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EXISTENCE FOR A FREE BOUNDARY PROBLEM IN GROUND FREEZING**

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Abstract

This paper deals with a free boundary problem that arises in ground freezing, where both the heat conduct and the mass transfer are taken into account. The authors obtain the local and global existence for the problem under some assumptions.

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§1. Introduction

The ground freezing process in cold regions can be described mathematically as a Stefan-type free boundary problem with both heat conduct and mass transfer in the porous media. The relevance of the problem with high technology engineering and human life has been stressed in a number of international conferences[1]. Here we are interested in the follwoing mathematical model, which is quoted from [3].

Let x denote the downwards-directed coordinate. Suppose the freezing front is given by an unknown function $x = \Gamma(t)$, the region $0 < x < \Gamma(t)$ is totally frozen, and no ice is present in the region $x > \Gamma(t)$. In the frozen region the temperature θ satisfies the heat equation

$$\rho_f c_f \frac{\partial \theta}{\partial t} = k_f \frac{\partial^2 \theta}{\partial x^2} \quad \text{in } \Omega^- = \{0 < x < \Gamma(t), t > 0\}, \tag{1.1}$$

where ρ, c, k denote the density, the specific heat, and the conductivity of the porous media, respectively, the suffix f refers to the frozen soils. In the following the suffix u will refer to unfrozen soils, i to pure ice, and w to pure water.

In the unfrozen region the movement of water leads to a convective term in the heat conduct equation

$$\rho_u c_u \frac{\partial \theta}{\partial t} = k_u \frac{\partial^2 \theta}{\partial x^2} - \frac{\partial}{\partial x}(\rho_w c_w \theta v) \ \text{ in } \Omega^+ = \{x > \Gamma(t), t > 0\}, \tag{1.2}$$

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where $v = v(x,t)$ is the volumetric velocity of the water which is usually small enough so that the Darcy's law holds,

$$v = -\frac{D\rho_w g_0}{\mu}\frac{\partial h}{\partial x}, \tag{1.3}$$

where $h = x + p_w/\rho_w g_0$ is the hydraulic head, p, g_0, D and μ stand for the pressure, the gravity, the soil permeability, and the viscosity, respectively. Equation (1.2) is accompanied with the mass conservation law

$$\frac{\partial(\varepsilon\rho_w)}{\partial t} + \frac{\partial(\rho_w v)}{\partial x} = 0, \tag{1.4}$$

where ε is the porosity. Moreover, the pressure p_w and the density ρ_w are assumed to have the relation

$$\rho = \rho_0 + \gamma p. \tag{1.5}$$

In the following we will denote $\rho = \rho_w$ and $p = p_w$, and suppose the parameters D, μ, ε, γ are positive constants, and $\gamma > 0$ is small. For simplicity we will also suppose ρ_f, c_f, k_f, ρ_u, c_u, k_u are constants. With these assumptions we get from (1.3)–(1.5) the equation for $p(x,t)$:

$$p_t = a_0 p_{xx} + c_0 \rho p_x \quad \text{in } \Omega^+, \tag{1.6}$$

where $a_0 = \frac{D\rho_0}{\mu\gamma\varepsilon}$, $c_0 = \frac{2Dg}{\mu\varepsilon}$.

On the free boundary $F = \{x = \Gamma(t)\}$ we have

$$\theta(\Gamma(t) - 0, t) = \theta(\Gamma(t) + 0, t) = 0. \tag{1.7}$$

Taking into account the volumetric velocity v on the freezing front F, the classical Stefan condition is replaced by

$$L\rho_i(\varepsilon\Gamma'(t) - v) = k_f\theta_x^- - k_u\theta_x^+, \tag{1.8}$$

where L is the latent heat. On the freezing front $F = \{x = \Gamma(t)\}$ the volumetric velocity v is determined via $\Gamma, \Gamma'(= d\Gamma/dt)$ and p by

$$v = \widetilde{g}(\Gamma', \Gamma, p, t), \tag{1.9}$$

where $\widetilde{g}$ is a given function. From (1.3), (1.5) and (1.9) we get

$$p_x(\Gamma(t) + 0, t) = g(\Gamma', \Gamma, p, t) =: -\frac{g_0}{\rho_0}\rho^2 - \frac{\mu\rho}{D\rho_0}\widetilde{g}. \tag{1.10}$$

The initial-boundary conditions are

$$\begin{cases} \theta(0,t) = \psi_0(t) < 0, \\ \theta(x,0) = \varphi(x), \\ p(x,0) = \eta_0(x). \end{cases} \tag{1.11}$$

Many efforts have been devoted to the above problem in experimental and numerical studies[1]. But theoretical analysis has been very limited because of the convective term $\frac{\partial}{\partial x}(\theta c\rho v)$ in the equation (1.2). If this term is neglected, existence and uniqueness have been proved by reducing the problem to a variational inequality, with the equation (1.6) linearized and (1.9) replaced by $p(\Gamma(t) + 0, t)$ =const.(see [8]). Guan and Shu[5] has also studied the above problem and obtained the local existence and continuous dependence for the problem, also neglecting the convective term in (1.2).

In this paper we discuss the local existence and global existence of solutions to the above problem. In Section 2 we present some preliminaries. In Sections 3 and 4 we prove the local and the global existence, respectively.

§2. Preliminaries

In this paper we will make use of the Schauder estimates repeatedly. For our purpose we state them as follows.

Let $\Omega = \{\Gamma(t) < x < X_0,\ 0 < t < T\}$, where $\Gamma(t) \ge 0$ and $\Gamma(t) \in C^\beta(0,T]$ with $\beta > \frac{1}{2}$. Denote $\|u\|_{0,\Omega} = \sup\limits_{\Omega} |u(x,t)|$,

$$\|u\|_{\alpha,\Omega} = \|u\|_{0,\Omega} + \sup_{\Omega} \frac{|u(x,t) - u(y,s)|}{((x-y)^2 + |t-s|)^{\alpha/2}},$$

$$\|u\|_{1+\alpha,\Omega} = \|u\|_{0,\Omega} + \|u_x\|_{0,\Omega} + \|u_x\|_{\alpha,\Omega} + \sup_{\Omega} \frac{|u(x,t) - u(x,s)|}{|t-s|^{(1+\alpha)/2}},$$

and

$$\|u\|_{2+\alpha,\Omega} = \sum_{i+2j\le 2} \|D_x^i D_t^j u\|_{0,\Omega} + \sum_{i+2j=2} \|D_x^i D_t^j u\|_{\alpha,\Omega},$$

where $0 < \alpha < 1$. For any $0 \le a_0 < a_1 < a_2$, the following interpolation inequality is well known:

$$\|u\|_{a_1,\Omega} \le C \|u\|_{a_0,\Omega}^{\gamma} \|u\|_{a_2,\Omega}^{1-\gamma}, \tag{2.1}$$

where $a_1 = a_0\gamma + a_2(1-\gamma)$, C depends only on a_0, a_1, a_2, β and $|\Gamma|_\beta$. The suffix Ω will be omitted if no confusion arises.

For any integer k and $0 < \alpha < 1$, we will also denote

$$\|u\|_{k+\alpha,\Omega}^{(b)} = \sup_{\delta>0} \delta^{k+\alpha+b} \|u\|_{k+\alpha,\Omega_\delta},$$

where $\Omega_\delta = \{P = (x,t),\ \mathrm{dist}(P,F) > \delta\}$. Note that the definition for Ω_δ here is a little different from the usual one, because we are only concerned with the behaviour of the free boundary F of the freezing problem. Noticing that $\beta > 1/2$, we have

$$\|u\|_b \le C \|u\|_a^{(-b)} \tag{2.2}$$

for any $a \ge b > 0$, b being not an integer, where C depends only on a, b, β, and $|\Gamma|_\beta$ (see Lemma 2.3 in [10]). (Lemma 2.3 holds if the parabolic boundary is Lipschitz in spatial variables and C^β in time for $\beta \ge 1/2$). From (2.1) it follows that

$$\|u\|_{a_1,\Omega}^{(b_1)} \le C \left(\|u\|_{a_0,\Omega}^{(b_0)}\right)^{\gamma} \left(\|u\|_{a_2,\Omega}^{(b_2)}\right)^{1-\gamma}, \tag{2.3}$$

provided $b_1 = b_0\gamma + b_2(1-\gamma)$ and $a_1 = a_0\gamma + a_2(1-\gamma)$.

