Chin. Ann. of Math.
19B: 2(1998),133-142.

PERIODIC CARDINAL INTERPOLATORY WAVELETS

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Abstract

The authors construct periodic interpolating wavelets and their duals from a periodic function $g(x)$ whose Fourier coefficients are positive. The corresponding decomposition and reconstruction algorithm is also given. The spline example shows that such kind of wavelets shares good localization with any desired regularity and symmetry. The construction depends essentially on the finite Fourier Transformation and the theory of circulant matrix.

Keywords Periodic wavelet, Multiresolution, Cardinal Interpolation, Dual basis, Circulant matrix
1991 MR Subject Classification 42C05, 41A63, 41A30, 41A15
Chinese Library Classification O174.2, O174.41

§1. Introduction

Periodic problems appear in various applications which motivated an extensive study of periodic wavelets in recent years.

Y. Meyer[8,11] studied periodic multiresolutions by periodizing known wavelets. Perrier and Basdevant[13] stated the construction and algorithm of periodic wavelets, their algorithm makes heavy use of the Fast Fourier Transformation (FFT). Chui and Mhasker[7] constructed the trigonometric wavelets. Plonka and Tasche[14,15] studied p-periodic wavelets for general periodic scaling functions. Their algorithms[16] are based on Fourier technique. The first author of this paper made a full study of periodic wavelets when the scaling functions are derived from different kinds of spline functions[1−5]. Each equation in the decomposition and reconstruction algorithms involves only two terms which do not depend on the regularity of the underlying wavelets. The error estimates are studied elaborately. The discret Fourier transform is used implicitly. Koh, Lee and Tan[10] gave a general framework of periodic wavelets, where two terms were obtained and the two-term algorithms operating on the frequency domain was also realized. Narcowich and Ward[12] investigated the periodic scaling functions and wavelets generated by continuously differentiable periodic functions with positive Fourier coefficients. They also discussed the localization of scaling functions and wavelets. The method of using the periodic wavelets, e.g., to denoise and to detect singularity, is also pointed out. Chen et el[6] construct a kind of real-valued periodic orthogonal wavelets. The relation between the periodic wavelets and the Fourier series is also discussed.

Manuscript received October 24, 1996. Revised August 8, 1997.
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∗∗Project supported by the National Natural Science Foundation of China.

In this paper, we construct periodic interpolatory wavelets and their duals from a periodic function $g(x)$ whose Fourier coefficients are positive. The wavelets as well as scaling function are symmetric with respect to some axes. The corresponding decomposition and reconstruction algorithm is also given. The spline example shows that such kind of wavelets shares good localization with any desired regularity. Our construction depends essentially on the finite Fourier Transformation and the theory of circulant matrix.

§2. Cardinal Interpolatory Scaling Functions

Let j be a nonnegative integer, K a positive integer. $K_j = 2^j K$, $h_j = \dfrac{2\pi}{K_j}$. Let $g(x)$ be a 2π-periodic, continuous differentiable function whose Fourier coefficients are positive, i.e., $g(x) \in \overset{\circ}{C}([0, 2\pi])$, and

$$g(x) = \sum_{n\in\mathbb{Z}} C_n \, e^{inx} \qquad \text{with} \quad C_n > 0 \quad \text{for any } n \in \mathbb{Z}. \tag{2.1}$$

For $f, g \in \overset{\circ}{C}[0, 2\pi)$ the inner product of f, g is defined by

$$\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\, dx.$$

Define

$$V_j := \operatorname{span}\{g(x), g(x - h_j), \cdots, g(x - (K_j - 1)h_j)\}. \tag{2.2}$$

Then from [11] we know that $\dim V_j = K_j$ and $V_j \subset V_{j+1}$.

