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MINIMUM PERIOD CONTROL PROBLEM FOR INFINITE DIMENSIONAL SYSTEM**

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Abstract

In order to solve the so-called minimum period control problem for a class of abstract evolutionary systems, the authors study an infinite dimensional time optimal control problem with mixed type target set. To the latter problem complete results are established, which then are applied to the former to derive the desirable answer.

Keywords Abstract evolutionary system, Minimum period control problem, Infinite dimensional time optimal control problem, Mixed type target set

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§1. Introduction

Let X be a separable reflexive Banach space and $e^{\cdot A}$ be a compact C_0-semigroup on X. Our main purpose of this paper is to solve the following so-called minimum period control problem:

Problem MPCP. Find a $\hat{T} \in (0, +\infty)$, an $\hat{x}_0 \in Q_1$ and a $\hat{u}(\cdot) \in \mathcal{U}_0 \equiv \{u(\cdot) : \mathrm{R}_+ \to U \mid u(\cdot)$ is measurable $\}$ such that

$$\hat{u}(t + \hat{T}) = \hat{u}(t), \qquad \text{a.e. } t \in \mathrm{R}_+, \tag{1.1}$$

$$x(t + \hat{T}; \hat{x}_0, \hat{u}(\cdot)) = x(t; \hat{x}_0, \hat{u}(\cdot)), \qquad \forall t \in \mathrm{R}_+, \tag{1.2}$$

$$\begin{aligned}\hat{T} = \inf\{T \in (0, +\infty) \mid \exists u(\cdot) \in \mathcal{U}_0, x_0 \in Q_1 \text{ such that } u(t + T) = u(t), \\ x(t + T; x_0, u(\cdot)) = x(t, x_0, u(\cdot)), \text{ a.e. } t \in \mathrm{R}_+\},\end{aligned} \tag{1.3}$$

where Q_1 is a bounded closed sphere with positive radius in X, U is a bounded subset of some Banach space Z, $U \neq \emptyset$, $B(\cdot) : Z \to X$ is a continuous map and

$$x(t; x_0, u(\cdot)) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}B(u(\tau))d\tau, \qquad t \in \mathrm{R}_+ \tag{1.4}$$

for any $(x_0, u(\cdot)) \in X \times \mathcal{U}_0$.

Denote

$$\begin{aligned}Q_{\mathrm{MPCP}} &= \{(y, y) \mid y \in Q_1\}, \\ R_{\mathrm{MPCP}}(t) &= \{(x_0, x(t; x_0, u(\cdot)) \mid x_0 \in Q_1, u(\cdot) \in \mathcal{U}_0\}, \qquad \forall t \in \mathrm{R}_+.\end{aligned} \tag{1.5}$$

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Definition 1.1. *Problem* MPCP *is called solvable if and only if*

$$\hat{T} \equiv \inf\{T \in (0,+\infty) \mid Q_{\text{MPCP}} \cap \overline{R_{\text{MPCP}}(T)} \neq \emptyset\} \in (0,+\infty) \tag{1.6}$$

(*here and hereafter, we always stipulate* $\inf \emptyset = +\infty$) *holds and*

$$Q_{\text{MPCP}} \cap R_{\text{MPCP}}(\hat{T}) \neq \emptyset. \tag{1.7}$$

Remark 1.1. If Problem MPCP is solvable, then from (1.7) there exist an $\hat{x}_0 \in Q_1$ and a $\hat{v}(\cdot) \in \mathcal{U}_0$ such that

$$x(\hat{T}; \hat{x}_0, \hat{v}(\cdot)) = \hat{x}_0. \tag{1.8}$$

Let

$$\hat{u}(t) = \hat{v}(t-(n-1)\hat{T}), \qquad (n-1)\hat{T} \le t < n\hat{T}, \qquad n = 1,\ 2,\ \cdots. \tag{1.9}$$

Then it is obvious that $\hat{u}(\cdot)$ is a minimum period control.

Definition 1.2. *Problem* MPCP *is called approximately solvable if and only if* (1.6) *holds and*

$$Q_{\text{MPCP}} \cap \overline{R_{\text{MPCP}}(\hat{T})} \neq \emptyset. \tag{1.10}$$

With the help of our results obtained in Section 4, we can deduce the following result to Problem MPCP (see Section 5).

Theorem 1.1. *Let the assumptions on* Q_1, U *and* $B(\cdot)$ *posed in the statement of Problem* MPCP *hold. Then Problem* MPCP *is approximately solvable if and only if the following equation*

$$f_0(t) = \inf\Big\{g_0(t,\xi,\eta) + \int_0^t \sup_{u\in U}\langle \eta, e^{(t-\tau)A}B(u)\rangle d\tau \mid \xi,\eta \in X^*, \|(\xi,\eta)\| \le 1\Big\} = 0 \tag{1.11}$$

admits of minimum positive root. In this case $\hat{T}$ *happens to equal the minimum positive root. Furthermore, if Problem* MPCP *is solvable and* $(\hat{x}_0, \hat{u}(\cdot)) \in Q_1 \times \mathcal{U}_0$ *is a solution of Problem* MPCP, *then there exists a nontrivial function* $\hat{\psi}(\cdot) \in C([0,\hat{T}], X^*)$ *which along with* $x(\cdot;\hat{x}_0,\hat{u}(\cdot))$ *satisfies*

$$\max_{u\in U}\langle \hat{\psi}(t), B(u)\rangle = \langle \hat{\psi}(t), B(\hat{u}(t))\rangle, \quad \text{a.e. } t \in [0,\hat{T}], \tag{1.12}$$

$$\hat{\psi}(t) = e^{(\hat{T}-t)A^*}\hat{\psi}(\hat{T}), \qquad \forall t \in [0,\hat{T}], \tag{1.13}$$

$$\begin{cases} \hat{x}_0 = x(\hat{T}; \hat{x}_0, \hat{u}(\cdot)), \\ \langle \hat{\psi}(0) - \hat{\psi}(\hat{T}), x_0 - \hat{x}_0\rangle \le 0, \qquad \forall x_0 \in Q_1 \end{cases} \tag{1.14}$$

where

$$g_0(t,\phi,\psi) \equiv \sup_{x_0,x\in Q_1}(\langle \phi, x_0 - x\rangle + \langle \psi, e^{tA}x_0 - x\rangle), \qquad \forall t \in \mathrm{R}_+,\ \phi,\psi \in X^*. \tag{1.15}$$

(In the above theorem and hereafter, we let $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the norms and the dual product on $X^* \times X$).

It is not hard to verify $R_{\text{MPCP}}(t)$ is closed for any $t \in \mathrm{R}_+$ if we further assume $B(\cdot)$ is affine and U is convex (see [12]). Hence Theorem 1.1 has the following consequence.

Theorem 1.2. *Let all assumptions of Theorem* 1.4 *hold and*

$$B(u) = Cu + h \tag{1.16}$$

where $C \in \mathcal{L}(Z, X)$ *and* $h \in X$. *Then Problem* MPCP *is solvable if and only if equation* (1.11) *admits of minimum positive root and all the other conclusions of Theorem* 1.1 *are valid.*

The remainder of this paper is devided into four parts. In Section 2 we state and explain an infinite dimensional time optimal control problem with mixed type target set. In Section 3 we give some preliminary results. In Section 4 we establish several results to the time optimal control problem posed in Section 2, Section 5 is devoted to the proof of Theorem 1.1 by using the results in Section 4.