If u is a function of single variable, we will denote $|u|_0 = \sup |u(x)|$,

$$|u|_{k+\alpha} = \sum_{0\le j\le k} \Big|\frac{d^j u}{dx^j}\Big|_0 + \sup \Big|\frac{d^k u}{dx^k}(x) - \frac{d^k u}{dx^k}(y)\Big| / |x-y|^\alpha.$$

Let $u(x,t)$ be the solution of the problem

$$\begin{cases} u_t - u_{xx} = f(x,t) & \text{in } \ \Omega = \{\Gamma(t) < x < X_0, 0 < t < T\}, \\ u(x,t) = \psi(t) & \text{on } \ F = \{x = \Gamma(t)\}, \\ u(X_0,t) = \eta(t), & u(x,0) = \varphi(x). \end{cases} \tag{2.4}$$

Since we are interested in the behaviour of the solution $u(x,t)$ near F, we always suppose η and φ are sufficiently smooth.

Lemma 2.1. *Suppos $\varphi(x) \in C^{2+\alpha}$, $\eta(t) \in C^{1+\alpha/2}$, and $\eta_t(0) = \varphi_{xx}(X_0) + f(X_0, 0)$, where $\alpha \in (0,1)$. Suppose $u(x,t)$ is the solution to the problem (2.4). We have*

(i) (*Interior Schauder estimates*).

$$\|u\|^{(0)}_{2+\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{2+\alpha} + |\eta|_{1+\alpha/2} + \|f\|^{(2)}_{\alpha,\Omega}), \tag{2.5}$$

where C depends only on α, X_0, T, and $\inf_t(X_0 - \Gamma(t))$.

(2) (*Intermediate Schauder estimates*). *If moreover $\psi(t) \in C^{(1+\alpha)/2}[0,T]$ and $\Gamma(t) \in C^\beta$ with $\beta > \frac{1+\alpha}{2}$, then*

$$\|u\|^{(-1-\alpha)}_{2+\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{2+\alpha} + |\eta|_{1+\alpha/2} + |\psi|_{(1+\alpha)/2} + \|f\|^{(1-\alpha)}_{\alpha,\Omega}), \tag{2.6}$$

$$\|u\|^{(-\alpha)}_{2+\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{2+\alpha} + |\eta|_{1+\alpha/2} + |\psi|_{\alpha/2} + \|f\|^{(2-\alpha)}_{\alpha,\Omega}), \tag{2.7}$$

where C depends only on $|\Gamma|_{(1+\alpha)/2}, X_0, T$, α and $\inf(X_0 - \Gamma(t))$.

(3) *If $\psi(t) \in C^{(1+\alpha)/2}[0,T]$ and $\Gamma(t) \equiv 0$, we have*

$$\|u\|_{1+\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{1+\alpha} + |\eta|_{(1+\alpha)/2} + |\psi|_{(1+\alpha)/2} + \|f\|^{(1-\alpha)}_{0,\Omega}). \tag{2.8}$$

If $\psi(t) \in C^{\alpha/2}[0,T]$ and $\Gamma(t) \equiv 0$, we have

$$\|u\|_{\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{\alpha} + |\eta|_{\alpha/2} + |\psi|_{\alpha/2} + \|f\|^{(2-\alpha)}_{0,\Omega}), \tag{2.9}$$

where C depends only on X_0, T and α.

(4) (*Intermediate Schauder estimates for oblique derivative problems*). *Suppose u is a solution to the problem (2.4) with the boundary condition $u = \psi(t)$ replaced by $u_x = \psi(t)$ on F. If $\psi(t) \in C^{\alpha/2}(0,T)$ and $\Gamma(t) \in C^\beta$ with $\beta \ge (1+\alpha)/2$, then*

$$\|u\|^{(-1-\alpha)}_{2+\alpha,\Omega} \le C(\|u\|_0 + |\varphi|_{2+\alpha} + |\eta|_{1+\alpha/2} + |\psi|_{\alpha/2} + \|f\|^{(1-\alpha)}_{\alpha,\Omega}), \tag{2.10}$$

where C depends only on $|\Gamma|_{(1+\alpha)/2}, X_0, T$, α and $\inf(X_0 - \Gamma(t))$.

For the proof of (2.5)–(2.10) we refer the reader to [10], where the classical and intermediate Schauder estimates for both the Dirichlet and the oblique derivative problems of linear parabolic equations were proved by means of the mollification of functions.

For later applications we give some a priori estimates for the problem

$$p_t - a_0 p_{xx} = c_0 \rho p_x \quad \text{in } \Omega = \{\Gamma(t) < x < X_0, 0 < t < T\}, \tag{2.11}$$

$$p_x = g(\Gamma', \Gamma, p, t) \quad \text{on } F = \{x = \Gamma(t)\}, \tag{2.12}$$

$$p(x,0) = \eta_0(x), \quad p(X_0, t) = \eta_1(t), \tag{2.13}$$

where $\rho = \rho_0 + \gamma p$, $\Gamma(t) \in C^1[0,T]$. We suppose $|\eta_i|_{C^3} \le M_0, i = 0, 1$, and $g(\Gamma', \Gamma, p, t)$ is Lipschitz continuous of its arguments.

Lemma 2.2. *Suppose there exist positive constants λ, C_1, and C_2 such that*

(H1) $g(y,z,p,t)p \ge C_1|p|^\lambda - C_2$ *for any* $y, z \in R, t \ge 0$.

Then the solution $p(x,t)$ satiafies

$$\|p\|_{L^\infty,\Omega} \le M_1, \tag{2.14}$$

where M_1 depends only on M_0, λ, C_1, and C_2.

Proof. $|p(x,t)|$ can not attain its maximum in Ω. If $|p(x,t)|$ attains its maximum on $\{t = 0\} \cup \{x = X_0\}$, then (2.14) follows from the initial-boundary condition. If $|p(x,t)|$ attains its maximum on F, then by the assumption (H1) we obtain (2.14).

Lemma 2.3. *Let $p(x,t)$ be the solution of* (2.11)–(2.13). *Suppose $\Gamma(t) \in C^1[0,T]$. Then*

$$\|p_x\|_{L^\infty,\Omega} \le M_2, \tag{2.15}$$

where M_2 depends only on M_0, M_1, $\sup|g(\Gamma',\Gamma,p,t)|$, and $\inf\limits_{t\in(0,T)} |X_0-\Gamma(t)|$.

Proof. Set $G = \log(1+p_x^2) + f(p)$, where

$$f(p) = \frac{1}{(M_1+1-p)^\gamma}, \tag{2.16}$$

$\gamma > 0$ is a constant to be determined. If G attains its maximum on $F = \{x = \Gamma(t)\}$, (2.15) follows from the boundary condition (2.12). If G attains its maximum on $x = X_0$, then it is standard to prove (2.15) by the method of barrier functions. If G attains its maximum at some point $P_0 = (x_0, t_0)$ in Ω, then at P_0 we have

$$0 = G_x = \frac{2p_xp_{xx}}{1+p_x^2} + f'(p)p_x, \quad \text{i.e.,} \quad p_{xx} = -\frac{1}{2}f'(p)(1+p_x^2), \tag{2.17}$$

$$0 \ge G_{xx} = \frac{2p_x}{1+p_x^2}p_{xxx} + \frac{2(1-p_x^2)}{(1+p_x^2)^2}p_{xx}^2 + f''(p)p_x^2 + f'(p)p_{xx}, \tag{2.18}$$

and

$$0 \le G_t = \frac{2p_x}{1+p_x^2}p_{xt} + f'(p)p_t. \tag{2.19}$$

Differentiating (2.11) we have

$$(p_x)_t - a_0(p_x)_{xx} = cp_x^2 + c\rho p_{xx}. \tag{2.20}$$

Combining (2.18) and (2.19), and noting that $p_{xx} = -\frac{1}{2}f'(p)(1+p_x^2)$, we get

$$\begin{aligned} 0 &\le \frac{2p_x}{1+p_x^2}(p_{xt} - a_0\, p_{xxx}) + f'(p)(p_t - a_0 p_{xx}) - \frac{2a_0(1-p_x^2)}{(1+p_x^2)^2}p_{xx}^2 - a_0 f''(p)p_x^2 \\ &\le C_1 + C_2|p_x| - a_0(f''(p) - \frac{1}{2}|f'(p)|^2)p_x^2, \end{aligned}$$

where C_1 and C_2 depend only on M_0 and M_1. Let γ be small enough so that $f''(p) - \frac{1}{2}|f'(p)|^2 > 0$. Then we get the estimate (2.15).