Definition 2.1. *For $\ell = 0, 1, \cdots, K_j - 1$, define*

$$Z_\ell^j(x) := \sum_{k=0}^{K_j-1} g(x + kh_j)\, e^{ik\ell h_j} = K_j \sum_{n\in\mathbb{Z}} C_{nK_j-\ell} \exp(i(nK_j - \ell)x),$$

$$\tilde{Z}_\ell^j(x) = \frac{Z_\ell^j(x)}{||Z_\ell^j||}.$$

It is easy to check that

$$\begin{aligned} &\langle \widetilde{Z}_{\ell_1}^j, \widetilde{Z}_{\ell_2}^j\rangle = \delta_{\ell_1\ell_2} \qquad \text{for} \quad 0 \le \ell_1, \ell_2 \le K_j - 1, \\ &Z_\ell^j(x + kh_j) = \exp(-i\ell kh_j)\, Z_\ell^j(x). \end{aligned} \tag{2.3}$$

Since $Z_\ell^j(0) = K_j \sum\limits_{n\in\mathbb{Z}} C_{nK_j-\ell} > 0$, we give the following definition.

Definition 2.2. *For $0 \le j \le K_j - 1$, define*

$$\varphi_j(x) = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1} \frac{Z_\ell^j(x)}{Z_\ell^j(0)}.$$

We have the following theorem.

Theorem 2.1. *Suppose $g(x)$ satisfies* (2.1). *Let V_j, φ_j be defined as* (2.2) *and Definition* 2.2 *respectively. Then*

(1) *$\varphi_j(x)$ possesses the cardinal interpolatory property, i.e., $\varphi_j(kh_j) = \delta_{0k}$ for $k = 0, 1, \cdots, K_j - 1$.*

(2) *$\{\varphi_j(\cdot - kh_j)\}_{k=0}^{K_j-1}$ is a basis for V_j,*

$$V_j = \operatorname{span}\{\varphi_j(\cdot - kh_j) : k = 0, 1, \cdots, K_j - 1\}.$$

(3) $\varphi_j(x)$ *satifies the following two-scale equation*

$$\varphi_j(x) = \varphi_{j+1}(x) + \sum_{\ell=0}^{K_j-1} \varphi_j((2\ell+1)h_{j+1})\varphi_{j+1}(x-(2\ell+1)h_{j+1}).$$

Proof. From (2.3), we know that $Z_\ell^j(kh_j) = \exp(-i\ell kh_j)\, Z_\ell^j(0)$. Hence

$$\varphi_j(kh_j) = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{Z_\ell^j(kh_j)}{Z_\ell^j(0)} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\exp(-i\ell kh_j) = \delta_{0k}.$$

Since $\varphi_j(\cdot - kh_j) \in V_j$, $\operatorname{span}\{\varphi_j(\cdot), \varphi_j(\cdot - h_j), \cdots, \varphi_j(\cdot - (K_j-1)h_j)\} \subset V_j$. But, the set of functions $\{\varphi_j(\cdot - kh_j)\}_{k=0}^{K_j-1}$ is a linearly independent system. Therefore $V_j = \operatorname{span}\{\varphi_j(\cdot - kh_j) : k = 0, 1, \cdots, K_j-1\}$, which completes the proof of the theorem since the two-scale equation can be deduced simply from the cardinal interpolatory property of $\varphi_j(x)$.

§3. Cardinal Interpolatory Wavelets

In this section, we shall construct the cardinal interpolatory wavelets.

For $\ell = 0, 1, \cdots, K_j-1$, define

$$R_\ell^j(x) := \{d_\ell^{j+1}\tilde{Z}_\ell^{j+1}(x) - d_{K_j+\ell}^{j+1}\widetilde{Z}_{K_j+\ell}^{j+1}(x)\}e^{i\ell h_{j+1}}, \tag{3.1}$$

where $d_\ell^{j+1} = \frac{||Z_{K_j+\ell}^{j+1}||}{||Z_\ell^j||}$.

We have the following lemma.