§2. Time Optimal Control Problem with Mixed Target Set

Consider the following abstract evolutionary controlled system on X

$$\begin{cases} x(t; x_0, u(\cdot)) = e^{tA}x_0 + \int_0^t e^{(t-\tau)A}u(\tau)d\tau, \quad u(\tau) \in U(\tau) \subseteq X, \ \tau \in [0, t], \quad t \in \mathrm{R}_+, \\ x_0 \in Q_0 \subseteq X \end{cases} \tag{2.1}$$

and the target set map

$$Q(\cdot) : \mathrm{R}_+ \to 2^{X \times X}, \tag{2.2}$$

where

(H1) X is a separable reflexive Banach space and $e^{\cdot A} : \mathrm{R}_+ \to \mathcal{L}(X)$ is a compact C_0-semigroup on X;

(H2) $2^Y \equiv \{S \subseteq Y \mid S$ is bounded and closed$\}$ for any normed space Y, $U(\cdot) : \mathrm{R}_+ \to 2^X$ is Hausdorff continuous and our admissible control function set is $\mathcal{U} \equiv \{u(\cdot) : \mathrm{R}_+ \to X \mid u(\cdot)$ is measurable, $u(t) \in U(t)$, a.e. $t \in \mathrm{R}_+\}$;

(H3) Q_0 is a bounded and closed convex subset of X, $Q(\cdot) : \mathrm{R}_+ \to 2^{X \times X}$ is Hausdorff continuous and $Q(t)$ is convex for any $t \in \mathrm{R}_+$.

Remark 2.1. $\mathcal{U} \neq \emptyset$ (according to Theorem 2 in [9]).

Let

$$R(t) \equiv \{(x_0, x(t; x_0, u(\cdot))) \mid x_0 \in Q_0, \ u(\cdot) \in \mathcal{U}\}, \qquad \forall t \in \mathrm{R}_+ \tag{2.3}$$

and $d(S_1, S_2) \equiv \inf\{\| y_1 - y_2 \| \mid y_i \in S_i, \ i = 1, 2\}$ for any normed space Y, $S_i \subseteq Y$, $i = 1, 2$. We consider the following time optimal control problem.

Problem P. Find an $\hat{x}_0 \in Q_0$ and a $\hat{u}(\cdot) \in \mathcal{U}$ such that

$$d((\hat{x}_0, x(t; \hat{x}_0, \hat{u}(\cdot))), Q(\hat{t})) = \hat{r} \equiv \inf_{t \in \mathrm{R}_+} d(R(t), Q(t)) \tag{2.4}$$

where

$$\hat{t} \equiv \inf\{t \in \mathrm{R}_+ \mid d(R(t), Q(t)) = \hat{r}\}. \tag{2.5}$$

Throughout this section, Section 3 and Section 4 we assume that

(H4) $r_0 \equiv d(\tilde{Q}_0, Q(0)) > \hat{r}$ where

$$\tilde{Q}_0 \equiv \{(x_0, x_0) \mid x_0 \in Q_0\}. \tag{2.6}$$

Definition 2.1. *We say Problem* P *is solvable if and only if* $\hat{t} \in (0, +\infty)$ *and there exists a pair* $(\hat{x}_0, \hat{u}(\cdot)) \in Q_0 \times \mathcal{U}$ *such that* (2.4) *holds.*

Definition 2.2. *We say Problem* P *is approximately solvable if and only if* $\hat{t} \in (0, +\infty)$ *and*

$$[Q(\hat{t}) + \hat{r}\bar{O}_1] \cap \overline{R(\hat{t})} \neq \emptyset \tag{2.7}$$

holds. (In (2.7) *and hereafter* $\bar{O}_1$ *denotes the closed unit ball with center at the origin in* $X \times X$*).*

Remark 2.2. It is easy to show (under (H1)–(H3)) $\overline{R(t)}$ is convex for any $t \in \mathrm{R}_+$ by means of a property of vector-valued measures (see [13]).

Remark 2.3. The following control system

$$\begin{cases} x(t; x_0, v(\cdot)) = e^{tA}x_0 + \displaystyle\int_0^t e^{(t-\tau)A}B(\tau, v(\tau))d\tau, \quad v(\tau) \in V(\tau) \subseteq Z,\ \tau \in [0, t], \quad t \in \mathrm{R}_+, \\ x_0 \in Q_0 \subseteq X, \end{cases} \tag{2.8}$$

where Z is a separable Banach space, $B(\cdot, \cdot) : \mathrm{R}_+ \times Z \to X$ is a continuous map and $V(\cdot) : \mathrm{R}_+ \to 2^Z$ is a Hausdorff continuous set-valued map, which seems to be more general than (2.1), can be referred to the latter actually. To see this, suffice it to set

$$U(t) = \{B(t, v) \mid v \in V(t)\}. \tag{2.9}$$

In fact, by Theorem 3 in [9], we immediately know

$$\mathcal{U} = \{B(\cdot, v(\cdot)) \mid v(\cdot) : \mathrm{R}_+ \to Z \ \text{ is measurable},\ v(t) \in V(t),\ \text{a.e. } t \in \mathrm{R}_+\} \tag{2.10}$$

to such $U(\cdot)$.

Problem P can be referred to the separated end type target set case by adding a suitable state equation like that has been done to finite dimensional optimal control problems with mixed endpoint constraints in [3]. Concretely speaking, one can directly verify that $(\hat{x}_0, \hat{u}(\cdot)) \in Q_0 \times \mathcal{U}$ is an optimal solution to Problem P if and only if $\left(\begin{pmatrix} I_X \\ I_X \end{pmatrix} \hat{x}_0, \begin{pmatrix} 0 \\ \hat{u}(\cdot) \end{pmatrix} \right)$ is optimal to the following problem.

Problem $\widetilde{\mathrm{P}}$. Find a pair $(\hat{\tilde{x}}_0, \hat{\tilde{u}}(\cdot)) \in \tilde{Q}_0(\hat{\tilde{t}}) \times \tilde{\mathcal{U}}$ such that

$$d(\tilde{x}(\hat{\tilde{t}}; \hat{\tilde{x}}_0, \hat{\tilde{u}}(\cdot)), Q(\hat{\tilde{t}})) = \hat{\tilde{r}} \equiv \inf_{t \in \mathrm{R}_+} d(\tilde{R}(t), Q(t)) \tag{2.11}$$

where

$$\tilde{U} \equiv \{\tilde{u}(\cdot) : \mathrm{R}_+ \to X \times X \mid \tilde{u}(\cdot) \text{ is measurable},\ \tilde{u}(t) \in \{0\} \times U(t),\ \text{a.e. } t \in \mathrm{R}_+\}, \tag{2.12}$$

$$\widetilde{R}(t) \equiv \{\tilde{x}(t; \tilde{x}_0, \tilde{u}(\cdot)) \mid \tilde{x}_0 \in \tilde{Q}_0,\ \tilde{u}(\cdot) \in \tilde{U}\}, \qquad t \in \mathrm{R}_+, \tag{2.13}$$

$$\hat{\tilde{t}} \equiv \inf\{t \in \mathrm{R}_+ \mid d(\tilde{R}(t), Q(t)) = \hat{\tilde{r}}\}, \tag{2.14}$$

$$\tilde{x}(t; \tilde{x}_0, \tilde{u}(\cdot)) = e^{t\tilde{A}}\tilde{x}_0 + \int_0^t e^{(t-\tau)\tilde{A}}\tilde{u}(\tau)d\tau, \quad \forall t \in \mathrm{R}_+,\ \tilde{x}_0 \in X \times X,\ \tilde{u}(\cdot) \in \tilde{U}, \tag{2.15}$$

where

$$\tilde{A} = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}. \tag{2.16}$$

However, $\begin{pmatrix} I_X & 0 \\ 0 & e^{\cdot A} \end{pmatrix}$, the semigroup generated by $\tilde{A}$, is not compact. Besides, the conditions of Theorem 4.4 and Theorem 4.5 (see Section 4) do not imply $Q(\hat{\tilde{t}})$ is finite codimensional in $X \times X$.

§3. Several Preliminary Results

In this section, we present several auxiliary results, which will play an important role in the discussions of the next section. Define

$$f(t,r)=\inf\Big\{g(t,r,\xi,\eta)+\int_0^t\sup_{u\in U(\tau)}\langle\eta,e^{(t-\tau)A}u\rangle d\tau\Big|\xi,\eta\in X^*,\ \|(\xi,\eta)\|=1\Big\},\quad\forall t,r\in\mathrm{R}_+,\tag{3.1}$$

where
$$g(t,r,\phi,\psi)=\sup\{\langle\phi,x_0-y\rangle+\langle\psi,e^{tA}x_0-z\rangle\mid x_0\in Q_0,\ (y,z)\in Q(t)+r\bar{O}_1\},\quad \forall t,r\in\mathrm{R}_+,\ \phi,\psi\in X^*.\tag{3.2}$$

Noticing that $e^{\cdot A}:\mathrm{R}_+\to\mathcal{L}(X)$ is a compact C_0-semigroup and $f(t,\cdot):\mathrm{R}_+\to\mathrm{R}^1$ is a Lip-1 function uniformly in $t\in\mathrm{R}_+$, by some meticulous estimations, we can obtain the following two lemmas (cf. the proofs of the similar results in [11]).

