Chin. Ann. of Math.
19B: 1(1998),129-132.

FIBONACCI SEQUENCE AND CANTOR'S TERNARY SET

TONG JINGCHENG* SAMONS, J.*

Abstract

Let **N** be the set of positive integers and **C** be Cantor's ternary set. A function $\xi : \mathbf{N} \to [0, 1]$ is established by the help of the Fibonacci sequence such that $\overline{\xi(\mathbf{N})}$, the closure of the set $\xi(\mathbf{N})$, is homeomorphic to the set **C**.

Keywords Fibonacci sequence, Cantor's ternary set, Homeomorphic
1991 MR Subject Classification 54H05
Chinese Library Classification O189.1

§1. Introduction

The Fibonacci sequence is a typical topic in discrete mathematics such as Elementary Number Theory[2] or Combinatorics[5], while Cantor's ternary set is a typical topic in continuous mathematics such as Real Analysis[1] or General Topology[3,4,6]. One can hardly find articles relating these two objects from two different fields in the literature. In this paper, using Zeckendorf's representation of natural numbers[7], we will establish a relation between the Fibonnaci sequence and Cantor's ternary set.

§2. Preliminaries

Let $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ be the Fibonacci sequence. It is well known[2] that a natural number n can be expressed as a sum of distinct Fibonacci numbers. The expression is not unique. Zeckendorf found[7] that if we have one more requirement that no consecutive Fibonacci numbers u_n, u_{n+1} can be used as addands, then the sum expression is unique. In summary we have the following fact.

Zeckendorf's Representation. For any given natural number n, there are Fibonacci numbers $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ such that

Z1. $i_1 < i_2 < \cdots < i_k$;

Z2. $|i_m - i_n| \geq 2$ for $m \neq n$ and $1 \leq m, n \leq k$;

Z3. $n = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$;

Z4. The representation in Z3 is unique, i.e., if $n = u_{j_1} + u_{j_2} + \cdots + u_{j_h}$ such that $j_1 < j_2 < \cdots < j_h$ and $|j_m - j_n| \geq 2$ for $m \neq n, 1 \leq m, n \leq h$, then $k = h$ and $i_m = j_m$ for $1 \leq m \leq k$.

Manuscript received September 12, 1994. Revised December 15, 1996.

*Department of Mathematics & Statistics, University of North Florida Jacksonville, Florida 32224, U.S.A.

Cantor's ternary set is a subset of the interval $[0,1]$, which is obtained by the following procedure. Deleting the middle third open interval of $[0,1]$, we have two closed subintervals. Deleting the middle third open intervals of the two subintervals respectively, we have four closed subintervals. Continuing deleting the open middle third intervals from each closed subintervals obtained in the previous procedure for countably infinitely many times, the set of all leftover points constitutes Cantor's ternary set. Another easier way to construct Cantor's set is to express every real number $x \in [0,1]$ in base three, $x = \sum a_n/3^n$, $a_n = 0, 1, 2$, then the set of all x in which a_n never equal to 1 is Cantor's ternary set. The following elegant result gives the characterization of Cantor's set (see [4] or [6]).

Characterization of Cantor's Ternary Set. A topological space $\mathcal{C}$ is homeomorphic to a Cantor's ternary set if and only if

C1. $\mathcal{C}$ is a metric space;

C2. $\mathcal{C}$ is compact;

C3. $\mathcal{C}$ is perfect;

C4. $\mathcal{C}$ is totally disconnected.

Here $\mathcal{C}$ is perfect means that every point $x \in \mathcal{C}$ is a limit point of a sequence of distinct point $x_n \in \mathcal{C}(n = 1, 2, \cdots)$. $\mathcal{C}$ is totally disconnected if every connected component of $\mathcal{C}$ is a single point.

The function $\xi(X)$ based on Zeckendorf's representation. Now we give the definition of the function $\xi(x)$ from the set $\mathbf{N}$ of natural numbers into the interval $[0,1]$. This function is based on Zeckendorf's representation. In next section we will prove that the closure of the set $\xi(\mathbf{N})$ is homeomorphic to Cantor's ternary set.

