beta} &= \tilde{\theta}^\alpha_\beta + \sqrt{-1}\tilde{\theta}^{\alpha^*}_\beta, & \Omega^\alpha_{\bar\beta} &= \widetilde{\Phi}^\alpha_\beta + \sqrt{-1}\widetilde{\Phi}^{\alpha^*}_\beta, \end{aligned} \tag{2.16}$$

we have from (2.13)–(2.15)

$$d\omega^\alpha = -\omega^\alpha_{\bar\beta} \wedge \omega^\beta + \Theta^\alpha, \quad \omega^\alpha_{\bar\beta} + \omega^{\bar\beta}_\alpha = 0,$$
$$d\omega^\alpha_{\bar\beta} = -\omega^\alpha_{\bar\gamma} \wedge \omega^\gamma_{\bar\beta} + \Omega^\alpha_{\bar\beta}, \tag{2.17}$$

where $\omega^{\bar\beta}_\alpha = \bar{\omega}^\beta_{\bar\alpha}$ as well as $\omega^{\bar\alpha} = \bar{\omega}^\alpha$ below.

By (2.6) and (2.8) we see easily that

$$J^\alpha_{\beta^*C} = J^{\alpha^*}_{\beta C}, \qquad J^{\alpha^*}_{\beta^*C} = -J^\alpha_{\beta C}. \tag{2.18}$$

Thus, if we set

$$P^\alpha_{\beta\gamma} = J^\alpha_{\beta\gamma} + \sqrt{-1}J^{\alpha^*}_{\beta\gamma}, \tag{2.19}$$

then the 2-form Θ^α defined as in (2.16) can be expressed as

$$\Theta^\alpha = \frac{\sqrt{-1}}{2}P^\alpha_{\beta\gamma}\omega^{\bar\beta} \wedge \omega^{\bar\gamma}. \tag{2.20}$$

In the case that $N = S^6$, i.e., the nearly Kähler unit 6-sphere, we have $n = 3$ and $\Phi^A_B = \theta^A \wedge \theta^B$. By [7] and (2.15), we have

$$\begin{aligned} \langle(\nabla J)(X,Y), (\nabla J)(Z,W)\rangle &= \langle X,Z\rangle\langle Y,W\rangle - \langle X,W\rangle\langle Z,Y\rangle \\ &\quad + \langle JX,Z\rangle\langle Y,JW\rangle - \langle JX,W\rangle\langle Y,JZ\rangle, \end{aligned} \tag{2.21}$$

which is equivalent to

$$J^A_{EB}J^E_{CD} = \delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD} + J^A_D J^C_B - J^A_C J^D_B, \tag{2.22}$$

$$\Omega^\alpha_{\bar\beta} = \frac{3}{4}\omega^\alpha \wedge \omega^{\bar\beta} - \frac{1}{4}\delta_{\alpha\beta}\sum_\gamma \omega^\gamma \wedge \omega^{\bar\gamma}. \tag{2.23}$$

Moreover, by (2.8), (2.18) and (2.22), we see that $P^\alpha_{\beta\gamma}$ is skew-symmetric with respect to any two indices and that

$$| P^3_{12} |^2 = 1. \tag{2.24}$$

By (2.4) we know easily that $\langle(\nabla J)(e_1, e_2), e_\lambda\rangle = 0$ for $\lambda = 1, 2, 1^*, 2^*$. By (2.21), we have $| (\nabla J)(e_1, e_2) |^2 = 1$. These allow us to be able to choose