Lemma 3.1. *Let* (H1)–(H3) *hold. Then*

$$f(\cdot,\cdot)\in C((0,+\infty)\times\mathrm{R}_+,\mathrm{R}^1).\tag{3.3}$$

Lemma 3.2. *Let* (H1)–(H3) *hold. Then* $\overline{R(\cdot)}:(0,+\infty)\to 2^{X\times X}$ *is Hausdorff continuous.*

Lemma 3.3. *Let* (H1)–(H4) *hold. Then for each* $r\in[0,r_0)$,

$$\exists t\in(0,+\infty),\ [Q(t)+r\bar{O}_1]\cap\overline{R(t)}\neq\emptyset\qquad\Longleftrightarrow\qquad r\in D_0\tag{3.4}$$

where
$$D_0\equiv\{r\in[0,r_0)\mid\exists t\in(0,+\infty)\ \textit{such that}\ f(t,r)=0\}.\tag{3.5}$$

Proof. First of all, we have

$$r_0\geq\hat{r}\geq 0\tag{3.6}$$

since (H4) holds. From the definition of r_0, we immediately know

$$[Q(0)+r\bar{O}_1]\cap\tilde{Q}_0=\emptyset,\qquad\forall r\in[0,r_0).\tag{3.7}$$

Then, according to the Hausdorff continuity of $Q(\cdot):\mathrm{R}_+\to 2^{X\times X}$ at 0, the boundedness of $U(t)$ for any $t\in\mathrm{R}_+$ and the Hausdorff continuity of $U(\cdot):\mathrm{R}_+\to 2^X$, we easily know there exists a positive integer $\nu_0(r)$ such that

$$\left.\begin{aligned}&\int_0^t\sup_{u\in U(\tau)}\langle\psi,e^{(t-\tau)A}u\rangle dt\leq\frac{1}{8}\rho_0(r),\quad\forall\psi\in\bar{O}_1^*,\\&[Q(t)+r\bar{O}_1]\cap\tilde{Q}_0=\emptyset,\end{aligned}\right.\quad t\in[0,\frac{1}{\nu_0(r)}],\ r\in[0,r_0),\tag{3.8}$$

where $\bar{O}_1^*$ denotes the closed unit ball with center at the origin in X^* and

$$\rho_0(r)\equiv\min_{t\in[0,\frac{1}{\nu_0(r)}]}d(Q(t)+r\bar{O}_1,\tilde{Q}_0),\qquad\forall r\in[0,r_0).\tag{3.9}$$

Take a positive integer $\nu_1(r)\geq\nu_0(r)$ being big enough so that

$$\begin{aligned}&\Big|\inf\{\langle\phi,y\rangle+\langle\psi,z\rangle\mid(y,z)\in Q(t)+r\bar{O}_1\}\\&-\inf\Big\{\langle\phi,y\rangle+\langle\psi,z\rangle\mid(y,z)\in Q\Big(\frac{1}{\nu_1(r)}\Big)+r\bar{O}_1\Big\}\Big|\leq\frac{1}{8}\rho_0(r),\\&\|(\phi,\psi)\|=1,\phi,\psi\in X^*,\quad t\in\Big[0,\frac{1}{\nu_1(r)}\Big],\ r\in[0,r_0).\end{aligned}\tag{3.10}$$

By Eidelheit's Theorem, there exist $\phi_1(r),\ \psi_1(r)\in X^*$, $\|\ (\phi_1(r),\psi_1(r))\ \|=1$ such that

$$\begin{aligned}&\frac{1}{2}+\sup\{\langle\phi_1(r)+\psi_1(r),x_0\rangle\mid x_0\in Q_0\}\\&\le\inf\Big\{\langle\phi_1(r),y\rangle+\langle\psi_1(r),z\rangle\mid(y,z)\in Q\Big(\frac{1}{\nu_0(r)}\Big)+r\bar{O}_1\Big\},\qquad\forall r\in[0,r_0).\end{aligned}\tag{3.11}$$

Now, take a positive integer $\nu_2(r)\ge\nu_1(r)$ being big enough so that

$$\|\ (e^{tA^*}-I_{X^*})\psi_0(r)\ \|<\frac{\rho_0(r)}{8(M_0+1)},\qquad\forall t\in[0,\frac{1}{\nu_2(r)}],\ r\in[0,r_0),\tag{3.12}$$

where

$$M_0\equiv\sup_{x_0\in Q_0}\|\ x_0\ \|\tag{3.13}$$

(the reflexivity of X guarantees $e^{\cdot A^*}:\mathrm{R}_+\to\mathcal{L}(X^*)$, the dual semigroup of $e^{\cdot A}:\mathrm{R}_+\to\mathcal{L}(X)$, is strongly continuous too[5]). The inequality in (3.8) and (3.10)-(3.12) imply

$$\begin{aligned}g(t,r,\phi_1(r),\psi_1(r))+\int_0^t\sup_{u\in U(\tau)}\langle\psi_1(r),e^{(t-\tau)A}u\rangle d\tau\le-\frac{1}{8}\rho_0(r)<0,\\\forall t\in[0,\frac{1}{\nu_2(r)}],\ r\in[0,r_0).\end{aligned}\tag{3.14}$$

Hence we have

$$f(t,r)\le-\frac{1}{8}\rho_0(r)<0,\qquad\forall t\in[0,\frac{1}{\nu_2(r)}],\ r\in[0,r_0).\tag{3.15}$$

For any $(t,r)\in(0,+\infty)\times[0,r_0)$, if

$$[Q(t)+r\bar{O}_1]\cap\overline{R(t)}\ne\emptyset,\tag{3.16}$$

then there exists a $(y_0,z_0)\in Q(t)+r\bar{O}_1$, two sequences $\{y_n\}\in Q_0$ and $\{u_n(\cdot)\}\subseteq\mathcal{U}$ such that

$$\|\ (y_n-y_0,x(t;y_n,u_n(\cdot))-z_0)\ \|<\frac{1}{n},\qquad n=1,2,\cdots,\tag{3.17}$$

which results in

$$\begin{aligned}f(t,r)&\ge\inf\Big\{\langle\phi,y_n-y_0\rangle+\langle\psi,\ e^{tA}y_n-z_0\rangle\\&\qquad+\int_0^t\langle\psi,e^{(t-\tau)A}u_n(\tau)\rangle d\tau\mid\phi,\psi\in X^*,\ \|\ (\phi,\psi)\ \|=1\}\\&=\inf\{\langle\phi,y_n-y_0\rangle+\langle\psi,x(t;y_n,u_n(\cdot))-z_0\rangle\mid\phi,\psi\in X^*,\ \|\ (\phi,\psi)\ \|=1\}\\&\ge-\frac{1}{n},\quad n=1,2,\cdots.\end{aligned}\tag{3.18}$$

Letting $n\to\infty$ in (3.18), we get

$$f(t,r)\ge0.\tag{3.19}$$

(3.15), (3.19) and the continuity of $f(\cdot,\cdot):(0,+\infty)\times\mathrm{R}_+\to\mathrm{R}^1$ (Lemma 3.1) ensure there exists a $t_r\in(0,t]$ such that $f(t_r,r)=0$. Thus $r\in D_0$.