Definition 2.1. *Let $\mathbf{N}$ be the set of all natural numbers and $x \in \mathbf{N}$. If the Zeckendorf's representation of x is $x = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$, then define*

$$\xi(x) = (2u_{i_1})^{-1} + (2^2 u_{i_2})^{-1} + \cdots + (2^k u_{i_k})^{-1}.$$

Let $d = \sum\limits_{m=1}^{\infty} (2^m u_{2m-1})^{-1} = 0.602636\cdots$. It is easily seen that $0 \le \xi(x) \le d$ holds for all $x \in \mathbf{N}$. We need a few lemmas to investigate the properties of the function $\xi(x)$.

Lemma 2.1. *Let u_i $(i = 1, 2, \cdots)$ be the Fibonacci sequence. If $i > j \ge 2$, then $u_j^{-1} \le (2/3)u_i^{-1}$.*

Proof. Since $i > j$ implies $i - 1 \ge j$, we have

$$3u_i = 3(u_{i-1} + u_{i-2}) \ge 3(u_j + u_{j-1}) = 3u_j + 3u_{j-1}.$$

Since $u_j = u_{j-1} + u_{j-2} \le 2u_{j-1}$, we have $3u_i \ge 3u_j + (3/2)u_j = (9/2)u_j$, $u_i^{-1} \le (2/3)u_j^{-1}$.

Lemma 2.2. *Let $x \in \mathbf{N}$. If $x = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ is Zeckendorf's representation of X, then $\xi(x) < 2/(3u_{i_1})$.*

Proof. It is easily seen that $u_{i_2} = u_{i_2-1} + u_{i_2-2} \ge 2u_{i_2-2} \ge 2u_{i_1}$. If $u_{i_m} \ge 2^{m-1}u_{i_1}$, then $u_{i_{m+1}} = u_{i_{m+1}-1} + u_{i_{m+1}-2} \ge 2u_{i_{m+1}-2} \ge 2u_{i_m} \ge 2^m u_{i_1}$. Therefore

$$\begin{aligned}\xi(x) &= (2u_{i_1})^{-1} + (2^2 u_{i_2})^{-1} + \cdots + (2^k u_{i_k})^{-1} \\ &\le (2u_{i_1})^{-1} + (2^3 u_{i_1})^{-1} + \cdots + (2^{2k-1} u_{i_1})^{-1} \\ &< (2u_{i_1})^{-1}(1 + 2^{-2} + \cdots + 2^{-2k+2} + \cdots) = 2/(3u_{i_1}).\end{aligned}$$

Lemma 2.3. *Let $x_1, x_2 \in \mathbf{N}$ and $x_1 = u_{i_1} + \cdots + u_{i_k}$, $x_2 = u_{j_1} + \cdots + u_{j_h}$ be Zeckendorf's representations of x_1, x_2, respectively. If furthermore $\xi(x_1) < \xi(x_2)$, then*

$$\xi(x_2) - \xi(x_1) > 0.1(2^h u_{j_h})^{-1}.$$

Proof. We discuss the following possible cases.

(1) $u_{i_m} = u_{j_m}$ for $m = 1, 2, \cdots, k$. Then $h \geq k+1$, otherwise $\xi(x_1) = \xi(x_2)$. It is easily seen that

$$\xi(x_2) - \xi(x_1) = (2^{k+1} u_{j_{k+1}})^{-1} + \cdots + (2^h u_{j_h})^{-1} \geq (2^h u_{j_h})^{-1} > 0.1(2^h u_{j_h})^{-1}.$$

(2) There is a natural number $g \leq k-1$ such that $u_{i_m} = u_{j_m}$ for $m = 1, 2, \cdots, g-1$, but $u_{i_g} \neq u_{j_g}$. We discuss two subcases.