$$e_3 = (\nabla J)(e_1, e_2), \quad e_{3^*} = Je_3, \tag{2.25}$$

so that $J^3_{12} = 1$ and $J^{3^*}_{12} = 0$. Hence,

$$P^3_{12} = \bar P^3_{12} = 1. \tag{2.26}$$

§3. Minimal Surfaces in the Nearly Kähler S^6

Let M be a connected oriented 2-dimensional Riemannian manifold and $x : M \to S^6$ be an isometric immersion of M into the nearly Kähler S^6. As is shown in [5], the Kähler angle α of x may be defined by $\cos\alpha = \langle Je_1, e_2\rangle$, where $\{e_1, e_2\}$ is an orthonormal basis on M. Thus, x is almost complex if and only if $\sin\alpha = 0$, while x is totally real if and only if $\cos\alpha = 0$. Assume that x is not almost complex. Then, in the open subset where $\sin\alpha \neq 0$, we may extend $\{e_i, Je_i\}$, $(i, j, \cdots = 1, 2)$, to a neighbourhood $U \subset S^6$. Let $\mathcal{D}_U = \mathrm{span}\{e_i, Je_i\}$ be a real 4-dimensional distribution on U. We now put

$$\begin{aligned} &(\sin\alpha)e_{1^*} = -(\cos\alpha)e_1 - Je_2, && (\sin\alpha)e_{2^*} = Je_1 - (\cos\alpha)e_2,\\ &\tilde e_1 = (\cos\tfrac{\alpha}{2})e_1 + (\sin\tfrac{\alpha}{2})e_{1^*}, && \tilde e_{1^*} = (\cos\tfrac{\alpha}{2})e_2 + (\sin\tfrac{\alpha}{2})e_{2^*},\\ &\tilde e_2 = (\sin\tfrac{\alpha}{2})e_1 - (\cos\tfrac{\alpha}{2})e_{1^*}, && \tilde e_{2^*} = -(\sin\tfrac{\alpha}{2})e_2 + (\cos\tfrac{\alpha}{2})e_{2^*}. \end{aligned}$$

Clearly, both $\{e_i, e_{i^*}\}$ and $\{\tilde e_i, \tilde e_{i^*}\}$ are the orthonormal bases of $\mathcal{D}_U$ and $J\tilde e_i = \tilde e_{i^*}$. By taking $\tilde e_3 = e_3 \in \mathcal{D}_U^\perp$, the orthogonal complement of $\mathcal{D}_U$, and $\tilde e_{3^*} = e_{3^*} = J\tilde e_3$, we obtain a unitary frame $\{\tilde e_\alpha, \tilde e_{\alpha^*}\}$ on $U \subset S^6$ where $\tilde e_3$ and $\tilde e_{3^*}$ are defined up to a unitary transformation. Particularly, if we choose $\tilde e_3 = (\nabla J)(\tilde e_1, \tilde e_2)$ and $\tilde e_{3^*} = J\tilde e_3$, then we obtain an adapted frame satisfying (2.26) on $U \subset S^6$.

On putting $\varphi = \theta^1 + \sqrt{-1}\theta^2$ where $\{\ \theta^\alpha\}$ is the coframe field dual to $\{e_\alpha\}$, the induced metric of M is written as $ds^2_M = \varphi\bar\varphi$. Thus, the structure equations of M are

$$d\varphi = \sqrt{-1}\theta^1_2 \wedge \varphi, \qquad d\theta^1_2 = \frac{\sqrt{-1}}{2}K\varphi \wedge \bar\varphi, \tag{3.1}$$

where θ^1_2 is the real connection 1-form and K is the Gauss curvature of M.

Let $\omega^\alpha = \tilde\theta^\alpha + \sqrt{-1}\tilde\theta^{\alpha^*}$ where $\{\tilde\theta^\alpha\}$ is the coframe field dual to $\{\tilde e_\alpha\}$ as described in §2. By restricting $\{\omega^\alpha\}$ to M, we have

$$\omega^1 = (\cos\tfrac{\alpha}{2})\varphi, \qquad \omega^2 = (\sin\tfrac{\alpha}{2})\bar\varphi, \qquad \omega^3 = 0, \tag{3.2}$$

which are equivalent to

$$(\cos\frac{\alpha}{2})\omega^1+(\sin\frac{\alpha}{2})\omega^{\bar 2}=\theta^1+\sqrt{-1}\theta^2,$$
$$(\sin\frac{\alpha}{2})\omega^1-(\cos\frac{\alpha}{2})\omega^{\bar 2}=\theta^{1^*}+\sqrt{-1}\theta^{2^*},$$
$$\omega^3=\theta^3+\sqrt{-1}\theta^{3^*}. \tag{3.3}$$