On the contrary, let $(t,r)\in(0,+\infty)\times\mathrm{R}_+$ such that

$$f(t,r)=0.\tag{3.20}$$

If

$$[Q(t)+r\bar{O}_1]\cap\overline{R(t)}=\emptyset,\tag{3.21}$$

then, by Ascoli-Mazur Theorem, we know

$$\rho(t,r) \equiv d(Q(t)+r\bar{O}_1, R(t)) > 0, \tag{3.22}$$

which yields

$$[Q(t)+r\bar{O}_1] \cap \Big[\overline{R(t)} + \frac{1}{2}\rho(t,r)\bar{O}_1\Big] = \emptyset. \tag{3.23}$$

Thus, by Eidelheit's Theorem, there exist $\bar{\phi}$, $\bar{\psi} \in X^*$ such that $\| (\phi,\psi) \| = 1$ and

$$\begin{aligned}&\sup\{\langle\bar{\phi}, x_0 + \frac{1}{2}\rho(t,r)e'\rangle + \langle\bar{\psi}, x_1 + \frac{1}{2}\rho(t,r)e"\rangle \mid (x_0,x_1) \in \overline{R(t)},\ (e',e") \in \bar{O}_1\}\\ &\le \inf\{\langle\bar{\phi}, y\rangle + \langle\bar{\psi}, z\rangle \mid (y,z) \in Q(t)+r\bar{O}_1\},\end{aligned} \tag{3.24}$$

which implies

$$\begin{aligned}&\frac{1}{2}\rho(t,r) + \langle\bar{\phi}, x_0\rangle + \langle\bar{\psi}, x(t;x_0,u(\cdot))\rangle\\ &\le \inf\{\langle\bar{\phi}, y\rangle + \langle\bar{\psi}, z\rangle \mid (y,z) \in Q(t)+r\bar{O}_1\}, \qquad \forall x_0 \in Q_0,\ u(\cdot) \in \mathcal{U}.\end{aligned} \tag{3.25}$$

It is obvious that for any $\Delta t \in (0,t)$,

$$\max_{v \in e^{\Delta tA}U(s)} \langle\bar{\psi}, e^{(t-\Delta A-s)A}v\rangle = \sup_{u \in U(s)} \langle\bar{\psi}, e^{(t-s)A}u\rangle, \qquad \forall s \in [0, t-\Delta t]. \tag{3.26}$$

By Theorem 3 in [9], we know there exists a $u_{\bar{\psi},\Delta t}(\cdot) \in \mathcal{U}$ such that

$$\langle\bar{\psi}, e^{(t-s)A}u_{\bar{\psi},\Delta t}(s)\rangle = \max_{v \in e^{\Delta tA}U(s)} \langle\bar{\psi}, e^{(t-\Delta t-s)A}u\rangle, \qquad s \in [0, t-\Delta t]. \tag{3.27}$$

(3.26) and (3.27) lead to (with the help of Lebesgue's Dominated Convergence Theorem)

$$\begin{aligned}\int_0^t \sup_{u\in U(s)} \langle\bar{\psi}, e^{(t-s)A}u\rangle ds &= \int_0^t \lim_{\Delta t\downarrow 0} \chi_{[0,t-\Delta t]}(s) \max_{v \in e^{\Delta tA}U(s)} \langle\bar{\psi}, e^{(t-\Delta t-s)A}v\rangle ds\\ &= \lim_{\Delta t\downarrow 0} \int_0^t \chi_{[0,t-\Delta t]}(s) \max_{v \in e^{\Delta tA}U(s)} \langle\bar{\psi}, e^{(t-\Delta t-s)A}v\rangle ds\\ &= \lim_{\Delta t\downarrow 0} \int_0^t \langle\bar{\psi}, e^{(t-s)A}\chi_{[0,t-\Delta t]}(s)u_{\bar{\psi},\Delta t}(s)\rangle ds\\ &\le \sup_{u(\cdot)\in\mathcal{U}} \int_0^t \langle\bar{\psi}, e^{(t-s)A}u(s)\rangle ds,\end{aligned} \tag{3.28}$$

while the converse inequality is true apparently. Hence we have

$$\sup_{u(\cdot)\in\mathcal{U}} \int_0^t \langle\bar{\psi}, e^{(t-s)A}u(s)\rangle ds = \int_0^t \sup_{u\in U(s)} \langle\bar{\psi}, e^{(t-s)A}u\rangle ds. \tag{3.29}$$

(3.25) together with (3.29) implies

$$g(t,r,\bar{\phi},\bar{\psi}) + \int_0^t \sup_{u\in U(s)} \langle\bar{\psi}, e^{(t-s)A}u\rangle ds \le -\frac{1}{2}\rho(t,r). \tag{3.30}$$

Therefore,

$$f(t,r) \le -\frac{1}{2}\rho(t,r) < 0, \tag{3.31}$$

which contradicts (3.20).

Remark 3.1. Let (H1)-(H4) hold. From the proof of Lemma 2.3 we know if $\hat{t} < +\infty$ then

$$\hat{t} \in \Big[\frac{1}{\nu_2(r)}, +\infty\Big) \subseteq (0,+\infty). \tag{3.32}$$

§4. Results on Problem P

Now, we would like to discuss the solvability of Problem P, give a formula for determining $(\hat{t}, \hat{r})$, derive a sufficient and a necessary optimality condition of Problem P, which will be used to prove Theorem 1.1 in Section 5.

Theorem 4.1. *Let* (H1)–(H4) *hold. Then*

$$\hat{r} = \inf D_0. \tag{4.1}$$

Proof. From (3.4) we know

$$\forall r \in D_0, \qquad \exists t \in (0, +\infty) \text{ such that } d(Q(t), R(t)) \le r, \tag{4.2}$$

which yields

$$\hat{r} \le r, \qquad \forall r \in D_0. \tag{4.3}$$

Therefore, we have

$$\hat{r} \le \inf D_0. \tag{4.4}$$

On the other hand, according to the definition of $\hat{r}$, there exists a sequence $\{t_n\} \subseteq (0, +\infty)$ such that

$$\hat{r} \le d(Q(t_n), R(t_n)) < \hat{r} + \frac{1}{n}, \qquad n = 1, 2, \cdots, \tag{4.5}$$

which apparently implies

$$\{Q(t_n) + [d(Q(t_n), R(t_n)) + \frac{1}{n}]\bar{O}_1\} \cap \overline{R(t_n)} \ne \emptyset, \qquad n = 1, 2, \cdots, \tag{4.6}$$

$$\lim_{n\to\infty} d(Q(t_n), R(t_n)) = \hat{r}. \tag{4.7}$$

From (H4), we have $\hat{r} < r_0$. Thus we can take a positive integer N_0 being big enough so that

$$d(Q(t_n), R(t_n)) + \frac{1}{n} < r_0, \quad n \ge N_0 \tag{4.8}$$

(recall (4.7)). (4.6) along with (4.8) means

$$d(Q(t_n), R(t_n)) + \frac{1}{n} \in D_0, \quad n \ge N_0. \tag{4.9}$$

Hence we have

$$d(Q(t_n), R(t_n)) + \frac{1}{n} \ge \inf D_0, \qquad n \ge N_0. \tag{4.10}$$

Finally, letting $n \to \infty$ in (4.10) we get

$$\hat{r} \ge \inf D_0. \tag{4.11}$$

(4.1) follows from (4.4) and (4.11) immediately.

To give the next results, we need the following definition.

Define an order on $\mathrm{R}_+ \times \mathrm{R}_+$: We say $(t_1, r_1) \preceq (t_2, r_2)$ if and only if $r_1 < r_2$, or $r_1 = r_2$ and $t_1 \le t_2$.

Obviously, Definition 2.2 has the following equivalent definition.

Definition 4.1. *We say Problem* P *is approximately solvable if and only if there exists a pair* $(\bar{t}, \bar{r}) \in \mathrm{R}_+ \times \mathrm{R}_+$ *such that*

$$\forall (t, r) \preceq (\bar{t}, \bar{r}),\ (t, r) \ne (\bar{t}, \bar{r}), \quad [Q(t) + r\bar{O}_1] \cap \overline{R(t)} = \emptyset, \tag{4.12}$$

$$[Q(\bar{t}) + \bar{r}\bar{O}_1] \cap \overline{R(\bar{t})} \ne \emptyset. \tag{4.13}$$

Remark 4.1. If such a $(\bar{t},\bar{r})$ as said in Definition 4.1 does exist, then it is not hard to see

$$\bar{t}=\hat{t},\ \bar{r}=\hat{r}. \tag{4.14}$$

Theorem 4.2. *Let* (H1)–(H4) *hold. Then Problem* P *is approximately solvable if and only if*

$$\inf D_0\in D_0 \tag{4.15}$$

holds. Moreover, in the case that (4.15) *holds,* $(\hat{t},\hat{r})$ *is the minimum* (*according to* "$\preceq$") *zero point of* $f(\cdot,\cdot)$ *in* $(0,+\infty)\times[0,r_0)$.