Lemma 3.1. *For* $\ell, \lambda = 0, 1, \cdots, K_j-1$,

$$\langle R_\ell^j(x), Z_\lambda^j(x)\rangle = 0.$$

Proof. By the definition of Z_λ^j , we have

$$\begin{aligned}
Z_\lambda^j(x) &= K_j\sum_{n\in\mathbb{Z}} C_{nK_j-\lambda}\exp(i(nK_j-\lambda)x)\\
&= \frac{K_{j+1}}{2}\sum_{n\in\mathbb{Z}} C_{2nK_j-\lambda}\exp(i(2nK_j-\lambda)x)\\
&\quad + \frac{K_{j+1}}{2}\sum_{n\in\mathbb{Z}} C_{(2n-1)K_j-\lambda}\exp(i((2n-1)K_j-\lambda)x)\\
&= \frac{1}{2}\left(Z_\lambda^{j+1}(x) + Z_{K_j+\lambda}^{j+1}(x)\right).
\end{aligned} \tag{3.2}$$

Then, from (3.1) , (3.2) and the orthogonality of Z_λ^{j+1}, we obtain

$$\begin{aligned}
&\langle R_\ell^j(x), Z_\lambda^j(x)\rangle\\
&= \frac{1}{2}\langle\{d_\ell^{j+1}\widetilde{Z}_\ell^{j+1}(x) - d_{K_j+\ell}^{j+1}\widetilde{Z}_{K_j+\ell}^{j+1}(x)\}e^{i\ell h_{j+1}}, Z_\lambda^{j+1}(x) + Z_{K_j+\lambda}^{j+1}(x)\rangle\\
&= \frac{1}{2}e^{i\ell h_{j+1}}\{d_\ell^{j+1}\cdot||Z_\lambda^{j+1}|| - d_{K_j+\ell}^{j+1}\cdot||Z_{K_j+\ell}^{j+1}||\}\cdot\delta_{\lambda,\ell} = 0.
\end{aligned}$$

Since

$$\begin{aligned}
R_\ell^j(h_{j+1}) &= \{d_\ell^{j+1}\widetilde{Z}_\ell^{j+1}(h_{j+1}) - d_{K_j+\ell}^{j+1}\widetilde{Z}_{K_j+\ell}^{j+1}(h_{j+1})\}e^{i\ell h_{j+1}}\\
&= d_\ell^{j+1}\widetilde{Z}_\ell^{j+1}(0) + d_{K_j+\ell}^{j+1}\widetilde{Z}_{K_j+\ell}^{j+1}(0) > 0,
\end{aligned}$$

we define

$$L_j(x) := \frac{1}{K_j} \sum_{\ell=0}^{K_j-1} \frac{R_\ell^j(x)}{R_\ell^j(h_{j+1})}; \tag{3.3}$$

since

$$\begin{aligned} R_\ell^j(kh_j + h_{j+1}) &= \{d_\ell^{j+1} \widetilde{Z}_\ell^{j+1}(kh_j + h_{j+1}) - d_{K_j+\ell}^{j+1} \widetilde{Z}_{K_j+\ell}^{j+1}(kh_j + h_{j+1})\} e^{i\ell h_{j+1}} \\ &= \{d_\ell^{j+1} \widetilde{Z}_\ell^{j+1}(h_{j+1}) e^{-i\ell kh_j} - d_{K_j+\ell}^{j+1} \widetilde{Z}_{K_j+\ell}^{j+1}(h_{j+1}) e^{-i(K_j+\ell)kh_j}\} e^{i\ell h_{j+1}} \\ &= e^{-i\ell kh_j} R_\ell^j(h_{j+1}), \end{aligned}$$

we have

$$L_j(kh_j + h_{j+1}) = \frac{1}{K_j} \sum_{\ell=0}^{K_j-1} \frac{R_\ell^j(kh_j + h_{j+1})}{R_\ell^j(h_{j+1})} = \frac{1}{K_j} \sum_{\ell=0}^{K_j-1} e^{-iklh_j} = \delta_{0k}.$$