Lemma 2.4. *Suppose $\Gamma(t) \in C^1$ and $p(x,t)$ is a solution of* (2.11)–(2.13). *Then*

$$\sup\{(x-\Gamma(t))|p_{xx}(x,t)|\} \le M_3, \tag{2.21}$$

where M_3 depends on $X_0, M_i, i = 0,1,2, |\Gamma|_1$ and $\inf(X_0 - \Gamma(t))$.

Proof. Since $\Gamma \in C^1$, for $P = (x,t) \in \Omega$ we have $x - \Gamma(t) \sim d(P,F)$. Since $|p_x| \le M_2$ and $p_x(x,t)$ satisfies (2.20), applying the interior Schauder estimates (2.5) and by the interpolation inequality (2.3) we have

$$\sup\{(x-\Gamma(t))^2|p_{xxx}(x,t)|\} \le M.$$

By the interpolation inequality (2.3) again we obtain

$$\sup\{(x-\Gamma(t))|p_{xx}(x,t)|\} \le C[\sup|p_x| \cdot \sup\{(x-\Gamma(t))^2|p_{xxx}(x,t)|\}]^{1/2} \le M_3.$$

Lemma 2.5. *Let $u(x,t)$ be a solution to the equation*

$$\begin{cases} u_t - u_{xx} - b(x,t)u_x - c(x,t)u = f(x,t) & \text{in } \Omega, \\ u = u_0(x,t) & \text{on } \partial^*\Omega, \end{cases} \tag{2.22}$$

where $\partial^*\Omega$ is the parabolic boundary of Ω. Suppose $|b(x,t)| \le C$,

$$\sup(x-\Gamma(t))|c(x,t)| \le C, \quad \sup\{(x-\Gamma(t))|f(x,t)|\} \le C\varepsilon, \quad |u_0(x,t)| \le C\varepsilon,$$

and $\Gamma(t) \in C^1$. *Then*

$$\sup |u(x,t)| \le C_1\varepsilon. \tag{2.23}$$

Proof. Let $y = x - \Gamma(t), s = t$. Then (2.22) becomes

$$u_s - u_{yy} - (b(y,s) + \Gamma'(s))u_y - c(y,s)u = f(y,s). \tag{2.24}$$

One easily verifies that $\overline{u} = C_1\varepsilon e^{\alpha s}y^\gamma + C\varepsilon$ and $\underline{u} = -\overline{u}$ are a supersolution and a subsolution to (2.24) respectively, provided $\alpha > 0$ is large enough and $\gamma > 0$ small enough.

§3. Local Existence of Solutions

In this section, we consider the ground freezing problem in the bounded domain $\Omega = \{0 < x < X_0, 0 < t < \tau\}$. We will denote

$$\Omega^- = \{0 < x < \Gamma(t), 0 < t < \tau\}, \quad \text{and} \quad \Omega^+ = \{\Gamma(t) < x < X_0, 0 < t < \tau\}.$$

From (1.1), (1.2), the temperature $\theta(x,t)$ satisfies

$$\theta_t - a_1\theta_{xx} = 0 \ \text{ in } \Omega^-, \tag{3.1}$$

$$\theta_t - a_2\theta_{xx} = [c_1\rho^2 + c_2p_x]\theta_x + (c_3\rho p_x + c_4p_{xx})\theta \ \text{ in } \Omega^+, \tag{3.2}$$

$$\theta(\Gamma(t)-0,t) = \theta(\Gamma(t)+0,t) = 0 \ \text{ on } F = \{x = \Gamma(t)\}, \tag{3.3}$$

$$\Gamma' + \widetilde{g}(\Gamma',\Gamma,p,t) = k_1\theta_x^- - k_2\theta_x^+ \ \text{ on } F \tag{3.4}$$

with initial-boundary conditions

$$\theta(0,t) = \psi_0(t) < 0, \quad \theta(X_0,t) = \psi_1(t) > 0, \tag{3.5}$$

$$\theta(x,0) = \varphi(x), \tag{3.6}$$

$$\Gamma(0) = b \in (0, X_0), \tag{3.7}$$

where a_i and c_i are positive constants, $\rho = \rho_0 + \gamma p$, and $\varphi(x) < 0$ in $(0,b)$, $\varphi(x) > 0$ in (b, X_0). From (1.6) and (1.10), $p(x,t)$ satisfies

$$p_t - a_0p_{xx} = c_0\rho p_x \ \text{ in } \Omega^+, \tag{3.8}$$

$$p_x = g(\Gamma',\Gamma,p,t) \ \text{ on } F, \tag{3.9}$$

$$p(x,0) = \eta_0(x), \quad p(X_0,t) = \eta_1(t), \tag{3.10}$$

where g and $\widetilde{g}$ satisfy (1.10). We suppose

(H2) φ, ψ_i, and η_i are bounded and sufficiently smooth;

(H3) $y + \widetilde{g}(y,z,p,t)$ is strictly increasing with respect to y.

Hence from (3.4) and (3.9) we have

$$\Gamma' = g_1(\Gamma,p,t,k_1\theta_x^- - k_2\theta_x^+), \tag{3.11}$$

$$p_x = g_2(\Gamma,p,t,k_1\theta_x^- - k_2\theta_x^+). \tag{3.12}$$

We suppose

(H4) $g, \widetilde{g}$, and g_i are Lipschitz continuous of its arguments.

For any given $\Gamma(t) \in C^{1+\delta}[0,\tau]$ with $\Gamma(0) = b$, where $\delta \in (0,\frac{1}{2})$ is fixed, let $p(x,t)$ be the solution of the problem (3.8)–(3.10), and $\theta(x,t)$ be the solution of (3.1)–(3.3), (3.5), and (3.6). We introduce a mapping T which is from $C^{1+\delta}[0,\tau]$ to $C^{1+\delta}[0,\tau]$, formally defined by

$$T\,\Gamma(t) = \int_0^t g_1(\Gamma(t), p(\Gamma(t),t), t, k_1\theta_x^- - k_2\theta_x^+)dt + b. \tag{3.13}$$

Let $\Sigma(M) = \{\Gamma(t) \in C^{1+\delta}[0,\tau], \Gamma(0) = b, \frac{b}{2} < \Gamma(t) < \frac{1}{2}(b+X_0),$ and $\|\Gamma(t)\|_{C^{1+\delta}[0,\tau]} < M\}$. We will prove that for $\tau > 0$ sufficiently small and M large enough, T is completely continuous from $\Sigma(M)$ into itself.

Remark 3.1. To preceed further let us first remark the boundedness of $\theta(x,t)$. $\theta(x,t)$ is obviously bounded in Ω^- by the heat equation (3.1). In Ω^+ the equation (3.2) can be written as

$$\theta_t - a_2\theta_{xx} - (c_4 p_x \theta)_x = (c_1\rho^2 + (c_2 - c_4)p_x)\theta_x + c_3\rho p_x \theta.$$

From Lemmas 2.2 and 2.3, the coefficients of the above equation are bounded, hence $\theta(x,t)$ is also bounded. By (2.2) and (2.10) we have the following Lemma 3.1.

Lemma 3.1. *If $p(x,t)$ is the solution of* (3.8)–(3.10)*, then*

$$|p|_{1+\delta,\Omega^+} \le C, \quad \sup_{\sigma>0} \sigma |p|_{2+\delta,\Omega^+_\sigma} < C,$$

where C depends on M, $\Omega^+_\sigma = \{(x,t) \in \Omega^+,\ \mathrm{dist}((x,t),F) > \sigma\}$.

Lemma 3.2. *Let $\theta(x,t)$ be the solution of* (3.1)–(3.3), (3.5), *and* (3.6). *Then for any $\alpha \in (0,1)$,*

$$\|\theta\|_{1+\alpha,\Omega^+} \text{ and } \|\theta\|_{1+\alpha,\Omega^-} \le M_4, \tag{3.14}$$

where M_4 depend only on $\alpha, |\Gamma|_{C^1}$ and $M_i, i = 0,1,2,3$.

Proof. Since $\theta(x,t)$ satisfies the heat equation $\theta_t - a_1\theta_{xx} = 0$ in Ω^- and $\Gamma(t) \in C^{1+\delta}[0,\tau]$, and since the initial-boundary condition of $\theta(x,t)$ is smooth, it follows that $\|\theta\|_{1+\alpha,\Omega^-} \le M_4$ for any $\alpha \in (0,1)$, where M_4 depends on α, but is independent of $\tau \in (0,1)$.