Proof. Let Problem P be approximately solvable, namely $\hat{t}<+\infty$ and

$$[Q(\hat{t})+\hat{r}\bar{O}_1]\cap\overline{R(\hat{t})}\neq\emptyset. \tag{4.16}$$

Then (noticing (H4)) we have

$$\hat{r}\in D_0 \tag{4.17}$$

according to Lemma 3.3. Thus, by Theorem 4.1 we immediately get (4.15). Conversely, let (4.15) hold. Then, also by Theorem 4.1, we know (4.17) holds. Consequently, we have

$$\exists t_0\in(0,+\infty),\ \text{such that } f(t_0,\hat{r})=0, \tag{4.18}$$

which implies (reviewing the proof of Lemma 3.3, we can find

$$\forall(t,r)\in(0,+\infty)\times\mathrm{R}_+,\ f(t,r)=0\quad\Longrightarrow\quad[Q(t)+r\bar{O}_1]\cap\overline{R(t)}\neq\emptyset \tag{4.19}$$

has been obtained there actually)

$$[Q(t_0)+\hat{r}\bar{O}_1]\cap\overline{R(t_0)}\neq\emptyset. \tag{4.20}$$

Therefore (recalling Remark 3.1), we have

$$\hat{t}\in(0,t_0]\subseteq(0,+\infty). \tag{4.21}$$

Hence, there exists a sequence $\{t_n\}\subseteq[\hat{t},+\infty)$ such that $t_n\downarrow\hat{t}$ $(n\to\infty)$ and

$$[Q(t_n)+\hat{r}\bar{O}_1]\cap\overline{R(t_n)}\neq\emptyset,\qquad n=1,2,\cdots. \tag{4.22}$$

By Lemma 3.3, from (4.22) we can conclude that

$$\exists s_n\in(0,t_n]\ \text{such that } f(s_n,\hat{r})=0,\quad n=1,2,\cdots. \tag{4.23}$$

By (4.19), we immediately know (4.23) implies

$$[Q(s_n)+\hat{r}\bar{O}_1]\cap\overline{R(s_n)}\neq\emptyset,\qquad n=1,2,\cdots. \tag{4.24}$$

This yields

$$s_n\geq\hat{t},\qquad n=1,2,\cdots. \tag{4.25}$$

Therefore, we have

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}t_n=\hat{t}(>0). \tag{4.26}$$

Thus, from (4.23) and the continuity of $f(\cdot,\cdot):(0,+\infty)\times\mathrm{R}_+\to\mathrm{R}^1$ (Lemma 3.1), we immediately get

$$f(\hat{t},\hat{r})=0, \tag{4.27}$$

which implies

$$[Q(\hat{t})+\hat{r}\bar{O}_1]\cap\overline{R(\hat{t})}\neq\emptyset \tag{4.28}$$

(here (4.19) is used again). The final conclusion of Theorem 4.2 is a natural consequence of (4.19), (4.27), the continuity of $f(\cdot,\cdot)$ and the optimality of $(\hat{t},\hat{r})$.

The next result we will give is a sufficient optimality condition to Problem P.

Theorem 4.3. *Let* (H1)–(H4) *hold. Let* $(\bar{t},\bar{r})$ *be the minimum* (*according to* "$\preceq$") *zero point of* $f(\cdot,\cdot)$ *in* $(0,+\infty)\times[0,r_0)$ *and there exist* $\bar{\psi}(\cdot)\in C([0,\bar{t}],X^*)$, $\bar{x}_0\in Q_0$, $\bar{u}(\cdot)\in\mathcal{U}$ *and* $(\bar{\xi},\bar{\eta})\in(\partial d_{Q(\bar{t})})((\bar{x}_0,x(\bar{t},\bar{x}_0,\bar{u}(\cdot))))$ *such that*

$$\bar{\psi}(t)=-e^{(\bar{t}-t)A^*}\bar{\eta},\qquad \forall t\in[0,\bar{t}], \tag{4.29}$$

$$\max_{u\in U(t)}\langle\bar{\psi}(t),u\rangle=\langle\bar{\psi}(t),\bar{u}(t)\rangle,\qquad \text{a.e. } t\in[0,\bar{t}], \tag{4.30}$$

$$\langle\bar{\psi}(0)-\bar{\xi},x_0-\bar{x}_0\rangle\le 0,\qquad \forall x_0\in Q_0. \tag{4.31}$$

Then $(\bar{x}_0,\bar{u}(\cdot))$ *is an optimal pair to Problem P.* (*In this theorem and hereafter, for any* y *belonging to a Banach space* Y, *we denote the generalized gradient*[4] *of* $d_S(\cdot)\equiv d(\{\cdot\},S):Y\to\mathrm{R}_+$ *at* y).

Proof. First, noticing that

$$(\bar{t},\bar{r})\in(0,+\infty)\times[0,r_0),\ f(\bar{t},\bar{r})=0\quad\Longrightarrow\quad\bar{r}\in D_0, \tag{4.32}$$

we immediately have

$$\bar{r}\ge\inf D_0. \tag{4.33}$$

If $\bar{r}>\inf D_0$, then we can take an $r\in(\inf D_0,\bar{r})$ and a $t\in(0,+\infty)$ such that

$$f(t,r)=0, \tag{4.34}$$

which contradicts the minimality of $(\bar{t},\bar{r})$. Therefore,

$$\bar{r}=\inf D_0, \tag{4.35}$$

(4.14) holds. Thus, according to Theorem 4.2, $(\hat{t},\hat{r})$ (also) is a minimum (in the sense of "$\preceq$") zero point of $f(\cdot,\cdot)$ in $(0,+\infty)\times[0,r_0)$. Hence we have

$$(\hat{t},\hat{r})=(\bar{t},\bar{r}); \tag{4.36}$$

Besides, we have

$$d(R(\hat{t}),Q(\hat{t}))=\hat{r}. \tag{4.37}$$

Second, we would like to show $(\bar{x}_0,\bar{u}(\cdot))$ is an optimal pair of Problem P. By the convexity of $Q(\bar{t})$ and the definition of generalized gradients (see [4]), we have

$$\begin{aligned}&d_{Q(\bar{t})}((x_0,x(\bar{t};x_0,u(\cdot))))\\ &\ge d_{Q(\bar{t})}((\bar{x}_0,x(\bar{t};\bar{x}_0,\bar{u}(\cdot))))+\langle\bar{\xi},x_0-\bar{x}_0\rangle\\ &\quad+\langle\bar{\eta},x(\bar{t};x_0,u(\cdot))-x(\bar{t};\bar{x}_0,\bar{u}(\cdot))\rangle,\quad\forall(x_0,u(\cdot))\in X\times\mathcal{U}.\end{aligned} \tag{4.38}$$

On the other hand, from (4.29)–(4.31) we have

$$\begin{aligned}&\langle\bar\xi, x_0-\bar x_0\rangle+\langle\bar\eta, x(\bar t;x_0,u(\cdot))-x(\bar t;\bar x_0,\bar u(\cdot))\rangle\\ &\ge\langle\bar\psi(0),x_0-\bar x_0\rangle-\langle\bar\psi(\bar t),x(\bar t;x_0,u(\cdot))-x(\bar t;\bar x_0,\bar u(\cdot))\rangle\\ &=\langle\bar\psi(0)-e^{\bar tA^*}\bar\psi(\bar t),x_0-\bar x_0\rangle-\int_0^{\bar t}\langle e^{(\bar t-s)A^*}\bar\psi(\bar t),u(s)-\bar u(s)\rangle ds\\ &=\int_0^{\bar t}[\max_{u\in U(s)}\langle\bar\psi(s),u\rangle-\langle\bar\psi(s),u(s)\rangle]ds\ge 0,\qquad\forall(x_0,u(\cdot))\in Q_0\times\mathcal U.\end{aligned}\tag{4.39}$$

Combining (4.38) and (4.39), we get

$$d_{Q(\bar t)}((x_0,x(\bar t;x_0,u(\cdot))))\ge d_{Q(\bar t)}((\bar x_0,x(\bar t;\bar x_0,\bar u(\cdot)))),\qquad\forall(x_0,u(\cdot))\in Q_0\times\mathcal U.\tag{4.40}$$

The optimality of $(\bar x_0,\bar u(\cdot))$ follows from (4.36), (4.37) and (4.40) immediately.