Now define

$$W_j = \operatorname{span}\{R_\ell^j(x) : \ell = 0, 1, \cdots, K_j - 1\}.$$

By Lemma 3.1, we know that

$$W_j \subset V_{j+1} \ominus V_j.$$

Note that for each $j, L_j(x - kh_j)$ is a linear combination of $\{R_\ell^j(x)\}_{\ell=0}^{K_j-1}$. We have

$$\operatorname{span}\{L_j(x - kh_j) : k = 0, 1, \cdots, K_j - 1\} \subset W_j \subset V_{j+1} \ominus V_j,$$

but $\dim\{\operatorname{span}\{L_j(x - kh_j) : k = 0, 1, \cdots, K_j - 1\}\} = K_j = \dim(V_{j+1} \ominus V_j)$. Therefore

$$W_j = V_{j+1} \ominus V_j = \operatorname{span}\{L_j(x - kh_j) : k = 0, 1, \cdots, K_j - 1\}.$$

From Proposition 4.1 in [11], we know that

$$\operatorname{Clos} \bigcup_{j \geq 0} V_j = \mathring{C}[0, 2\pi).$$

We can summarize the discussion of this section as the following theorem.

Theorem 3.1. *The set*

$$\{\varphi_0(x)\} \bigcup_{j \geq 0} \{L_j(x - \ell h_j)\}_{\ell=0}^{K_j-1}$$

is an interpolatory basis for $\mathring{C}[0, 2\pi)$ *and* $L_j(x)$ *satisfies the following two-scale relation*

$$L_j(x) = \varphi_{j+1}(x - h_{j+1}) + \sum_{\ell=0}^{K_j-1} L_j(\ell h_j) \varphi_{j+1}(x - \ell h_j).$$

§4. Symmetry of Scaling Functions and Wavelets

From now on, we shall assume that $g(x)$ is real-valued and symmetric about the origin, i.e., $g(x) = g(-x) = \overline{g(x)}$, which implies that $C_n = C_{-n} = \overline{C_n}$ for $n \in \mathbb{Z}$.

We have the following Lemma.

Lemma 4.1. *Suppose that* Z_ℓ^j *is defined as Definition* 2.1, d_ℓ^{j+1} *in* (3.1). *Then, the following equalities are valid:*

$$\begin{aligned} \overline{Z_\ell^j(x)} &= Z_{-\ell}^j(x) = Z_{K_j-\ell}^j(x) = Z_\ell^j(-x), \\ d_\ell^{j+1} &= d_{-\ell}^{j+1} = d_{K_{j+1}-\ell}^{j+1}. \end{aligned}$$

Proof.

$$\begin{aligned}\overline{Z_\ell^j(x)} &= K_j \sum_{n\in\mathbb{Z}} C_{nK_j-\ell}\exp(-i(nK_j-\ell)x)\\ &= K_j \sum_{n\in\mathbb{Z}} C_{-nK_j-\ell}\exp(-i(-nK_j-\ell)x)\\ &= K_j \sum_{n\in\mathbb{Z}} C_{nK_j+\ell}\exp(i(nK_j+\ell)x) = Z_{-\ell}^j(x),\end{aligned}$$

the proofs of others are similar.

Now, we can state the main results of this section.

Theorem 4.1. *Suppose that* $\varphi_j(x)$ *is defined as Definitioin* 2.2, $L_j(x)$ *as* (3.3), *and* $g(x)$ *is real-valued with* $C_n = C_{-n}$. *Then*

(1) $\varphi_j(x)$ *is real-valued and* $\varphi_j(-x) = \varphi_j(x)$;

(2) $L_j(x)$ *is real and* $L_j(h_{j+1}+x) = L_j(h_{j+1}-x)$.