In the domain Ω^+, $\theta(x,t)$ satisfies

$$\theta_t - a_2\theta_{xx} = (c_1\rho^2 + c_2 p_x)\theta_x + (c_3\rho p_x + c_4 p_{xx})\theta =: h.$$

By Schauder estimates we have

$$\begin{aligned}
\|\theta\|_{1+\alpha,\Omega^+} &\le C(1 + \|\theta\|_0 + \|h(x,t)\|_0) \\
&\le C(1 + \|\theta_x\|_0 + \|\theta p_{xx}(x,t)\|_0) \\
&\le C(1 + \|\theta_x\|_0 + \|\theta_x\|_0 \sup(x - \Gamma(t))|p_{xx}(x,t)|).
\end{aligned}$$

By the interpolation inequality $\|\theta\|_1 \le C\|\theta\|_{1+\alpha}^{1/(1+\alpha)}\ \|\theta\|_0^{\alpha/(1+\alpha)}$ it follows that

$$\|\theta\|_{1+\alpha,\Omega^+} \le C[1 + \|\theta\|_0 + \sup(x-\Gamma(t))|p_{xx}(x,t)|]^{(1+\alpha)/\alpha}.$$

By Lemma 2.4, $\sup(x - \Gamma(t))|p_{xx}(x,t)| \le M_3$, we therefore obtain (3.14).

Remark 3.2. From (3.12) and (3.14) we therefore obtain $\|p\|_{1+\alpha} \le C$ for any $\alpha \in (0,1)$; and by (2.10), $\sup\limits_{\delta>0} \delta \|p\|_{2+\alpha,\Omega^+_\delta} \le C$.

From Lemma 3.2 and Remark 3.2 it follows that the mapping T is compact. Next we show that if τ is small enough and M is large enough, the mapping is injective. Let $K = (k_1 + k_2)(|\varphi_x(b-)| + |\varphi_x(b+)| + 1)$, and let

$$M = 1 + \sup\{|g_1(\Gamma, p, t, y)|;\ |\Gamma| < X_0, t \in (0,1), |p| < M_0, \text{and } |y| < K\},$$

where g_1 is the function in (3.11). For any $s(t) \in \Sigma(M)$, if τ is small enough, by (3.13) we have $\frac{b}{2} < T\,s(t) < \frac{1}{2}(b + X_0)$. Note that

$$\frac{d}{dt} Ts(0) = g_1(s(0), \eta(0), 0, k_1\varphi(b-) - k_2\varphi(b+)).$$

From Lemma 3.2 we have

$$\Big|\frac{d}{dt}Ts(t)\Big| \le \Big|\frac{d}{dt}Ts(0)\Big| + Ct^{\alpha/2} \le M,$$

and

$$\Big|\frac{d}{dt}T\,s(t) - \frac{d}{dt}T\,s(t')\Big|/|t-t'|^{\delta} \le C|t-t'|^{\frac{\alpha}{2}-\delta} \le M,$$

provided $\alpha > 2\delta$ and τ is small enough. Hence T is injective.

To see the continuity of T, for any given $\Gamma_0(t) \in \Sigma(M)$ be let $\Gamma_1(t) \in \Sigma(M)$ so that $|\Gamma_1 - \Gamma_0|_{1+\delta} \le \varepsilon$. Suppose θ_i and p_i are the solutions corresponding to $\Gamma_i, i = 0, 1$, respectively. Let

$$\Omega_i^- = \{0 < x < \Gamma_i(t), \quad 0 < t < \tau\} \quad \text{and} \quad \Omega_i^+ = \{X_0 > x > \Gamma_i(t), \quad 0 < t < \tau\}.$$

In Ω_0^-, we have by the maximum principle

$$\sup_{\Omega_0^-} |\theta_1 - \theta_0| \le \sup\Big|\frac{\partial}{\partial x}\theta_i\Big| \cdot \sup|\Gamma_1 - \Gamma_0| \le C\varepsilon. \tag{3.15}$$

Let $\widetilde{\theta}_1^-(x,t) = \theta_1^-(x \cdot \frac{\Gamma_1(t)}{\Gamma_0(t)}, t)$. Then $\widetilde{\theta}_1^-(x,t)$ is also defined on the domain Ω_0^- with $\widetilde{\theta}_1^- = 0$ on $\Gamma_0(t)$, and we still have

$$\sup_{\Omega_0^-} |(\widetilde{\theta}_1 - \theta_0)(x,t)| \le C\varepsilon.$$

And by Lemma 3.2, $\|\widetilde{\theta}_1^-\|_{1+\alpha,\Omega_0^-} < CM_4$. Therefore

$$|\widetilde{\theta}_1 - \theta_0|_{1+\alpha/2,\Omega_0^-} \le (|\widetilde{\theta}_1 - \theta_0|_{0,\Omega_0^-})^{\alpha/(2+2\alpha)} \cdot (|\widetilde{\theta}_1 - \theta_0|_{1+\alpha,\Omega_0^-})^{(1+\alpha/2)/(1+\alpha)} \le C\varepsilon^{\alpha/(2+2\alpha)}.$$

Hence

$$\Big|\frac{\partial}{\partial x}\theta_0(\Gamma_0(t)-0,t) - \frac{\partial}{\partial x}\theta_1(\Gamma_1(t)-0,t)\Big| \le \delta(\varepsilon) \tag{3.16}$$

with $\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next we consider the estimates for p_i and θ_i^+. Without loss of generality we may suppose

$$\frac{\partial}{\partial p}g(\Gamma', \Gamma, p, t) \ge 1. \tag{3.18}$$

Indeed, if (3.18) is not true, let

$$K_0 = 1 + \sup\Big\{\Big|\frac{\partial}{\partial p}g(\Gamma', \Gamma, p, t)\Big|;\ \Gamma \in \Sigma(M), |p| \le M_0, 0 \le t \le \tau\Big\},$$

and let $p = e^{-K_0x}\widetilde{p}$. Then (3.8)-(3.10) are equivalent to

$$\begin{cases} \widetilde{p}_t - a_0\widetilde{p}_{xx} = F(\widetilde{p}, \widetilde{p}_x, x) & \text{in } \Omega^+, \\ \widetilde{p}_x = K_0\widetilde{p} + e^{-K_0x}g(\Gamma', \Gamma, p, t) \equiv: g_3(\Gamma', \Gamma, \widetilde{p}, t) & \text{on } \{x = \Gamma(t)\}, \\ \widetilde{p}(x,0) = e^{K_0x}\eta_0(x), \quad \widetilde{p}(X_0,t) = C_0\eta_1(t), \end{cases} \tag{3.19}$$

where

$$F(\widetilde{p}, \widetilde{p}_x, x) = a_0K_0^2\widetilde{p} - 2a_0K_0\widetilde{p}_x + c_0(\rho_0 + \gamma e^{-K_0x}\widetilde{p})(\widetilde{p}_x - K_0\widetilde{p}).$$

By the choice of K_0,

$$\frac{\partial}{\partial \widetilde{p}}g_3(\Gamma', \Gamma, \widetilde{p}, t) = K_0 + e^{-K_0x}\frac{\partial g}{\partial p}(\Gamma', \Gamma, p, t) \ge 1. \tag{3.18'}$$

Hence we may suppose (3.18) holds.

Let $\widetilde{p}_1(y,s)=p_1(x,t)$, and $\widetilde{\theta}_1(y,s)=\theta_1(x,t)$, where

$$y=x-\Gamma_1(t)+\Gamma_0(t),\ \ s=t. \tag{3.20}$$

Then $\widetilde{p}_1$ and $\widetilde{\theta}_1$ are defined on $\Omega_0^+\cap\{x<X_0-\varepsilon\}$. From (3.8), the function $\widetilde{p}_1-p_0$ satisfies (with (y,s) written as (x,t))

$$(\widetilde{p}_1-p_0)_t-a_0(\widetilde{p}_1-p_0)_{xx}=(\Gamma_1'-\Gamma_0')\widetilde{p}_{1x}+\mathcal{B} \tag{3.21}$$

on $\Omega_0^+\cap\{x<X_0-\varepsilon\}$, and

$$(\widetilde{p}_1-p_0)_x=g(\Gamma_1',\Gamma_1,\widetilde{p}_1,t)-g(\Gamma_0',\Gamma_0,p_0,t)\text{ on }\{x=\Gamma_0(t)\}, \tag{3.22}$$

$$\widetilde{p}_1-p_0=0\ \text{ on }\{t=0\}, \tag{3.23}$$

where by the right-hand side of (3.8), and (1.5),

$$\mathcal{B}=c_0(\rho_0+\gamma\widetilde{p}_1)(\widetilde{p}_1-p_0)_x+c_0\gamma p_{0x}(\widetilde{p}_1-p_0).$$