By taking the exterior derivatives of (3.2) and using (2.17) and (2.23), we get

$$\begin{aligned}\tfrac{1}{2}\{d\alpha-(\sin\alpha)(\omega_1^1+\omega_2^2)\}&=a\varphi+b\bar\varphi,\\ -\omega_{\bar 2}^1&=b\varphi+c\bar\varphi,\end{aligned} \tag{3.4}$$

$$\begin{aligned}-(\cos\tfrac{\alpha}{2})\omega_1^3+\tfrac{\sqrt{-1}}{4}(\sin\alpha)P_{12}^3\bar\varphi&=a'\varphi+b'\bar\varphi,\\ -(\sin\tfrac{\alpha}{2})\omega_{\bar 2}^3-\tfrac{\sqrt{-1}}{4}(\sin\alpha)P_{12}^3\varphi&=b'\varphi+c'\bar\varphi,\end{aligned} \tag{3.5}$$

where a,b,c,a',b' and c' are smooth complex valued functions defined locally on M. The condition for M to be minimal is that $b=b'\equiv 0$. As is proved in [5], when $x:M\to S^6$ is minimal which is not almost complex, the Kähler angle of x is a smooth function on M everywhere except at some isolated points.

For a minimal immersion $x:M\to S^6$ which is not almost complex, by taking exterior derivatives of (3.3) and making use of (2.17) and (3.2)–(3.5), we get

$$\theta_2^1=\sqrt{-1}\{(\cos\tfrac{\alpha}{2})^2\omega_1^1-(\sin\tfrac{\alpha}{2})^2\omega_{\bar 2}^{\bar 2}\}, \tag{3.6}$$

$$\begin{aligned}\theta_1^{1^*}+\sqrt{-1}\theta_2^{1^*}&=-(\bar a+c)\bar\varphi, &\quad \theta_1^{2^*}+\sqrt{-1}\theta_2^{2^*}&=-\sqrt{-1}(\bar a-c)\bar\varphi,\\ \theta_1^3+\sqrt{-1}\theta_2^3&=-(\bar a'+c')\bar\varphi, &\quad \theta_1^{3^*}+\sqrt{-1}\theta_2^{3^*}&=-\sqrt{-1}(\bar a'-c')\bar\varphi.\end{aligned} \tag{3.7}$$

By differentiating (3.6), we can get the Gauss equation of M:

$$K=1-2(|\,a\,|^2+|\,c\,|^2+|\,a'\,|^2+|\,c'\,|^2). \tag{3.8}$$

By taking exterior derivative of $(3.4)_1$ and using (2.17) and (3.1), we have

$$da+\sqrt{-1}a\theta_2^1=a_1\varphi+a_2\bar\varphi,$$
$$\text{with}\quad a_2=(\cot\alpha)\,|\,a\,|^2-(\tan\frac{\alpha}{2})\,|\,a'\,|^2+(\cot\frac{\alpha}{2})\,|\,c'\,|^2, \tag{3.9}$$
$$dc-3\sqrt{-1}c\theta_2^1=c_1\varphi+c_2\bar\varphi,$$
$$\text{with}\quad c_1=-(\cot\alpha)ac+\frac{\sqrt{-1}}{2}(P_{12}^3\bar a'-\bar P_{12}^3c'). \tag{3.10}$$