Proof. (i) Since

$$\begin{aligned}\overline{\varphi_j(x)} &= \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{\overline{Z_\ell^j(x)}}{\overline{Z_\ell^j(0)}} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{Z_{K_j-\ell}^j(x)}{Z_{K_j-\ell}^j(0)}\\ &= \frac{1}{K_j}\sum_{\ell=1}^{K_j}\frac{Z_\ell^j(x)}{Z_\ell^j(0)} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{Z_\ell^j(x)}{Z_\ell^j(0)} = \varphi_j(x),\end{aligned}$$

where we use the relation $Z_{K_j}^j(x) = Z_0^j(x)$,

$$\varphi_j(-x) = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{\overline{Z_\ell^j(-x)}}{\overline{Z_\ell^j(0)}} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{\overline{Z_{-\ell}^j(x)}}{\overline{Z_{-\ell}^j(0)}} = \varphi_j(x),$$

which completes the proof of the first part of the theorem.

(ii) To prove the second part of this theorem, we first note that

$$\begin{aligned}\overline{R_\ell^j(x)} &= \{d_\ell^{j+1}\overline{\widetilde{Z}_\ell^{j+1}(x)} - d_{K_j+\ell}^{j+1}\overline{\widetilde{Z}_{K_j+\ell}^{j+1}(x)}\}e^{-i\ell h_{j+1}}\\ &= \{d_\ell^{j+1}\widetilde{Z}_{-\ell}^{j+1}(x) - d_{K_j+\ell}^{j+1}\widetilde{Z}_{-K_j-\ell}^{j+1}(x)\}e^{-i\ell h_{j+1}}\\ &= \{d_{-\ell}^{j+1}\widetilde{Z}_{-\ell}^{j+1}(x) - d_{K_j-\ell}^{j+1}\widetilde{Z}_{K_j-\ell}^{j+1}(x)\}e^{-i\ell h_{j+1}}\\ &= R_{-\ell}^j(x) = R_{K_j-\ell}^j(x).\end{aligned}$$

Hence

$$\begin{aligned}\overline{L_j(x)} &= \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{\overline{R_\ell^j(x)}}{\overline{R_\ell^j(h_{j+1})}} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{R_{K_j-\ell}^j(x)}{R_{K_j-\ell}^j(h_{j+1})}\\ &= \frac{1}{K_j}\sum_{\ell=1}^{K_j}\frac{R_\ell^j(x)}{R_\ell^j(h_{j+1})} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{R_\ell^j(x)}{R_\ell^j(h_{j+1})} = L_j(x),\end{aligned}$$

where the relation $R_{K_j}^j(x) = R_0^j(x)$ is used.

Pay attention to $R_{-\ell}^j(h_{j+1}-x) = R_\ell^j(h_{j+1}+x)$. We have

$$\begin{aligned}L_j(h_{j+1}-x) &= \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{R_\ell^j(h_{j+1}-x)}{R_\ell^j(h_{j+1})} = \frac{1}{K_j}\sum_{\ell=0}^{K_j-1}\frac{R_{-\ell}^j(h_{j+1}+x)}{R_{-\ell}^j(h_{j+1})}\\ &= \overline{L_j(h_{j+1}+x)} = L_j(h_{j+1}+x),\end{aligned}$$

which implies that $L_j(x)$ is symmetric about the point h_{j+1}. The proof of the theorem is finished.

§5. Dual Scaling Functions and Dual Wavelets

In this section, we shall construct the dual scaling functions and dual wavelets where circulant matrix and its properties are used heavily.

The following lemma is important for the construction of dual scaling functions.