Namely, $\widetilde{p}_1-p_0$ satisfies the equation

$$u_t-u_{xx}-c_0(\rho_0+\gamma\widetilde{p}_1)u_x-c_0\gamma p_{0x}u=(\Gamma_1'-\Gamma_0')\widetilde{p}_{1x}.$$

On $\{x=X_0-\varepsilon\}$ we have

$$|\widetilde{p}_1-p_0|\le C\varepsilon.$$

From (3.18) and (3.22) we see that if $\widetilde{p}_1-p_0$ attains its maximum on $\{x=\Gamma_0(t)\}$, then

$$\|\widetilde{p}_1-p_0\|_0\le C(|\Gamma_1'-\Gamma_0'|+|\Gamma_1-\Gamma_0|)\le C\varepsilon. \tag{3.24}$$

Applying the parabolic maximum principle to the equation (3.21)-(3.23) and by virtue of (3.24), we obtain

$$\|\widetilde{p}_1-p_0\|_0\le C\varepsilon+C\sup|\Gamma_1'-\Gamma_0'|\cdot|\widetilde{p}_{1x}|\le C\varepsilon.$$

By Remark 3.2 we therefore obtain for any $0<\alpha<\alpha'<1$,

$$|\widetilde{p}_1-p_0|_{1+\alpha,\Omega_0^+}\le(|\widetilde{p}_1-p_0|_{0,\Omega_0^+})^{\gamma_1}\cdot(|\widetilde{p}_1-p_0|_{1+\alpha',\Omega_0^+})^{1-\gamma_1}\le\delta_1(\varepsilon). \tag{3.25}$$

Again by Remark 3.2

$$\begin{aligned}\sup_\sigma\sigma\|\widetilde{p}_1-p_0\|_{2+\delta/2,\Omega_0^+(\sigma)}&\le C(\|\widetilde{p}_1-p_0\|_{1+\alpha,\Omega_0^+})^{\gamma_2}\cdot(\sup_\sigma\sigma\|\widetilde{p}_1-p_0\|_{2+\alpha,\Omega_0^+(\sigma)})^{1-\gamma_2}\\&\le\delta_2(\varepsilon),\end{aligned} \tag{3.26}$$

where

$$\Omega_0^+(\sigma)=\{(x,t)\in\Omega_0^+,\ \mathrm{dist}\{(x,t),F_0\}>\sigma\},\quad F_0=\{x=\Gamma_0(t)\},$$

and γ_1,γ_2 are suitable constants.

For the function $\widetilde{\theta}_1(y,s)=\theta_1(x,t)$ defined above, we have

$$\widetilde{\theta}_{1t}-a_2\widetilde{\theta}_{1xx}=[c_1\rho_1^2+c_2\widetilde{p}_{1x}]\widetilde{\theta}_{1x}+(c_3\rho_1\widetilde{p}_{1x}+c_4\widetilde{p}_{1xx})\widetilde{\theta}+(\Gamma_1'-\Gamma_0')\widetilde{\theta}_{1x}.$$

Hence $\widetilde{\theta}=\widetilde{\theta}_1-\theta_0$ satisfies

$$\widetilde{\theta}_t-a_2\widetilde{\theta}_{xx}=[c_1\rho^2+c_2p_{0x}]\widetilde{\theta}_x+(c_3\rho p_{0x}+c_4p_{0xx})\widetilde{\theta}+G(p_0,\widetilde{p}_1,\theta_0,\widetilde{\theta}_1)+(\Gamma_1'-\Gamma_0')\widetilde{\theta}_{1x},$$

where $\rho=\rho_0+\gamma p_0(x,t)$, $\rho_1=\rho_0+\gamma\widetilde{p}_1(x,t)$, and

$$|G(p_0,\widetilde{p}_1,\theta_0,\widetilde{\theta}_1)|\le C\{\|\widetilde{p}_1-p_0\|_1\cdot(\|\widetilde{\theta}_{1x}\|_0+1)+|(\widetilde{p}_1-p_0)_{xx}(x,t)\widetilde{\theta}_1(x,t)|\}.$$

By (3.25), (3.26), and Lemma 3.2, we have

$$\begin{aligned}|G(p_0,\widetilde{p}_1,\theta_0,\widetilde{\theta}_1)| &\le C(\delta_1(\varepsilon)+|(\widetilde{p}_1-p_0)_{xx}(x,t)\widetilde{\theta}_1(x,t)|)\\ &\le C(\delta_1(\varepsilon)+C_1\sup(x-\Gamma_0(t))\cdot|(\widetilde{p}_1-p_0)_{xx}(x,t)|\\ &\le C_2(\delta_1(\varepsilon)+\delta_2(\varepsilon)),\end{aligned}$$

where C_1 and C_2 depend on $\sup|\widetilde{\theta}_{1x}|$ and $|\Gamma_0|_1$. Note that

$$\sup(x-\Gamma_0(t))\cdot|p_{0xx}|\le C\delta_2(\varepsilon),$$

and on $\{x=X_0-\varepsilon\}$ we have $|\widetilde{\theta}|\le C\varepsilon$. Hence by Lemma 2.5 we obtain $\|\widetilde{\theta}\|_0\le C\delta_3(\varepsilon)$ with $\delta_3(\varepsilon)\to 0$ as $\varepsilon\to 0$. By means of the interpolation inequality and Lemma 3.2 we therefore conclude $\|\widetilde{\theta}\|_{1+\alpha/2,\Omega_0^+}\to 0$ as $\varepsilon\to 0$. Hence

$$\Big|\frac{\partial}{\partial x}\theta_0(\Gamma_0(t)+0,t)-\frac{\partial}{\partial x}\theta_1(\Gamma_1(t)+0,t)\Big|\to 0 \tag{3.27}$$

as $\varepsilon\to 0$. Combining (3.16) and (3.27) we obtain

$$\|T\Gamma_1-T\Gamma_0\|_{C^1[0,\tau]}\to 0 \ \text{ as } \ \|\Gamma_1-\Gamma_0\|_{C^{1+\delta}[0,\tau]}\to 0.$$

From Lemma 3.2 and by the interpolation inequality we conclude that for any $\delta<\alpha<1$,

$$\|T\Gamma_1-T\Gamma_0\|_{C^{1+\delta}[0,\tau]}\le C\|T\Gamma_1-T\Gamma_0\|^{\gamma}_{C^1[0,\tau]}\cdot\|T\Gamma_1-T\Gamma_0\|^{1-\gamma}_{C^{1+\alpha}[0,\tau]}\to 0.$$

Hence T is completely continuous. From the above arguments we obtain

Theorem 3.1. *Under the hypotheses* (H1)–(H4)*, there exists a local solution to the problem* (3.1)–(3.10)*.*

Remark 3.3. We have not proved the uniqueness because we are unable to show that T is a contraction mapping.

§4. Existence of Global Solutions

For any given $T>0$ let

$$\begin{aligned}\Omega&=\{0<x<X_0,0<t<T\},\\ \Omega^-&=\{0<x<\Gamma(t),0<t<T\},\\ \Omega^+&=\{\Gamma(t)<x<X_0,0<t<T\}.\end{aligned}$$

To prove the global existence of solutions to the problem (3.1)-(3.10) we will establish the a priori estimates for θ_x. We suppose

(H5) $p_x=g(\Gamma,p,t),\quad g(\Gamma,p,t)\in C^2(R\times R\times R^+)$.

We also suppose (H1) and (H2) hold.

Theorem 4.1. *Under the above hypotheses, there exists a global solution to the problem* (3.1)–(3.10)*.*

Remark 4.1. By global solution we mean either $\Gamma(t)\in C^1[0,T]$, or there is a $t_0\in(0,T]$ such that $\Gamma(t_0)=0$ or $\Gamma(t_0)=X_0$, and $\Gamma(t)\in C^1[0,t_0)$.

Proof. We argue by contradiction. If the conclusion is not true, then for some $\tau\in(0,T]$ we have

$$\overline{\lim_{t\to\tau}}|\Gamma'(t)|=+\infty \quad\text{and}\quad \inf_{t\in(0,\tau)}\min\{\Gamma(t),X_0-\Gamma(t)\}\ge\delta>0.$$

Without loss of generality we may suppose $\tau = 1$. We will prove $\sup\{|\theta_x(x,t)|;\ t \in (0, 1-\varepsilon)\} < C$ for some $C > 0$ independent of $\varepsilon > 0$, which is in conflict with $\overline{\lim\limits_{t\to\tau}}|\Gamma'(t)| = +\infty$.