Let $\triangle$ be the Laplacian on M. By (3.2), (3.6)–(3.9), a standard computation gives

$$\tfrac{1}{4}\triangle\alpha=(\cot\alpha)\,|\,a\,|^2-(\tan\tfrac{\alpha}{2})\,|\,a'\,|^2+(\cot\tfrac{\alpha}{2})\,|\,c'\,|^2\,. \tag{3.11}$$

If we set $\theta_i^r=h_{ij}^r\theta^j$, $r=1^*,2^*,3^*,3$, and $H^r=h_{11}^r+\sqrt{-1}h_{12}^r$, then, by virtue of (3.7) and the minimality, we see easily that

$$Q=\sum_r(\bar H^r)^2\varphi^4=4(a\bar c+a'\bar c')\varphi^4 \tag{3.12}$$

is a holomorphic form of degree 4 on M as pointed out in [4]. Following R. Bryant[1], the minimal immersion $x:M\to S^6$ is called superminimal if the form Q defined by (3.12) vanishes identically.

Let T_xM and T_xS^6 be the tangent spaces to M and S^6, respectively, at a point $x\in M$. We denote by $\mathcal{H}_xM$ the almost complex subspace in T_xS^6 generated by T_xM, called the almost complex tangent space of M at x. Clearly, $\mathcal{H}_xM$ =span$\{e_1, e_2, e_{1^*}, e_{2^*}\}$ when x is not almost complex. Thus, T_xS^6 can be decomposed as

$$T_xS^6 = \mathcal{H}_xM \oplus \mathcal{H}_x^{\perp}M, \tag{3.13}$$

where $\mathcal{H}_x^{\perp}M$ =span$\{e_3, e_{3^*}\}$ is the orthogonal complement of $\mathcal{H}_xM$ in T_xS^6.

On the other hand, let $T_x^{(2)}M$ be the second osculating space of M at x(see [4]). Then, the first normal space of M at x, denoted by $\mathcal{N}_x^{(1)}M$, is the orthogonal complement of T_xM in $T_x^{(2)}M$, so that

$$T_xS^6 = T_x^{(2)}M \oplus \mathcal{N}_x^{(c)}M = T_xM \oplus \mathcal{N}_x^{(1)}M \oplus \mathcal{N}_x^{(c)}M, \tag{3.14}$$

where $\mathcal{N}_x^{(c)}M$ is the orthogonal complement of $T_x^{(2)}M$ in T_xS^6. By the minimality of M, the real dimension of $\mathcal{N}_x^{(1)}M$ is not larger than two and the second fundamental form of M with respect to any $\xi \in \mathcal{N}_x^{(c)}M$ vanishes identically.

Before concluding this section we show a topological property for minimal surfaces in the nearly Kähler S^6. Define globally a real canonical 1-form σ on M by

$$\sigma = \sqrt{-1}\sum_{\alpha=1}^{3}\omega_{\bar{\alpha}}^{\alpha}. \tag{3.15}$$

Proposition 3.1. *Let M be a minimal surface in the nearly Kähler S^6. Then the canonical* 1*-form σ on M defined by* (3.15) *is closed. Hence, σ defines a canonical cohomology class on M, namely, $[\sigma] \in H^1(M, R)$.*

Proof. If M is not almost complex, then, by (2.17), (3.4) and (3.5), a straightforward computation gives $d\sigma = 0$.

If M is almost complex, then, by using the Frenet frame along the complex curve $x : M \to S^6$ (see §5 below), a similar computation gives $d\sigma = 0$.

§4. Minimal Surfaces with Constant Kähler Angle

In this section we will consider minimal surfaces in the nearly Kähler S^6 with constant Kähler angle $\alpha \neq 0, \pi$. From $(3.4)_1$ and its complex conjugation it follows that α is constant if and only if $a \equiv 0$. Thus, in the case that α =const., the holomorphic form Q defined by (3.12) is reduced to

$$Q = 4a'\bar{c}'\varphi^4. \tag{4.1}$$

By introducing the local complex coordinate z on M such that $\varphi = \lambda dz$, we have from (4.1) the locally holomorphic function $F = 4a'\bar{c}'\lambda^4$ on M which satisfies

$$\triangle \log |F|^2 = 0 \tag{4.2}$$

for $F \neq 0$. On the other hand, in the isothermal net $K = -\triangle \log \lambda$. Hence we have

$$\triangle \log |a'\bar{c}'|^2 = 8K \tag{4.3}$$

for $F \neq 0$.