Lemma 5.1. *Suppose that $\varphi_j(x)$ is defined in Definition 2.2,*

$$\omega := e^{\frac{2\pi i}{K_j}} = e^{ih_j}, \quad F := \frac{1}{K_j}(\omega^{\ell k})_{\ell,k=0}^{K_j-1}, \quad G_j := (\langle \varphi_j(\cdot - \ell h_j), \varphi_j(\cdot - kh_j)\rangle)_{\ell,k=0}^{K_j-1},$$

$$P_j(z) = \langle \varphi_j, \varphi_j\rangle + \langle \varphi_j(\cdot), \varphi_j(\cdot - h_j)\rangle z + \cdots + \langle \varphi_j(\cdot), \varphi_j(\cdot - (K_j-1)h_j)\rangle z^{K_j-1}.$$

Then, G_j is an invertible circulant matrix, and $G_j^{-1} = F\Lambda^{-1}F^*$, where

$$\Lambda = \operatorname{diag}\{P_j(1), P_j(\omega), \cdots, P_j(\omega^{K_j-1})\},$$

the star denotes the complex conjugate.

Proof. By the periodicity of $\varphi_j(x)$, we know that

$$\langle \varphi_j(x-\ell h_j), \varphi_j(x-kh_j)\rangle = \langle \varphi_j(x), \varphi_j(x-(k-\ell)h_j)\rangle,$$

which implies that G_j is a circulant matrix.

From [8], we know that G_j can be diagonalized by F, i.e. $G_j = F\Lambda_j F^*$, where $\Lambda = \operatorname{diag}\{P_j(1), P_j(\omega), \cdots, P_j(\omega^{K_j-1})\}$.

Now, we need to check that $P_j(\omega^r) \neq 0$ for $r = 0, 1, \cdots, K_j-1$.

From Definition 2.2 and the orthogonality of Z_ℓ^j, we have

$$\begin{aligned}
\langle \varphi_j(x), \varphi_j(x-kh_j)\rangle &= \frac{1}{{K_j}^2}\sum_{\ell,n=0}^{K_j-1}\left\langle \frac{Z_\ell^j(x)}{Z_\ell^j(0)}, \frac{Z_n^j(x-kh_j)}{Z_n^j(0)}\right\rangle \\
&= \frac{1}{{K_j}^2}\sum_{\ell,n=0}^{K_j-1}\left\langle \frac{Z_\ell^j(x)}{Z_\ell^j(0)}, \frac{Z_n^j(x)}{Z_n^j(0)}\right\rangle e^{-iknh_j} \\
&= \frac{1}{{K_j}^2}\sum_{\ell=0}^{K_j-1}\frac{||Z_\ell^j||^2}{|Z_\ell^j(0)|^2}e^{-ik\ell h_j}.
\end{aligned}$$

Hence, for $r = 0, 1, \cdots, K_j-1$,

$$\begin{aligned}
P_j(\omega^r) &= \sum_{k=0}^{K_j-1}\langle \varphi_j(\cdot), \varphi_j(\cdot - kh_j)\rangle\omega^{rk} \\
&= \frac{1}{{K_j}^2}\sum_{\ell=0}^{K_j-1}\frac{||Z_\ell^j||^2}{|Z_\ell^j(0)|^2}\sum_{k=0}^{K_j-1}e^{i(r-\ell)kh_j} \\
&= \frac{1}{K_j}\frac{||Z_r^j||^2}{|Z_r^j(0)|^2} > 0.
\end{aligned}$$

Therefore, G_j is an invertible cirulant matrix with $G_j^{-1} = F\Lambda_j^{-1}F^*$.

The proof of the Lemma is finished.

Theorem 5.1. *Suppose that Λ_j and F are defined in Lemma 5.1, $e=(1,0,\cdots,0)\in I\!R^{K_j}$ is a K_j-dimensional unit vector, and*

$$\tilde{\varphi}_j(x) := eF\Lambda_j^{-1}F^* \begin{pmatrix} \varphi_j(x) \\ \varphi_j(x-h_j) \\ \vdots \\ \varphi_j(x-(K_j-1)h_j) \end{pmatrix} \in V_j.$$

Then $\{\tilde{\varphi}_j(x-kh_j)\}_{k=0}^{K_j-1}$ is a dual basis for $\{\varphi_j(x-kh_j)\}_{k=0}^{K_j-1}$, i.e.,

$$\langle \tilde{\varphi}_j(x-kh_j), \varphi_j(x-\ell h_j)\rangle = \delta_{k\ell} \qquad \text{for} \quad k,\ell = 0,1,\cdots,K_j-1.$$