First recall that by Lemmas 2.2 and 2.3, we have

$$\|p\|_0 + \|p_x\|_0 \le C. \tag{4.1}$$

Denote $\Omega^- = \{0 < x < \Gamma(t), 0 < t < 1-\varepsilon\}, \Omega^+ = \{X_0 > x > \Gamma(t), 0 < t < 1-\varepsilon\}$.

Let $\theta^+ = e^{-Ax}\widetilde{\theta}^+,\ \theta^- = \widetilde{\theta}^-$. Then from (3.1)- (3.4), $\widetilde{\theta}$ satisfies

$$\widetilde{\theta}_t - a_1\widetilde{\theta}_{xx} = 0 \quad \text{in } \Omega^- \tag{4.2}$$

$$\widetilde{\theta}_t - a_2\widetilde{\theta}_{xx} = [c_1\rho^2 + c_2 p_x - 2a_2A]\widetilde{\theta}_x + c(x,t)\,\widetilde{\theta} \quad \text{in } \Omega^+, \tag{4.3}$$

$$\widetilde{\theta}(\Gamma(t)-0,t) = \widetilde{\theta}(\Gamma(t)+0,t) = 0 \quad \text{on } F = \{x = \Gamma(t)\}, \tag{4.4}$$

$$\Gamma' + \widetilde{g}(\Gamma,p,t) = k_1\theta_x^- - k_2\theta_x^+ = k_1\widetilde{\theta}_x^- - k_2 e^{-A\Gamma(t)}\widetilde{\theta}_x^+ \text{ on } F, \tag{4.5}$$

where

$$c(x,t) = c_3\rho p_x + c_4 p_x x - (c_1\rho^2 + c_2p_x)A + A^2, \quad \text{and} \quad \rho = \rho_0 + \gamma p.$$

Since $|p|_0 \le C$ and $|p_x|_0 \le C$, we may take A suitably large so that

$$c_1\rho^2 + c_2p_x - 2a_2A \le -1. \tag{4.6}$$

Since $\widetilde{\theta}_x^+$ and $\widetilde{\theta}_x^-$ are positive on F, we need only to estimate $\sup\widetilde{\theta}_x$ for if $\sup\widetilde{\theta}_x \le C$, we have $|\Gamma'| \le C_1$. By Lemmas 3.1 and 3.2 we have $|\theta|_{1+\alpha} \le C_2$ (recall that the constants M_i in Lemmas 3.1 and 3.2 depend only on $|\Gamma|_1$) and hence $|\Gamma|_{1+\alpha/2} \le C_2$. By Schauder estimates we therefore obtain $|\theta|_{2+\alpha} \le C$.

Let

$$\psi(x,t) = \begin{cases} k_1\widetilde{\theta}_x^-(x,t), & (x,t) \in \Omega^-, \\ k_2e^{-A\Gamma(t)}\widetilde{\theta}_x^+(x,t) + \widetilde{g}(\Gamma(t),p(x,t),t), & (x,t) \in \Omega^+, \end{cases}$$

where k_1, k_2 and $\widetilde{g}$ are as in (4.5). By the maximum principle we see that $\psi(x,t)$ attains its maximum either (i): on the parabolic boundary of the domain Ω; or (ii): on the free boundary F; or (iii): in Ω^+.

In case (i), from the smoothness of the initial-boundary conditions and from (4.1) we have $\psi(x,t) \le C$ and hence $\sup\widetilde{\theta}_x \le C$.

In case (ii), if

$$\sup\psi(x,t) = \psi(\Gamma(t_0)-,t_0) = k_1\widetilde{\theta}_x^-(\Gamma(t_0)-,t_0) \quad \text{for some } \ 0 < t_0 \le 1-\varepsilon,$$

by thy strong maximum principle we have $\widetilde{\theta}_{xx}^- > 0$ at this point, hence $\widetilde{\theta}_t^- > 0$ at $x = \Gamma(t_0) - 0$. Differentiating $\widetilde{\theta}^-(\Gamma(t),t) = 0$ we get $\widetilde{\theta}_x^-\Gamma' + \widetilde{\theta}_t^- = 0$. Hence at $x = \Gamma(t_0)$ we have $\widetilde{\theta}_x^-\Gamma' < 0$. Since $\widetilde{\theta}_x^- > 0$ at $x = \Gamma(t_0)$, we conclude that $\Gamma'(t_0) < 0$. But

$$\Gamma'(t_0) = \psi(\Gamma(t_0)-,t_0) - \psi(\Gamma(t_0)+,t_0) \ge 0,$$

we obtain a contradiction.

If

$$\sup\psi(x,t) = \psi(\Gamma(t_0)+,t_0) = k_2e^{-A\Gamma(t_0)}\widetilde{\theta}_x^+(\Gamma(t_0)+,t_0) + \widetilde{g}(\Gamma(t),p(\Gamma(t),t),t)_{|t=t_0} \quad \text{for some} \quad 0 < t_0 \le 1-\varepsilon,$$

then at this point we have

$$k_2e^{-A\Gamma(t)}\widetilde{\theta}_{xx}^+ + \widetilde{g}_p'\,p_x \le 0, \quad \text{i.e.,} \quad \widetilde{\theta}_{xx}^+ \le C\,e^{A\Gamma(t)} \le C_1 \ \text{at} \ \ x = \Gamma(t_0) + 0.$$

If $\widetilde{\theta}_x^+ \leq C_1 a_2$, then we are through. If $\widetilde{\theta}_x^+ > C_1 a_2$, then from (4.6) and the equation (4.3) we obtain $\widetilde{\theta}_t^+ < 0$ at $x = \Gamma(t_0) + 0$, which implies $\Gamma'(t_0) > 0$. We therefore conclude that

$$\psi(\Gamma(t_0)+, t_0) = \psi(\Gamma(t_0)-, t_0) - \Gamma'(t_0) < \psi(\Gamma(t_0)-, t_0),$$

also a contradiction.

The treatment for the case (iii) is somewhat complicated. Let

$$G(x,t) = \log \psi(x,t) + f(\widetilde{\theta}), \quad (x,t) \in \Omega^+ \cap \{\psi(x,t) > 0\},$$

where $f(\widetilde{\theta}) = 1/(M - \widetilde{\theta})^\alpha$, $M = 1 + \|\widetilde{\theta}\|_{L^\infty}$, and $\alpha > 0$ is small enough so that

$$f''(\widetilde{\theta}) - |f'(\widetilde{\theta})|^2 \geq C = C(M, \alpha).$$

Since $\widetilde{\theta} = 0$ on F and f is increasing, it follows that $G(x,t)$ attains its maximum either on $\partial^*\Omega^+ \backslash F$ or in Ω^+, where $\partial^*\Omega^+$ denotes the parabolic boundary of Ω^+. If $G(x,t)$ attains its maximum on $\partial^*\Omega^+ \backslash F$, then by the smoothness of the initial value and by (4.1) we have $\sup\limits_{\Omega^+} G(x,t) \leq C$.

If $G(x,t)$ attains its maximum at some point P_0 in Ω^+, then at P_0 we have

$$0 = G_x = \frac{\psi_x}{\psi} + f'(\widetilde{\theta})\widetilde{\theta}_x, \tag{4.7}$$

$$0 \geq G_{xx} = \frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2} + f''(\widetilde{\theta})\widetilde{\theta}_x^2 + f'(\widetilde{\theta})\widetilde{\theta}_{xx}, \tag{4.8}$$

$$0 \leq G_t = \frac{\psi_t}{\psi} + f'(\widetilde{\theta})\widetilde{\theta}_t. \tag{4.9}$$

From (4.7) we have

$$\widetilde{\theta}_{xx} = \frac{1}{k_2} e^{A\Gamma(t_0)}(\psi_x - \widetilde{g}_p' p_x) = -f'(\widetilde{\theta})\widetilde{\theta}_x^2 - \frac{1}{k_2} e^{A\Gamma(t_0)}(f'(\widetilde{\theta})\widetilde{\theta}_x \widetilde{g} + \widetilde{g}_p' p_x). \tag{4.10}$$

From (4.8) and (4.9) we get

$$0 \geq a_2 G_{xx} - G_t = \frac{1}{\psi}(a_2 \psi_{xx} - \psi_t) + a_2(f''(\widetilde{\theta}) - |f'(\widetilde{\theta})|^2)\widetilde{\theta}_x^2 + f'(\widetilde{\theta})(a_2 \widetilde{\theta}_{xx} - \widetilde{\theta}_t). \tag{4.11}$$