From (4.3) we have the following

Theorem 4.1. *Let $x : M \to S^6$ be a minimal immersion with constant Kähler angle α $(0 < \alpha < \pi)$. If M is complete and has nonnegative Gauss curvature K, then either $K \equiv 1$ and x is totally geodesic; or $K \equiv 0$ and x is either totally real or superminimal. In the last* *case, with respect to an adapted local field of unitary coframes* $\{\omega^\alpha\}$ *in the nearly Kähler* S^6*, the almost complex connection forms* $\{\omega^\alpha_{\bar\beta}\}$ *of* S^6 *restricted to* M *are given locally by*

$$\begin{pmatrix} 0 & -\frac{\sqrt{2}}{2}e^{i\rho}d\bar z & \frac{i}{2}(\sin\frac{\alpha}{2})dz \\ \frac{\sqrt{2}}{2}e^{-i\rho}dz & 0 & -\frac{i}{2}(\cos\frac{\alpha}{2})d\bar z \\ \frac{i}{2}(\sin\frac{\alpha}{2})d\bar z & -\frac{i}{2}(\cos\frac{\alpha}{2})dz & 0 \end{pmatrix}, \tag{4.4}$$

where $i=\sqrt{-1}$*,* ρ *is a real constant, and* z *is the local complex coordinate on* M.

Proof. The proof will be separated into some steps.

1st Step. From (4.3) it follows that

$$\triangle \mid a'\bar c' \mid^2 = 8 \mid a'\bar c' \mid^2 K + 4 \parallel \mathrm{grad} \mid a'\bar c' \mid\parallel^2, \tag{4.5}$$

which holds on M globally. Under the hypothesis of the theorem, we see from (4.5) that $\mid a'\bar c' \mid^2$ is a subharmonic function on M. By (3.8) and the assumption that $K \ge 0$ we have

$$\mid a'\bar c' \mid \le \frac{1}{2}(\mid a' \mid^2 + \mid c' \mid^2) \le \frac{1}{4}(1-K) \le \frac{1}{4},$$

namely, $\mid a'\bar c' \mid^2$ is bounded above on M. Then, by Liouville's theorem, $\mid a'\bar c' \mid^2$ is a constant, so that either $\mid a'\bar c' \mid^2 = 0$ or, by (4.3), $K \equiv 0$ on M.

2nd Step. We will prove that if $K \ne 0$ on M, then $K \equiv 1$ and x is totally geodesic.

Since $a = 0$, (3.11) gives

$$(\tan\tfrac{\alpha}{2}) \mid a' \mid^2 = (\cot\tfrac{\alpha}{2}) \mid c' \mid^2 . \tag{4.6}$$

Thus, in the case that $\mid a'\bar c' \mid^2 = 0$, (4.6) deduces that

$$a' = 0, \qquad c' = 0. \tag{4.7}$$

We now consider the following cubic form

$$P = \omega^1\omega^{\bar 1}_2\omega^{\bar 2} = -\frac{1}{2}(\sin\alpha)\bar c\varphi^3 = -\frac{1}{2}(\sin\alpha)\bar c\lambda^3 dz^3. \tag{4.8}$$

By vertue of (2.17), (3.4), (3.5), (3.10) and (4.5), we see easily that the cubic form P is holomorphic on M, as shown similarly in [5]. Then, $\bar c\lambda^3$ is a holomorphic function on M. In the same manner as in the first step, we then have

$$\triangle \mid c \mid^2 = 6 \mid c \mid^2 K + 4 \parallel \mathrm{grad} \mid c \mid\parallel^2 . \tag{4.9}$$

Since $K \ne 0$, by using Liouville's theorem again, we obtain from (4.9) that $c \equiv 0$, so that $a = c = a' = c' = 0$ on M identically, i.e., x is totally geodesic and $K \equiv 1$ according to (3.8).