Proof. Let

$$\Pi := \mathrm{Circ}(0,1,0,\cdots,0) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Note that $F\Lambda_j^{-1}F^*$ is a circulant matrix, $F\Lambda_j^{-1}F^*\Pi = \Pi F\Lambda_j^{-1}F^*$. Hence

$$\begin{aligned} \tilde{\varphi}_j(x-kh_j) &= eF\Lambda_j^{-1}F^*(\varphi_j(x-kh_j), \varphi_j(x-(k+1)h_j), \cdots, \varphi_j(x-(k+K_j-1)h_j))^T \\ &= eF\Lambda_j^{-1}F^* \cdot \Pi^k \cdot (\varphi_j(x), \varphi_j(x-h_j), \cdots, \varphi_j(x-(K_j-1)h_j))^T \\ &= e\Pi^k F\Lambda_j^{-1}F^* \cdot (\varphi_j(x), \varphi_j(x-h_j), \cdots, \varphi_j(x-(K_j-1)h_j))^T. \end{aligned}$$

Therefore

$$\begin{aligned} &\langle \tilde{\varphi}_j(x-kh_j), \varphi_j(x-\ell h_j)\rangle \\ &= e\Pi^k F\Lambda_j^{-1}F^* \begin{pmatrix} \langle \varphi_j(x), \varphi_j(x-\ell h_j)\rangle \\ \langle \varphi_j(x-h_j), \varphi_j(x-\ell h_j)\rangle \\ \vdots \\ \langle \varphi_j(x-(K_j-1)h_j), \varphi_j(x-\ell h_j)\rangle \end{pmatrix}, \end{aligned}$$

or equivalently,

$$(\langle \tilde{\varphi}_j(x-kh_j), \varphi_j(x-\ell h_j)\rangle)_{k,\ell=0}^{K_j-1} = I \cdot F\Lambda_j^{-1}F^* \cdot G_j = I,$$

which implies that $\{\tilde{\varphi}_j(x-kh_j)\}_{k=0}^{K_j-1}$ is a dual basis for $\{\varphi_j(x-kh_j)\}_{k=0}^{K_j-1}$.

Analogously, we can give the dual wavelets as follows.

Theorem 5.2. *Let*

$$Q_j(z) = \sum_{k=0}^{K_j-1} \langle L_j(\cdot), L_j(\cdot - kh_j)\rangle z^k,$$

$$\tilde{L}_j(x) = e\,F\,\mathrm{diag}\{(Q_j(1))^{-1}, (Q_j(\omega))^{-1}, \cdots, (Q_j(\omega^{K_j-1}))^{-1}\}F^* \begin{pmatrix} L_j(x) \\ L_j(x-h_j) \\ \vdots \\ L_j(x-(K_j-1)h_j) \end{pmatrix}.$$

Then $\langle \tilde{L}_j(x-kh_j), L_j(x-\ell h_j)\rangle = \delta_{k\ell}$ for $k,\ell=0,1,\cdots,K_j-1$.

Since $Q_j(\omega^r) = K_j \frac{(d_r^{j+1})^2 + (d_{K_j+r}^{j+1})^2}{|R_r^j(h_{j+1})|^2} > 0$, the proof of this theorem is similar to that of Theorem 5.1.

In applications, symmetry is very important. Since φ_j and L_j are symmetric, we have the following theorem.