(i) $i_g > j_g$. Similar to Lemma 2.2, we can prove by induction that $u_{i_{g+s}} \geq 2^s u_{i_g}$. Then

$$\begin{aligned}\xi(x_1) - \sum_{m=1}^{g-1} (2^m u_{i_m})^{-1} &= (2^g u_{i_g})^{-1} + (2^{g+1} u_{i_{g+1}})^{-1} + \cdots + (2^k u_{i_k})^{-1} \\ &\leq (2^g u_{i_g})^{-1} + (2^{g+2} u_{i_g})^{-1} + \cdots + (2^{2k-g} u_{i_k})^{-1} \\ &< (2^g u_{i_g})^{-1}(1 + 2^{-2} + \cdots + 2^{-2k+2g} + \cdots) \\ &\leq (4/3)(2^g u_{i_g})^{-1}.\end{aligned}$$

Since $i_g > j_g$, by Lemma 2.1, we have $(4/3)(2^g u_{i_g})^{-1} \leq (8/9)(2^g u_{j_g})^{-1}$. Therefore

$$\begin{aligned}\xi(x_2) - \xi(x_1) &\geq (2u_{j_1})^{-1} + \cdots + (2^g u_{j_g})^{-1} - \xi(x_1) \\ &= (2^g u_{j_g})^{-1} - \Big(\xi(x_1) - \sum_{m=1}^{g-1} (2^m u_{i_m}\Big)^{-1}) \\ &\geq (2^g u_{j_g})^{-1} - (8/9)(2^g u_{j_g})^{-1} \\ &> 0.1(2^g u_{j_g})^{-1} \geq 0.1(2^h u_{j_h})^{-1}.\end{aligned}$$

(ii) $i_g < j_g$. This case is impossible because if discussing $\xi(x_2) - \sum\limits_{m=1}^{g-1} (2^m u_{j_m})^{-1}$ in (i) instead of $\xi(x_1) - \sum\limits_{m=1}^{g-1} (2^m u_{i_m})^{-1}$, we may get $\xi(x_1) - \xi(x_2) > 0$, which is a contradiction to the condition $\xi(x_2) > \xi(x_1)$.

The proof of Lemma 2.3 is completed.

Let $d = \sum\limits_{m=1}^{\infty} (2^m u_{2m-1})^{-1} = 0.602636\cdots$. We give an important property of the function $\xi(x)$.

Theorem 2.1. *The function $\xi(x)$ is an injection from $\mathbf{N}$ into $(0, d)$.*

Proof. Suppose that there are $x_1, x_2, \in \mathbf{N}$ such that $\xi(x_1) = \xi(x_2)$. We are going to prove $x_1 = x_2$. Let $x_1 = u_{i_1} + \cdots + u_{i_k}$ and $x_2 = u_{j_1} + \cdots + u_{j_h}$ be Zeckendorf's representations of x_1, x_2 respectively. If $x_1 \neq x_2$, then there is a natural number g such that $u_{i_g} \neq u_{j_g}$. Using the method in the proof of Lemma 2.3, we know that $i_g > j_g$ implies $\xi(x_2) > \xi(x_1)$, while $i_g < j_g$ implies $\xi(x_1) > \xi(x_2)$, both are contradictory to the condition $\xi(x_1) = \xi(x_2)$.

§3. Main Result

Now we prove the main result.

Theorem 3.1. *The closure of the set* $\xi(\mathbf{N})$ *is homeomorphic to Cantor's ternary set.*

Proof. Let $\overline{\xi(\mathbf{N})}$ be the closure of the set $\xi(\mathbf{N})$. It is easily seen that $\overline{\xi(\mathbf{N})}$ is a subset of the interval $[0, d]$. Naturally it inherits a distance function from the metric space of real numbers. As a matter of fact we know that $d(u, v) = |u - v|$ for $u, v \in \overline{\xi(\mathbf{N})}$, where $|\ |$ means the absolute value. Therefore $\overline{\xi(\mathbf{N})}$ is a metric space.

As a subset of $[0, d]$, $\overline{\xi(\mathbf{N})}$ is bounded. Therefore $\overline{\xi(\mathbf{N})}$ is compact since it is a bounded closed set in the space of real numbers.