From (4.3) we have

$$|a_2 \widetilde{\theta}_{xx} - \widetilde{\theta}_t| \leq C(1 + |\widetilde{\theta}_x| + \widetilde{\theta}|p_{xx}|).$$

Differentiating (4.3) and using (4.10) we obtain

$$\begin{aligned} |\widetilde{\theta}_{xt} - a_2 \widetilde{\theta}_{xxx}| &\leq C(|\widetilde{\theta}_{xx}| + |\widetilde{\theta}_x| + |p_{xx}| \cdot |\widetilde{\theta}_x| + |p_{xxx}| \cdot \widetilde{\theta} + 1) \\ &\leq C(|\widetilde{\theta}_x|^2 + |p_{xx}| \cdot |\widetilde{\theta}_x| + |p_{xxx}| \cdot \widetilde{\theta} + 1). \end{aligned}$$

Since

$$\begin{aligned} |a_2 \psi_{xx} - \psi_t| &\leq |a_2 \widetilde{\theta}_{xxx} - \widetilde{\theta}_{xt}| + \Big| a_2 \frac{\partial^2}{\partial x^2} \widetilde{g} - \frac{\partial}{\partial t} \widetilde{g} + k_2 A \Big| \Gamma'(t_0)| \cdot |\widetilde{\theta}_x| \\ &\leq |a_2 \widetilde{\theta}_{xxx} - \widetilde{\theta}_{xt}| + C(1 + |p_{xx}| + |\Gamma|_1(1 + |\widetilde{\theta}_x|)), \end{aligned}$$

from (4.11) we obtain

$$0 \geq a_2(f''(\widetilde{\theta}) - |f'(\widetilde{\theta})|^2)\widetilde{\theta}_x^2 - \frac{C}{\psi(P_0)}(1 + |\widetilde{\theta}_x|^2 + |p_{xx}|\,|\widetilde{\theta}_x| + |p_{xxx}| \cdot \widetilde{\theta} + |\Gamma|_1(|\widetilde{\theta}_x| + 1)).$$

Since $G(x,t)$ attains its maximum at P_0, we may suppose $\psi(P_0) \geq \delta_0 \widetilde{\theta}_x$ at P_0 for some $\delta_0 > 0$ small. Noting that $\|\Gamma\|_1 \leq C\|\widetilde{\theta}_x\|_0$, we therefore conclude that

$$0 \geq a_2(f''(\widetilde{\theta}) - |f'(\widetilde{\theta})|^2)\widetilde{\theta}_x^2 - C(1 + |\widetilde{\theta}_x|^2 + |p_{xx}| \cdot \|\widetilde{\theta}_x\|_0 + \widetilde{\theta} \cdot p_{xxx})/\|\widetilde{\theta}_x\|_0.$$

That is

$$\|\widetilde{\theta}_x\|_0 \le C(1 + \|p_{xx}\|_0^{1/2} + \sup|p_{xxx}(x,t)|^{1/3}). \tag{4.12}$$

It remains to estimate $\sup|p_{xx}|$ and $\sup|p_{xxx}|$. By the interior Schauder estimates we have

$$|p_{xx}(x,t)| \le C_0, \quad |p_{xxx}(x,t)| \le C_0 \ \text{ for } \ x \ge \Gamma(t) + \delta,$$

where $\delta > 0$ is a constant. Let $y = x - \Gamma(t), \quad s = t$ and for convenience we still denote (y,s) as (x,t). From (3.8)–(3.10) one sees that p satisfies

$$\begin{cases} p_t - a_0 p_{xx} = c\rho p_x + \Gamma'(t) p_x & \text{in } \ \widetilde{\Omega} = \{0 < x < \widetilde{X}_0, 0 < t < 1 - \varepsilon\} \\ p_x(0,t) = g(\Gamma, p, t), \\ p(x,0) = \eta_0(x), \end{cases} \tag{4.13}$$

where $0 < \widetilde{X}_0 \le \inf(X_0 - \Gamma(t))$. Differentiating the equation above we get

$$\begin{cases} v_t - a_0 v_{xx} = h =: cv^2 + c\rho v_x + \Gamma'(t) v_x & \text{in } \ \widetilde{\Omega}, \\ v(0,t) = g(\Gamma, p, t), \\ v(x,0) = \frac{\partial}{\partial x}\eta_0(x), \end{cases} \tag{4.14}$$

where $v = p_x$. By the intermediate Schauder estimates and note that v is smooth in $\{\frac{1}{2}X_0 < x < X_0\}$, we have

$$\|v\|_{1+\beta,Q} \le C(|g|_{(1+\beta)/2} + \|h\|_{L^\infty} + \|v\|_0 + \|\eta_0\|_{2+\beta}), \tag{4.15}$$

where $Q = \{0 < x < \frac{1}{2}\widetilde{X}_0, 0 < t < 1 - \varepsilon\}$, and

$$\begin{aligned} |g|_{(1+\beta)/2} &\le \left|\frac{\partial g}{\partial \Gamma}\right| \cdot |\Gamma|_{(1+\beta)/2} + |g_p(p_x \cdot |\Gamma|_{(1+\beta)/2} + p_t)| + |g_t| \\ &\le C(1 + |\Gamma|_1 + |p_t|). \end{aligned}$$

From the equation (4.13) we have

$$|p_t| \le a_0|p_{xx}| + c|\rho p_x| + |\Gamma'(t) p_x| \le a_0|v_x| + C(1 + |\Gamma|_1).$$

Hence

$$|g|_{(1+\beta)/2} \le C(1 + |\Gamma|_1 + \|v_x\|_{L^\infty}). \tag{4.16}$$

Since $v = p_x$ is bounded, from the equation (4.14) we have

$$\|h\|_{L^\infty} \le C(1 + \|v_x\|_{L^\infty} + |\Gamma|_1 \|v_x\|_{L^\infty}).$$

By (4.15) it therefore follows that

$$\|v\|_{1+\beta,Q} \le C(1 + |\Gamma|_1)(1 + \|v_x\|_{L^\infty}). \tag{4.17}$$

By the interpolation inequality

$$\sup_t |v_x(\cdot,t)|_{L^\infty} \le C \sup_t (|v(\cdot,t)|_{L^\infty}^{\beta/(1+\beta)} |v(\cdot,t)|_{1+\beta}^{1/(1+\beta)} \le C\|v\|_{1+\beta}^{1/(1+\beta)}.$$

We thus conclude that

$$\|v\|_{1+\beta,Q} \le C(1 + |\Gamma|_1^{(1+\beta)/\beta}), \tag{4.18}$$

and

$$\sup_Q |v_x| \le C(1 + \|v\|_{1+\beta,Q}^{1/(1+\beta)}) \le C(1 + |\Gamma|_1^{1/\beta}), \tag{4.19}$$

where $\beta \in (\frac{1}{2}, 1)$ will be determined below, and $C = C(\beta)$.

Next we estimate $\sup_Q |p_{xxx}| = \sup_Q |v_{xx}|$. By Schauder estimates we have

$$\|v\|_{2+\alpha,Q} \le C(|g|_{1+\alpha/2} + \|h\|_\alpha + \|\eta_0\|_{3+\alpha} + \|v\|_0), \tag{4.20}$$

where

$$\begin{aligned} |g|_{1+\alpha/2} &\le C\Big(\sup|g| + \Big|\frac{d}{dt}g\Big|_{\alpha/2}\Big) \\ &\le C(1 + |g_\Gamma\Gamma' + g_t|_{\alpha/2} + |g_p(p_x\Gamma' + p_t)|_{\alpha/2}). \end{aligned} \tag{4.21}$$

The first term on the right hand side of (4.21) is estimated as follows. Since g is C^2 continuous, we have

$$\begin{aligned} |g_\Gamma\Gamma' + g_t|_{\alpha/2} &\le C(1 + |g_{\Gamma\Gamma}| \cdot |\Gamma|_1 \cdot |\Gamma|_{\alpha/2} + |g_{\Gamma p}||\Gamma|_1\|p\|_\alpha + |g_{\Gamma t}||\Gamma|_1 \\ &\quad + |g_\Gamma| \cdot |\Gamma|_{1+\alpha/2} + |g_{t\Gamma}| \cdot |\Gamma|_1 + |g_{tp}|\|p\|_\alpha) \\ &\le C(1 + |\Gamma|_1^2 + |\Gamma|_{1+\alpha/2} + |\Gamma|_1 \cdot \|p\|_\alpha). \end{aligned}$$