Let

$$Q^0(t)\equiv\{y\in X\mid\exists z\in X\text{ such that }(y,z)\in Q(t)+\hat r\bar O_1\},\qquad\forall t\in\mathrm R_+.\tag{4.41}$$

Theorem 4.4. *Let* (H1)–(H4) *hold and Problem* P *be approximately solvable. Let* $Q^0(\hat t)-Q_0$ *and*

$$\mathcal S\equiv\{z-x(\hat t;y,u(\cdot))\mid y\in Q_0,\ (y,z)\in Q(\hat t)+\hat r\bar O_1,\ u(\cdot)\in\mathcal U\}\tag{4.42}$$

be finite codimensional in X. *Then there exist* $\hat\phi$, $\hat\psi\in X^*$ *such that*

$$\|(\hat\phi,\hat\psi)\|=1,\tag{4.43}$$

$$g(\hat t,\hat r,\hat\phi,\hat\psi)+\int_0^{\hat t}\sup_{u\in U(t)}\langle\hat\psi,e^{\hat t-t)A}u\rangle dt=0.\tag{4.44}$$

Proof. Since Problem P is approximately solvable, according to Theorem 4.2, we have $f(\hat t,\hat r)=0$. Hence we can take a sequence $\{(\hat\phi_n,\hat\psi_n)\}\subseteq X^*\times X^*$ such that

$$\|(\hat\phi_n,\hat\psi_n)\|=1,\tag{4.45}$$

$$g(\hat t,\hat r,\hat\phi_n,\hat\psi_n)+\int_0^{\hat t}\sup_{u\in U(t)}\langle\hat\psi_n,e^{(\hat t-t)A}u\rangle dt<\frac1n,\qquad n=1,2,\cdots,\tag{4.46}$$

which implies

$$\begin{aligned}&\langle\hat\phi_n+e^{\hat tA^*}\hat\psi_n,y-x_0\rangle+\langle\hat\psi_n,z-x(\hat t;y,u(\cdot))\rangle>-\frac1n,\\ &\forall x_0\in Q_0,\ (y,z)\in Q(\hat t)+\hat r\bar O_1,\ u(\cdot)\in\mathcal U,\ n=1,2,\cdots.\end{aligned}\tag{4.47}$$

(4.47) yields

$$\langle\hat\psi_n,z-x(\hat t;y,u(\cdot))\rangle>-\frac1n,\quad\forall(y,z)\in[Q(\hat t)+\hat r\bar O_1]\cap(Q_0\times X),\ u(\cdot)\in\mathcal U,\ n=1,2,\cdots.\tag{4.48}$$

Besides, from

$$\overline{R(\hat t)}\cap[Q(\hat t)+\hat r\bar O_1]\ne\emptyset\tag{4.49}$$

we immediately know

$$0\in\bar{\mathcal S}(\subseteq\overline{co}S).\tag{4.50}$$

Thus, if

$$\limsup_{n\to\infty}\|\hat\psi_n\|>0,\tag{4.51}$$

then by Lemma 3.2 in [13] (noting that we have assumed $\mathcal{S}$ is finite codimensional), there exists a subsequence of $\{\hat{\psi}_n\}$ (we still denote it by $\{\hat{\psi}_n\}$ for simplicity) and a nonzero $\hat{\psi} \in X^*$ such that

$$\hat{\psi}_n \xrightarrow{w} \hat{\psi} \ (n \to \infty). \tag{4.52}$$

If (4.51) does not hold, namely

$$\lim_{n\to\infty} \parallel \hat{\psi}_n \parallel = 0, \tag{4.53}$$

then (notice (4.45)) we have

$$\liminf_{n\to\infty} \parallel \hat{\phi}_n \parallel > 0. \tag{4.54}$$

And, from (4.47) we know there exists a sequence $\{\epsilon_n\} \subseteq \mathrm{R}_+$ such that $\epsilon_n \to 0$ $(n \to \infty)$ and

$$\langle \hat{\phi}_n, y - x_0 \rangle \geq -\epsilon_n, \qquad \forall y \in Q^0(\hat{t}),\ x_0 \in Q_0,\ n = 1, 2, \cdots. \tag{4.55}$$

Also due to (4.49),

$$0 \in \overline{Q^0(\hat{t}) - Q_0} \subseteq \overline{co}[Q^0(\hat{t}) - Q_0]. \tag{4.56}$$

Again using Lemma 3.2 in [13] (note that it has been assumed that $Q^0(\hat{t}) - Q_0$ is finite codimensional), we get a subsequence of $\{\hat{\phi}_n\}$ (still denote it by $\{\hat{\phi}_n\}$ for simplicity) and a nonzero $\hat{\phi} \in X^*$ such that

$$\hat{\phi}_n \xrightarrow{w} \hat{\phi} \ (n \to \infty). \tag{4.57}$$

Correspondingly, to the case that (4.51) (resp. (4.54)) holds, take a subsequence of $\{\hat{\phi}_n\}$ (resp. $\{\hat{\psi}_n\}$) (still denote it by $\{\hat{\phi}_n\}$ (resp. $\{\hat{\psi}_n\}$)) for which there exists a nonzero $\hat{\phi}$ (resp. $\hat{\psi}$) $\in X^*$ such that

$$\hat{\phi}_n \xrightarrow{w} \hat{\phi} \ (\text{resp.} \ \ \hat{\psi}_n \xrightarrow{w} \hat{\psi}) \ (n \to \infty). \tag{4.58}$$

Summarizing the above discussions, we know there exists a subsequence of $\{(\hat{\phi}_n, \hat{\psi}_n)\}$ (still denote it by $\{(\hat{\phi}_n, \hat{\psi}_n)\}$ for simplicity) and $\hat{\phi}$, $\hat{\psi} \in X^*$ such that

$$(\hat{\phi}_n, \hat{\psi}_n) \xrightarrow{w} (\hat{\phi}, \hat{\psi}) \ (n \to \infty), \qquad (\hat{\phi}, \hat{\psi}) \neq 0 \tag{4.59}$$

whether or no; And, obviously, we may demand $(\hat{\phi}, \hat{\psi})$ satisfies (4.43). Letting $n \to \infty$ in (4.47), we get

$$\begin{aligned} &\langle \hat{\phi} + e^{\hat{t}A^*}\hat{\psi}, y - x_0 \rangle + \langle \hat{\psi}, z - x(\hat{t}; y, u(\cdot)) \rangle \geq 0, \\ &\qquad \forall x_0 \in Q_0,\ (y, z) \in Q(\hat{t}) + \hat{r}\bar{O}_1,\ u(\cdot) \in \mathcal{U}, \end{aligned} \tag{4.60}$$

which is equivalent to

$$g(\hat{t}, \hat{r}, \hat{\phi}, \hat{\psi}) + \int_0^{\hat{t}} \sup_{u \in U(t)} \langle \hat{\psi}, e^{(\hat{t}-t)A} u \rangle dt \leq 0 \tag{4.61}$$

(see the proof of Lemma 3.3). On the other hand, from (4.49) we know that the left hand side of (4.61) is no less than 0. Combining this and (4.61) we immediately get (4.44).

Theorem 4.4 has the following corollary (such sort of results are called Pontryagin type maximum principles usually).