3rd Step. We now consider the case that $K \equiv 0$ on M. By the first step, we have seen that $\mid a'\bar c' \mid^2$ is a constant, which together with (4.6) implies that both $\mid a' \mid^2$ and $\mid c' \mid^2$ are constant. Moreover, both $\mid a' \mid^2$ and $\mid c' \mid^2$ are either zero or nonzero simultaneously. In such a case we see from (3.8) that

$$\mid c \mid^2 = \frac{1}{2} - \mid a' \mid^2 - \mid c' \mid^2, \tag{4.10}$$

which is also a constant. In the following we will consider two cases separately.

4th Step. Suppose that $\mid a'\bar c' \mid^2 \ne 0$ on M. From (3.7), (3.13) and (3.14) we see that $\mathcal{N}^{(1)}_x M \cap \mathcal{H}^\perp_x M \ne \emptyset$ for any point $x \in M$. In fact, if $\mathcal{N}^{(1)}_x M \cap \mathcal{H}^\perp_x M = \emptyset$, then $\mathcal{H}^\perp_x M \subseteq \mathcal{N}^{(c)}_x M$, which implies that $\theta^3_j = \theta^{3^*}_j = 0$ for $j = 1, 2$, namely, $a' = c' = 0$ according to (3.7). This is impossible. Hence, we consider only two cases as follows.

Case (i) $\mathcal{N}_x^{(1)}M \cap \mathcal{H}_x M = \emptyset$ for a point $x \in M$. In such a case, we have $\mathcal{N}_x^{(1)}M \subseteq \mathcal{H}_x^{\perp}M$, which implies that $\theta_j^{1^*} = \theta_j^{2^*} = 0$ for $j = 1, 2$, namely, $c = 0$ at the point x. Since $|\, c\,|^2$ is constant, $c \equiv 0$ on M. We now have from (3.10) that $P_{12}^3 \bar{a}' = \bar{P}_{12}^3 c'$. By choosing suitably $\{e_3, e_{3^*} = Je_3\}$ such that (2.26) holds, we have $\bar{a}' = c'$, so that $|\, a'\,|^2 = |\, c'\,|^2 \neq 0$. From (4.4) it follows that $\tan\frac{\alpha}{2} = \cot\frac{\alpha}{2}$, namely, $\alpha = \pi/2$ and $x : M \to S^6$ is totally real.

Case (ii) $\mathcal{N}_x^{(1)}M \cap \mathcal{H}_x M \neq \emptyset$ for a point $x \in M$. By taking $\xi \in \mathcal{N}_x^{(1)}M \cap \mathcal{H}_x^{\perp}M$ and $\eta \in \mathcal{N}_x^{(1)}M \cap \mathcal{H}_x M$, we have $\mathcal{N}_x^{(1)}M = \mathrm{span}\{\xi, \eta\}$ because $\langle \xi, \eta \rangle = 0$. We now choose $e_3 = \xi / \|\, \xi\,\|$ and $e_{3^*} = Je_3$. Since $\mathcal{H}_x^{\perp}M$ is almost complex, $e_{3^*} = J\xi / \|\, \xi\,\|$ lies in $\mathcal{H}_x^{\perp}M$, so that $\langle e_{3^*}, \eta \rangle = 0$. Clearly, $\langle e_{3^*}, \xi \rangle = 0$. Thus, $e_{3^*} \notin \mathcal{N}_x^{(1)}M$, i.e., $e_{3^*} \in \mathcal{N}_x^{(c)}M$. In such a case, we have $\theta_1^{3^*} = \theta_2^{3^*} = 0$, so that $\bar{a}' = c'$ according to (3.7). In the same manner, from (4.4) it follows that $\alpha = \pi/2$ and the immersion x is totally real.