From the equation (4.13) we have

$$\begin{aligned} \|p\|_\alpha &\le C(1 + \|p\|_2)^{\alpha/2} = C(1 + \|p_{xx}\|_0 + \|p_t\|_0)^{\alpha/2} \\ &\le C(1 + |\Gamma|_1 + \|p_{xx}\|_0)^{\alpha/2} = C(1 + |\Gamma|_1 + \|v_x\|_0)^{\alpha/2}. \end{aligned} \tag{4.22}$$

Hence from (4.19) and letting $\alpha < \beta$, we obtain

$$|g_\Gamma\Gamma' + g_t|_{\alpha/2} \le C(1 + |\Gamma|_1^2 + |\Gamma|_{1+\alpha/2}). \tag{4.23}$$

For the second term on the right hand side of (4.21) we have (by the equation (4.13))

$$\begin{aligned} |g_p(p_x\Gamma' + p_t)|_{\alpha/2} &\le |g_p| \cdot |p_x\Gamma' + p_t|_{\alpha/2} + (|g_{p\Gamma}| \cdot |\Gamma|_{\alpha/2} + |g_{pt}|) \cdot |p_x\Gamma' + p_t|_0 \\ &\quad + |g_{pp}|(|p_x||\Gamma|_{\alpha/2} + |p|_\alpha)(|p_x\Gamma' + p_t|_0 + 1) \\ &\le C[1 + \|p_x\|_\alpha|\Gamma|_1 + |\Gamma|_{1+\alpha/2} + \|p_{xx}\|_\alpha + (1 + |\Gamma|_{\alpha/2})(|\Gamma|_1 + \|p_{xx}\|_0)] \\ &\quad + (|\Gamma|_{\alpha/2} + |p|_\alpha)(|\Gamma|_1 + |p_{xx}|_0 + 1) \\ &\le C[1 + (|\Gamma|_1 + |v_x|_0)^2 + \|v\|_\alpha|\Gamma|_1 \\ &\quad + |\Gamma|_{1+\alpha/2} + \|v\|_{1+\alpha} + (1 + |\Gamma|_1)(|\Gamma|_1 + \|v_x\|_0). \end{aligned}$$

Since for $\alpha \le \beta$,

$$\|v\|_{1+\alpha} \le C(1 + \|v\|_{1+\beta}), \quad \text{and} \quad \|v\|_\alpha \le C\|v\|_{1+\beta}^{\alpha/(1+\beta)},$$

from (4.18) and (4.19) we obtain

$$|g_p(p_x\Gamma' + p_t)|_{\alpha/2} \le C(1 + |\Gamma|_1^{2/\beta} + |\Gamma|_{1+\alpha/2}). \tag{4.24}$$

Combining (4.23) and (4.24) we therefore conclude that

$$|g|_{1+\alpha/2} \le C(1 + |\Gamma|_1^{2/\beta} + |\Gamma|_{1+\alpha/2}). \tag{4.25}$$

Next we estimate the term $\|h\|_\alpha$ in (4.20).

$$\begin{aligned} |h|_\alpha &\le C(\|v\|_\alpha + \|p\|_\alpha\|v_x\|_0 \\ &\quad + \|p\|_0\|v_x\|_\alpha + |\Gamma|_{1+\alpha/2}\|v_x\|_0 + |\Gamma|_1\|v_x\|_\alpha) \\ &\le C(1 + \|v\|_{1+\beta}^{\alpha/(1+\beta)} + \|p\|_\alpha\|v_x\|_0 + |\Gamma|_{1+\alpha/2}\|v_x\|_0 + |\Gamma|_1\|v\|_{1+\alpha}). \end{aligned}$$

By virtue of (4.18), (4.19) and (4.22), and noting that for $\alpha \le \beta$,

$$\|v\|_{1+\alpha} \le C(1 + \|v\|_{1+\beta}),$$

we obtain

$$\|h\|_\alpha \le C(1 + |\Gamma|_1^{2+1/\beta} + |\Gamma|_{1+\alpha/2}|\Gamma|_1^{1/\beta}).$$

Hence (4.20) reduces to

$$\|v\|_{2+\alpha,Q} \le C[1 + |\Gamma|_1^{2+1/\beta} + |\Gamma|_{1+\alpha/2}(|\Gamma|_1^{1/\beta} + 1)]. \tag{4.26}$$

By the interpolation inequality we thus obtain

$$\|v_{xx}\|_0 \le C\|v\|_{2+\alpha}^{2/(2+\alpha)} \le C[1 + |\Gamma|_1^{2+1/\beta} + |\Gamma|_{1+\alpha/2}(|\Gamma|_1^{1/\beta} + 1)]^{2/(2+\alpha)}. \tag{4.27}$$

Suppose for a moment that

$$|\Gamma|_{1+\alpha/2} \le C(1 + \|\theta_x\|_0^2). \tag{$**$}$$

Noticing that $|\Gamma|_1 \le C\|\theta_x\|_0 + C$, we reduce (4.27) to

$$\|v_{xx}\|_0 \le C(1 + \|\theta_x\|_0^{(1+2\beta)/\beta})^{2/(2+\alpha)}. \tag{4.28}$$

From (4.12) and by (4.19), (4.28), we conclude that

$$\|\theta_x\|_0 \le C(1 + \|\widetilde{\theta}_x\|_0) \le C(1 + \|\theta_x\|_0^{1/2\beta} + \|\theta_x\|_0^{2(1+2\beta)/3\beta(2+\alpha)}).$$

Let $\beta = \frac{5}{6}$ and let $\alpha = \frac{1}{2}$. We thus obtain $\|\theta_x\|_0 \le C$.

Now we need only to prove ($**$). By Schauder estimates we see that θ^-, θ^+ and p are smooth in $\Omega \backslash N_\delta(F)$, where $N_\delta(F) = \{P = (x,t);\ \mathrm{dist}(P,F) < \delta\}$. Let

$$(y,s) = T(x,t) = (x - \Gamma(t), t). \tag{4.29}$$

Then θ^+ satisfies

$$\theta_s - a_2\theta_{yy} = h =: [c_1\rho^2 + c_2p_y]\theta_y + (c_3\rho p_y + c_4p_{yy})\theta + \Gamma'(s)\theta_y.$$

Hence in the domain $\{0 < y < \delta_0\} \times \{0 < t < 1 - \varepsilon\}$ we have

$$\begin{aligned}\|\theta^+(y,s)\|_{1+\alpha} &\le C(1 + \|\theta\|_0 + \sup|h|)\\ &\le C(1 + \|\theta_y\|_0 + \|p_{yy}\|_0 + |\Gamma|_1\|\theta_y\|_0).\end{aligned}$$

By virtue of (4.19) and noticing that $|\Gamma|_1 \le C\|\theta_y\|_0$, we obtain

$$\|\theta^+(y,s)\|_{1+\alpha} \le C(1 + \|\theta_y\|_0^2 + \|\theta_y\|_0^{1/\beta}). \tag{4.30}$$

In the domain $\{-\delta_0 < y < 0\} \times \{0 < t < 1 - \varepsilon\}$, similarly we have

$$\|\theta^-(y,s)\|_{1+\alpha} \le C(1 + \|\theta_y\|_0^2 + \|\theta_y\|_0^{1/\beta}). \tag{4.31}$$

From the Stefan condition

$$\Gamma' = k_1\theta_x^- - k_2\theta_x^+ - \widetilde{g}(\Gamma, p, t) \tag{3.4'}$$

we conclude that, under the new coordinates (y,s),

$$\begin{aligned}|\Gamma|_{1+\alpha/2} &\le |\widetilde{g}|_{\alpha/2} + C\sup\frac{\theta_y^\pm(0,s_2) - \theta_y^\pm(0,s_1)}{|s_2 - s_1|^{\alpha/2}}\\ &\le |\widetilde{g}|_{\alpha/2} + C\|\theta^\pm(y,s)\|_{1+\alpha}.\end{aligned}$$

Hence ($**$) holds.

Remark 4.2. Since $g(\Gamma,p,t) \in C^{1,1}$, the free boundary $\Gamma(t)$ is indeed $C^{2+\alpha}(0,T)$ for any $\alpha \in (0,1)$. If $g(\Gamma,p,t) \in C^\infty$, then $\Gamma(t) \in C^\infty$. For the proof we refer the reader to [6] or [9]. We also remark that Theorem 4.1 can be extended to the unbounded situation $\Omega = \{0 < x < \infty, 0 < t < T\}$.

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