Theorem 4.5. *Let* (H1)–(H4) *hold. Let* $(\hat{x}_0, \hat{u}(\cdot))$ *be an optimal pair of Problem* P. *Then there exist* $\hat{\phi}$, $\hat{\psi} \in X^*$ *such that*

$$\| (\hat{\phi}, \hat{\psi}) \| = 1, \qquad -(\hat{\phi}, \hat{\psi}) \in (\partial d_{Q(\hat{t})+\hat{r}\bar{O}_1})((\hat{x}_0, x(\hat{t}; \hat{x}_0, \hat{u}(\cdot)))), \tag{4.62}$$

$$\max_{u\in U(t)} \langle \hat{\psi}(t), u\rangle = \langle \hat{\psi}(t), \hat{u}(t)\rangle, \qquad \text{a.e. } t \in [0, \hat{t}], \tag{4.63}$$

where

$$\hat{\psi}(t) = e^{(\hat{t}-t)A^*}\hat{\psi}, \qquad \forall t \in [0, \hat{t}], \tag{4.64}$$

$$\langle \hat{\psi}(0) + \hat{\phi}, x_0 - \hat{x}_0\rangle \le 0, \qquad \forall x_0 \in Q_0. \tag{4.65}$$

Proof. According to Theorem 4.4, we know there exist $\bar{\phi}$, $\bar{\psi} \in X^*$ such that (4.43) and (4.44) hold, while (4.44) implies

$$\begin{aligned} &\langle \hat{\phi}, x_0 - y\rangle + \langle \hat{\psi}, e^{\hat{t}A}x_0 - z\rangle + \int_0^{\hat{t}} \sup_{u\in U(t)} \langle \hat{\psi}, e^{(\hat{t}-t)A}u\rangle dt \le 0, \\ &\qquad \forall x_0 \in Q_0,\ (y, z) \in Q(\hat{t}) + \hat{r}\bar{O}_1. \end{aligned} \tag{4.66}$$

Taking $x_0 = \hat{x}_0$ and $(y, z) = (\hat{x}_0, x(\hat{t}; \hat{x}_0, \hat{u}(\cdot)))$ in (4.66), we get

$$\int_0^{\hat{t}} [\sup_{u\in U(t)} \langle \hat{\psi}, e^{(\hat{t}-t)A}u\rangle - \langle \hat{\psi}, e^{(\hat{t}-t)A}\hat{u}(t)\rangle] dt \le 0, \tag{4.67}$$

which immediately yields

$$\max_{u\in U(t)} \langle \hat{\psi}, e^{(\hat{t}-t)A}u\rangle = \langle \hat{\psi}, e^{(\hat{t}-t)A}\hat{u}(t)\rangle, \qquad \text{a.e. } t \in [0, \hat{t}]. \tag{4.68}$$

Therefore, (4.66) implies

$$\langle -\hat{\phi}, y - \hat{x}_0\rangle + \langle -\hat{\psi}, z - x(\hat{t}; \hat{x}_0, \hat{u}(\cdot))\rangle \le 0, \qquad \forall (y, z) \in Q(\hat{t}) + \hat{r}\bar{O}_1. \tag{4.69}$$

Next, note that there exists a sequence $\{(y_n, z_n)\} \subseteq Q(\hat{t}) + \hat{r}\bar{O}_1$ such that

$$\| (y_n - y, z_n - z) \| < d_{Q(\hat{t})+\hat{r}\bar{O}_1}((y, z)) + \frac{1}{n}, \qquad n = 1, 2, \cdots \tag{4.70}$$

for any $(y, z) \in X \times X$. (4.43) and (4.69)-(4.70) result in

$$\begin{aligned} &\langle -\hat{\phi}, y - \hat{x}_0\rangle + \langle -\hat{\psi}, z - x(\hat{t}; \hat{x}_0, \hat{u}(\cdot))\rangle \\ &\le \langle -\hat{\phi}, y - y_n\rangle + \langle -\hat{\psi}, z - z_n\rangle \le \| (y - y_n, z - z_n) \| \\ &\to d_{Q(\hat{t})+\hat{r}\bar{O}_1}((y, z)) \ (n \to \infty) \end{aligned} \tag{4.71}$$

for any $(y, z) \in X \times X$. Besides, noticing that

$$d_{Q(\hat{t})}((\hat{x}_0, x(\hat{t}; \hat{x}_0, \hat{u}(\cdot)))) = \hat{r}, \tag{4.72}$$

from (4.71) we immediately know the second expression in (4.62) holds. Finally, taking $(y, z) = (\hat{x}_0, x(\hat{t}; \hat{x}_0, \hat{u}(\cdot)))$ in (4.66), we get

$$\langle \hat{\phi} + e^{\hat{t}A^*}\hat{\psi}, x_0 - \hat{x}_0\rangle \le 0, \qquad \forall x_0 \in Q_0. \tag{4.73}$$

Thus, let $\hat{\psi}(\cdot)$ be the function defined by (4.64), then (4.63) and (4.65) follow from (4.68) and (4.73) respectively.

§5. Proof of Theorem 1.1

We return to Problem MPCP. First of all, we have

$$f_0(t) \le 0, \qquad \forall t \in \mathrm{R}_+, \tag{5.1}$$

$$\forall t \in \mathrm{R}_+, \quad Q_{\mathrm{MPCP}} \cap \overline{R_{\mathrm{MPCP}}(t)} \ne \emptyset \quad \Longrightarrow \quad f_0(t) \ge 0, \tag{5.2}$$

$$\forall t \in \mathrm{R}_+, \quad f_0(t) = 0 \quad \Longrightarrow Q_{\mathrm{MPCP}} \cap \overline{R_{\mathrm{MPCP}}(t)} \ne \emptyset. \tag{5.3}$$

In fact, one easily see (5.1) holds from (1.11). The proofs of (5.2) and (5.3) are completely similar to the ones of (3.19) and (4.19) respectively (cf. the proof of Lemma 3.3). From (5.1)-(5.3) we immediately get

$$\forall t \in \mathrm{R}_+, \quad Q_{\mathrm{MPCP}} \cap \overline{R_{\mathrm{MPCP}}(t)} \ne \emptyset \quad \Longleftrightarrow \quad f_0(t) = 0. \tag{5.4}$$

Contrasting Definition 1.2 with (5.4), we immediately know Problem MPCP is approximately solvable if and only if $f_0(\cdot)$ admits of minimum positive zero point, and, in this case the minimum positive zero point is just the minimum positive period.

Secondly, let Problem MPCP be solvable. Then $\hat{T} \in (0, +\infty)$. Thus we can take a positive integer $\hat{N}$ such that

$$\hat{T} \in \Big(\frac{1}{\hat{N}}, \hat{N}\Big). \tag{5.5}$$

Let $(\hat{x}_0, \hat{u}(\cdot)) \in Q_1 \times \mathcal{U}_0$ be a solution of Problem MPCP. Then it is obvious that $(\hat{x}_0, \hat{u}(\cdot))$ is an optimal pair to the following problem:

Problem (MPCP)$_{\hat{N}}$. Find a $\hat{T}_{\hat{N}} \in [\frac{1}{\hat{N}}, \hat{N}]$, an $\hat{x}_0^{(\hat{N})} \in Q_1$ and a $\hat{u}_{\hat{N}}(\cdot) \in \mathcal{U}_0$ such that

$$\hat{u}_{\hat{N}}(t + \hat{T}_{\hat{N}}) = \hat{u}_{\hat{N}}(t), \quad \text{a.e. } t \in \mathrm{R}_+, \tag{5.6}$$

$$x(t + \hat{T}_{\hat{N}}; \hat{x}_0^{(\hat{N})}, \hat{u}_{\hat{N}}(\cdot)) = x(t; \hat{x}_0^{(\hat{N})}, \hat{u}_{\hat{N}}(\cdot)), \qquad \forall t \in \mathrm{R}_+, \tag{5.7}$$

$$\begin{aligned} \hat{T}_{\hat{N}} = \inf \Big\{ T \in \Big[\frac{1}{\hat{N}}, \hat{N}\Big] \Big| \exists u(\cdot) \in \mathcal{U}_0,\ x_0 \in Q_1 \text{ such that } u(t+T) = u(t), \\ x(t+T; x_0, u(\cdot)) = x(t; x_0, u(\cdot)),\ \forall t \in \mathrm{R}_+ \}; \end{aligned} \tag{5.8}$$

$$\hat{T}_{\hat{N}} = \hat{T}. \tag{5.9}$$

It is not hard to see that Problem (MPCP)$_{\hat{N}}$ can be referred to a special case of Problem P; and, for this case

$$\hat{r} = 0, \tag{5.10}$$

the dynamic operator of the system is

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} : \mathcal{D}(A) \times \mathrm{R}^1 \to X \times \mathrm{R}^1 \tag{5.11}$$