5th Step. Suppose that $|\, a'\bar{c}'\,|^2 = 0$ on M. It means that $a' = c' = 0$ on M identically. Then, the form Q defined by (3.12) vanishes on M, namely, $x : M \to S^6$ is superminimal. From (4.8) it follows that $|\, c\,|^2 = 1/2$, which together with (3.10) yields

$$dc - 3\sqrt{-1}c\theta_2^1 = 0. \tag{4.11}$$

Since $K \equiv 0$, we can choose the local complex coordinate z on M such that $\varphi = dz$, so that $\theta_2^1 = 0$. Thus, (4.11) implies that

$$c = \frac{\sqrt{2}}{2} e^{\sqrt{-1}\rho} \quad \text{for some real constant } \rho. \tag{4.12}$$

$(3{,}4)_2$ and (4.12) yield that

$$\omega_2^1 = -\frac{\sqrt{2}}{2} e^{\sqrt{-1}\rho} d\bar{z}. \tag{4.13}$$

By choosing $\{e_3, e_{3^*} = Je_3\}$ such that (2.26) holds, we have from (3.5)

$$\omega_1^3 = \frac{\sqrt{-1}}{2}(\sin\frac{\alpha}{2})d\bar{z}, \quad \omega_2^3 = -\frac{\sqrt{-1}}{2}(\cos\frac{\alpha}{2})dz. \tag{4.14}$$

From $(3.4)_1$ and (3.6) it follows that

$$\omega_1^1 = \omega_2^2 = 0. \tag{4.15}$$

By taking exerior derivatives of (4.14) and using the structure equations, we can find that $\omega_3^2 \wedge dz = 0$ and $\omega_3^3 \wedge d\bar{z} = 0$, which imply $\omega_3^3 = 0$. (4.4) now follows directly from (4.12)–(4.15).

Hence, the theorem is proved completely.

Remark 4.1. The partial conclusion of this theorem was shown also by X. Li[11] in a different way, which is a generalization of [6], where M is compact. (4.4) gives an example of flat superminimal surfaces in the nearly Kähler S^6 with constant Kähler angle $\alpha \neq 0, \pi$.

If M is a topological 2-sphere, then the forms (4.1) and (4.8) vanish on M identically. Hence, we have directly

Corollary 4.1. *If $x : S^2 \to S^6$ is a minimal immersion with constant Kähler angle α $(0 < \alpha < \pi)$, then x is totally geodesic.*

Remark 4.2. This corollary was proved in [3] by a different approach.

§5. Almost Complex Curves in the Nearly Kähler S^6

In this section We consider the case that $x : M \to S^6$ is almost complex, i.e., its differential

dx is complex linear. Thus, x is necessarily a branched minimal immersion and is called an almost complx curve.

Let $\{u_\alpha\}$ be a local field of unitary frames in S^6 and $\widetilde{\nabla}$ be the almost complex affine connection of S^6 as mentioned in §2. The covariant differential of u_α is given by $\widetilde{\nabla} u_\alpha = \omega_{\bar\alpha}^{\beta} u_\beta$. A unitary frame $\{u_\alpha\}$ of S^6 along an almost complex curve $x : M \to S^6$ is called a Frenet frame if, at each point of M, u_1 is tangent to M and

$$\widetilde{\nabla} u_\alpha = \omega_{\bar\alpha}^{\alpha-1} u_{\alpha-1} + \omega_{\bar\alpha}^{\alpha} u_\alpha + \omega_{\bar\alpha}^{\alpha+1} u_{\alpha+1} \quad \text{(indices do not sum up)}, \tag{5.1}$$

where $\omega_{\bar\alpha}^{\alpha+1}$ is a (1,0)-form and $\omega_{\bar 1}^{0} = \omega_{\bar 3}^{4} = 0$ (see [13]). We have the following