To show that $\overline{\xi(\mathbf{N})}$ is a perfect set, we have to prove that every point in $\xi(\mathbf{N})$ is a limit point of $\xi(\mathbf{N})$.

Suppose $p \in \xi(\mathbf{N})$. Then there is an $x \in \mathbf{N}$ such that $p = \xi(x)$. If $x = u_{i_1} + \cdots + u_{i_k}$ is Zeckendorf's representation of x, then consider the sequence $x_r (r = 1, 2, \cdots)$ defined as follows:

$$x_r = (u_{i_1} + \cdots + u_{i_k}) + u_{i_k + r + 1} = x + u_{i_k + r + 1}.$$

It is easily seen that $x_r \in \mathbf{N}$ and the above expression is Zeckendorf's representation of x_r since $i_k + r + 1 \geq i_k + 2$. A direct enumeration gives

$$\xi(x_r) = \xi(x) + (2^{k+1} u_{i_k + r + 1})^{-1} = p + (2^{k+1} u_{i_k + r + 1})^{-1}.$$

Therefore $\lim\limits_{r \to \infty} \xi(x_r) = p$. The set $\overline{\xi(\mathbf{N})}$ is perfect.

To show that $\overline{\xi(\mathbf{N})}$ is totally disconnected, we have to prove that each connected component of $\overline{\xi(\mathbf{N})}$ is a single point.

Let $\mathcal{C}$ be a connected component of $\overline{\xi(\mathbf{N})}$ which is not a single point. Since $\mathcal{C}$ is a connected set and connected sets in the space of real numbers are intervals[3], we know that there are two real numbers a, b such that $a < b$ and the interval $(a, b) \subseteq \mathcal{C}$. Let $p \in (a, b)$. Then p is a limit point of $\xi(x_r)$ for a sequence $x_r \in \xi(\mathbf{N})$. Hence there is an x_{r_0} such that $\xi(x_{r_0}) \in (a, b)$.

Let $x_{r_0} = u_{j_1} + \cdots + u_{j_h}$ be Zeckendorf's representation of x_{r_0}.Then consider the interval $I = (\xi(x_{r_0}) - 0.1(2^h u_{j_h})^{-1},\ \xi(x_{r_0}))$. We have $\xi(\mathbf{N}) \cap I = \emptyset$ since $x \in \mathbf{N}$ and $\xi(x) < \xi(x_{r_0})$ imply $x \notin I$ by Lemma 2.3. Noticing that I is an open set, we have $\overline{\xi(\mathbf{N})} \cap I = \emptyset$. Hence $\mathcal{C} \cap I = \emptyset$, and $(a, b) \cap I = \emptyset$. This is impossible because letting $c = \max(a, \xi(x_{r_0}) - 0.1(2^h u_{j_h})^{-1})$, we have $(c, \xi(x_{r_0})) \subset (a, b) \cap I$. This contradiction proves that $\mathcal{C}$ must be a single point.

By the discussion above we know that $\overline{\xi(\mathbf{N})}$ is homeomorphic to Cantor's ternary set. The proof is completed.

References

[1] Bartle, R. G., The elements of real analysis, 2nd edition, John Wiley and Sons, New York, 1964.
[2] Burton, D., Elementary number theory, 2nd Edition, Wm. C. Brown Fublishers, Dubuque, 1989.
[3] Croom, F. H., Principles of topology, Saunders College Publishing, Philadelphia, 1989.
[4] Jocking, J. G. & Young, G. S., Topology, Addison-Wesley Publ. Co. Inc., Reading, Mass., 1961.
[5] Straight, H. J., Combinatorics, Brooks/Cole Publ. Co., Pacific Grove, Calif., 1993.
[6] Willard, S., General topology, Addison-Wesley, Publ. Co. Inc, Reading, Mass. 1970.
[7] Zeckendorf, E., Reprensentations des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bulletin de la Societe Royale des Sciences de Liege, 179–182, 1972.