($\mathcal{D}(A)$ denotes the domain of A) and

$$\begin{aligned} U(t) = \begin{pmatrix} B(U) \\ 1 \end{pmatrix}, \quad Q_0 = \Big\{ \begin{pmatrix} y \\ 0 \end{pmatrix} \Big| y \in Q_1 \Big\}, \quad Q(t) = \Big\{ \Big(\begin{pmatrix} y \\ s_0 \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix} \Big) \Big| y \in Q_1, \\ s_0 \in [-1, -1],\ s \in \Big[\frac{1}{\hat{N}}, \hat{N}\Big] \Big\}, \quad \forall t \in \mathrm{R}_+. \end{aligned} \tag{5.12}$$

It is easy to verify that (H1)–(H4) hold,

$$\hat{t} = \hat{T}, \qquad Q^0(\hat{T}) = Q_1 \times [-1, 1], \tag{5.13}$$

$$\begin{aligned}\mathcal{S} = \Big\{ \begin{pmatrix} y \\ s \end{pmatrix} - \Big[e^{\hat{T}\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} y \\ 0 \end{pmatrix} + \int_0^{\hat{T}} e^{(\hat{T}-\tau)\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}} \begin{pmatrix} B(u(\tau)) \\ 1 \end{pmatrix} d\tau \Big] \Big| y \in Q_1,\\ s \in \Big[\frac{1}{\hat{N}}, \hat{N}\Big] \Big\} \qquad (5.14)\\ = \Big\{ \begin{pmatrix} (I_X - e^{\hat{T}A})y - \int_0^{\hat{T}} e^{(\hat{T}-\tau)A} B(u(\tau)) d\tau \\ s - \hat{T} \end{pmatrix} \Big| y \in Q_1,\ s \in \Big[\frac{1}{\hat{N}}, \hat{N}\Big] \Big\}.\end{aligned}$$

The compactness of $e^{\hat{T}A}$ implies that $X_0 \equiv (I_X - e^{\hat{T}A})X$ is closed and finite codimensional in X. By Banach's Open Mapping Theorem, we know there exists a positive number ϵ_0 such that

$$\bar{O}_{\epsilon_0}(\equiv \{x \in X \mid \| x \| \le \epsilon_0\}) \cap X_0 \subseteq (I_X - e^{\hat{T}A})(Q_1 - y_0), \qquad (5.15)$$

where y_0 is the centre of Q_1. Hence, to this special case of Problem P, all hypotheses posed on Problem P hold. Thus, applying Theorem 4.5 to it, we immediately know there exist $\hat{\alpha}$, $\beta \in \mathrm{R}^1$ and $\hat{\phi}$, $\hat{\psi} \in X^*$ such that

$$\Big\| \Big(\begin{pmatrix} \hat{\phi} \\ \hat{\alpha} \end{pmatrix}, \begin{pmatrix} \hat{\psi} \\ \hat{\beta} \end{pmatrix} \Big) \Big\| = 1, \qquad (5.16)$$

$$\Big(\begin{pmatrix} \hat{\phi} \\ \hat{\alpha} \end{pmatrix}, \begin{pmatrix} \hat{\psi} \\ \hat{\beta} \end{pmatrix} \Big) \in \Big(\partial d_{\{ (\begin{pmatrix} y \\ s_0 \end{pmatrix}, \begin{pmatrix} y \\ s \end{pmatrix}) | y \in Q_1\ s_0 \in [-1,1],\ s \in [\frac{1}{\hat{N}}, \hat{N}] \}} \Big) \Big(\Big(\begin{pmatrix} \hat{x}_0 \\ 0 \end{pmatrix}, \begin{pmatrix} \hat{x}_0 \\ \hat{T} \end{pmatrix} \Big) \Big), \qquad (5.17)$$

$$\max_{u \in U} \Big\langle \begin{pmatrix} \hat{\psi}(t) \\ -\hat{\beta} \end{pmatrix}, \begin{pmatrix} B(u) \\ 1 \end{pmatrix} \Big\rangle = \Big\langle \begin{pmatrix} \hat{\psi}(t) \\ -\hat{\beta} \end{pmatrix}, \begin{pmatrix} B(\hat{u}(t)) \\ 1 \end{pmatrix} \Big\rangle \quad \text{a.e. } t \in [0, \hat{T}], \qquad (5.18)$$

$$\Big\langle \begin{pmatrix} \hat{\psi}(0) \\ -\hat{\beta} \end{pmatrix} - \begin{pmatrix} \hat{\phi} \\ \hat{\alpha} \end{pmatrix}, \begin{pmatrix} x_0 - \hat{x}_0 \\ 0 \end{pmatrix} \Big\rangle \le 0, \quad \forall x_0 \in Q_1, \qquad (5.19)$$

where
$$\hat{\psi}(t) = -e^{(\hat{T}-t)A^*} \hat{\psi}, \qquad \forall t \in [0, \hat{T}]. \qquad (5.20)$$

Noting (5.5) and $0 \in (-1, 1)$, we easily derive

$$(\hat{\phi}, \hat{\psi}) \in (\partial d_{\{(y,y) | y \in Q_1\}})((\hat{x}_0, \hat{x}_0)), \qquad \hat{\alpha} = \hat{\beta} = 0 \qquad (5.21)$$

from (5.17). Combining (5.16) and the second expression in (5.21) we immediately have

$$\| (\hat{\phi}, \hat{\psi}) \| = 1. \qquad (5.22)$$

Substituting the second expression of (5.21) into (5.18)-(5.19), we get

$$\max_{u \in U} \langle \hat{\psi}(t), B(u) \rangle = \langle \hat{\psi}(t), B(\hat{u}(t)) \rangle, \qquad \text{a.e. } t \in [0, \hat{T}], \qquad (5.23)$$

$$\langle \hat{\psi}(0) - \hat{\phi}, x_0 - \hat{x}_0 \rangle \le 0, \qquad \forall x_0 \in Q_1. \qquad (5.24)$$

Then, noticing that

$$\forall \phi, \psi \in X^*, \qquad (\phi, \psi) \in (\partial d_{\{(y,y) | y \in Q_1\}})((\hat{x}_0, \hat{x}_0)) \implies \frac{\phi + \psi}{\sqrt{2}} \in (\partial d_{Q_1})(\hat{x}_0), \qquad (5.25)$$

we can get

$$\langle \hat{\phi} - \hat{\psi}(\hat{T}), x_0 - \hat{x}_0 \rangle \le 0, \qquad \forall x_0 \in Q_1 \qquad (5.26)$$

from (5.20), the first expression of (5.21) and (5.24), which together with (5.26) yields

$$\langle \hat{\psi}(0) - \hat{\psi}(\hat{T}), x_0 - \hat{x}_0 \rangle \le 0, \quad \forall x_0 \in Q_1. \tag{5.27}$$

Finally, we show

$$\hat{\psi}(\cdot) \ne 0. \tag{5.28}$$

If (5.28) does not hold, then from (5.20),(5.22) and (5.24) and (5.26) we immediately have

$$\hat{\phi} \ne 0, \tag{5.29}$$

$$\langle \hat{\phi}, x_0 - \hat{x}_0 \rangle = 0, \qquad \forall x_0 \in Q_1. \tag{5.30}$$

While the interior of Q_1 is nonempty, from (5.30) we immediately see

$$\hat{\phi} = 0, \tag{5.31}$$

which contradicts (5.29).

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