Theorem 5.1. *The nearly Kähler S^6 has a Frenet frame along every almost complex curve $x : M \to S^6$.*

Proof. We begin by choosing u_1 tangent to M and completing to a local unitary frame $\{u_1, u_2, u_3\}$. Then we have

$$\omega^1 = \varphi = \lambda dz, \quad \omega^2 = \omega^3 = 0 \tag{5.2}$$

along x, where z is the local complex coordinate on M.

By taking exterior derivative of $(5.2)_2$ and using the structure equations, we have

$$\omega_{\bar 1}^{r} = h^r dz \quad (r, s = 2, 3) \tag{5.3}$$

for some complex valued functions h^r on M. By virtue of (2.17) and (5.2), the differentiation of (5.3) gives

$$\Big(dh^r - h^r \omega_{\bar 1}^{1} + \sum_s h^s \omega_{\bar s}^{r}\Big) \wedge dz = 0.$$

The expression in parenthesis is therefore a multiple of dz, which means that functions $h^r(z)$ satisfy locally the following differential equation system

$$\frac{\partial h^r}{\partial \bar z} = \sum_s a_s^r(z) h^s(z) \quad (r, s = 2, 3), \tag{5.4}$$

where a_s^r are complex valued functions. By a theorem of [4], (5.4) shows that either $h^r \equiv 0$ or h^r have only isolated zeroes. If $h^r \equiv 0$ for all r, then $\{u_\alpha\}$ is trivially a Frenet frame; if not we can make a unitary change of u_2 and u_3 so that $\widetilde{\nabla} u_1 = \omega_{\bar 1}^{1} u_1 + \omega_{\bar 1}^{2} u_2$, where $\omega_{\bar 1}^{2}$ is a (1,0)-form. Now, since $\omega_{\bar 1}^{3} = 0$, (2.17) yields $\omega_{\bar 2}^{3} \wedge \omega_{\bar 1}^{2} = 0$, which implies that $\omega_{\bar 2}^{3}$ is also a (1,0)-form. Hence, (5.1) holds and the theorem is proved.

Remark 5.1. Another development of the theory of almost complex curves in S^6 is given in [2].

By using a Frenet frame for S^6 along an almost complex curve $x : M \to S^6$, we have

$$\omega_{\bar 1}^{2} = p\varphi, \qquad \omega_{\bar 1}^{3} = 0, \qquad \omega_{\bar 2}^{3} = q\varphi \tag{5.5}$$

for some complex valued functions p and q on M. By taking exterior derivative of $(5.2)_1$ and using (5.5) and (3.1), we have $\omega_{\bar 1}^{1} = -\sqrt{-1}\theta_2^1$ and

$$K = 1 - 2 \mid p \mid^2 . \tag{5.6}$$

From (5.5) and (5.6) we can find

$$\frac{1}{4} \triangle \log \mid p \mid^2 = \frac{3}{2}\Big(K - \frac{1}{6}\Big) + \mid q \mid^2, \tag{5.7}$$

for $p \neq 0$. Similarly, we have

$$\frac{1}{4}\triangle \log \mid q \mid^2 = \frac{1}{2} - 2 \mid q \mid^2 \tag{5.8}$$

for $q \neq 0$.

From (5.7) and (5.8) we can obtain easily the following result proved by Dillen Verstraeien Vraneken[7] in different way.

Proposition 5.1. *Let M be a compact surface and $x : M \to S^6$ be an almost complex curve with the Gauss curvature K related to the induced metric.*

(i) *If $K \geq 1/6$, then $K \equiv 1/6$ or $K \equiv 1$ on M;*

(ii) *If $0 \leq K \leq 1/6$, then $K \equiv 0$ or $K \equiv 1/6$ on M.*

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