Theorem 5.3. *Suppose that $g(x)$ is symmetric about the origin point. Then $\tilde{\varphi}_j(x)$ is symmetric about the origin and $\tilde{L}_j(x)$ is symmetric about the origin.*

Proof. For simplicity, we rewrite $\tilde{\varphi}_j(x)$ as $\tilde{\varphi}_j(x) = \sum\limits_{k=0}^{K_j-1} c_k\varphi_j(x-kh_j)$. Then, by the symmetry and periodicity of φ_j, we have

$$\begin{aligned}\tilde{\varphi}_j(-x) &= e\sum_{k=0}^{K_j-1} c_k\varphi_j(-x-kh_j) = c_0\varphi_j(x) + \sum_{k=1}^{K_j-1} c_k\varphi_j(x+kh_j)\\ &= c_0\varphi_j(x) + \sum_{k=1}^{K_j-1} c_k\varphi_j(x-(K_j-k)h_j) = c_0\varphi_j(x) + \sum_{k=1}^{K_j-1} c_{K_j-k}\varphi_j(x-kh_j).\end{aligned}$$

Since $\mathrm{Circ}(c_0, c_1, \cdots, c_{K_j-1}) = F\Lambda^{-1}F^*$ and $\Lambda = \mathrm{diag}\{P_j(1), P_j(\omega), \cdots, P_j(\omega^{K_j-1})\}$ is real, we see that $\mathrm{Circ}(c_0, c_1, \cdots, c_{K_j-1})$ is a real Hermitian matrix, so that $c_{K_j-k} = c_k$.

Therefore $\tilde{\varphi}_j(-x) = \sum\limits_{k=0}^{K_j-1} c_k\varphi_j(x-kh_j) = \tilde{\varphi}_j(x)$.

A similar discussion gives $\tilde{L}_j(x) = \tilde{L}_j(-x)$, which completes the proof of the theorem.

§6. Algorithms and Examples

In this section, we shall give the decomposition and reconstruction algorithms of this kind of periodic wavelets.

To this end, let $f(x) \in V_{j+1}$. Then we can rewrite $f(x)$ as

$$f(x) = \sum_{k=0}^{K_{j+1}-1} f(kh_{j+1})\varphi_{j+1}(x-kh_{j+1}) = \sum_{k=0}^{K_{j+1}-1} c_k^{j+1}\varphi_{j+1}(x-kh_{j+1}).$$

Let

$$\varphi_{j+1}(x) = \sum_{\ell=0}^{K_j-1} \{a_\ell^j\varphi_j(x-\ell h_j) + b_\ell^j L_j(x-\ell h_j)\}, \tag{6.1}$$

$$\varphi_{j+1}(x-h_{j+1}) = \sum_{\ell=0}^{K_j-1} \{p_\ell^j\varphi_j(x-\ell h_j) + q_\ell^j L_j(x-\ell h_j)\}. \tag{6.2}$$

Then

$$\begin{aligned}\varphi_{j+1}(x-2kh_{j+1}) &= \sum_{\ell=0}^{K_j-1} \{a_{\ell-k}^j\varphi_j(x-\ell h_j) + b_{\ell-k}^j L_j(x-\ell h_j)\}\\ \varphi_{j+1}(x-(2k+1)h_{j+1}) &= \sum_{\ell=0}^{K_j-1} \{p_{\ell-k}^j\varphi_j(x-\ell h_j) + q_{\ell-k}^j L_j(x-\ell h_j)\}.\end{aligned}$$

Therefore

$$\begin{aligned}f(x) = &\sum_{k=0}^{K_j-1} c_{2k}^{j+1}\sum_{\ell=0}^{K_j-1} \{a_{\ell-k}^j\varphi_j(x-\ell h_j) + b_{\ell-k}^j L_j(x-\ell h_j)\}\\ &+ \sum_{k=0}^{K_j-1} c_{2k+1}^{j+1}\sum_{\ell=0}^{K_j-1} \{p_{\ell-k}^j\varphi_j(x-\ell h_j) + q_{\ell-k}^j L_j(x-\ell h_j)\}\end{aligned}$$

$$= \sum_{\ell=0}^{K_j-1} \Big\{ \sum_{k=0}^{K_j-1} \Big[a^j_{\ell-k} c^{j+1}_{2k} + p^j_{\ell-k} c^{j+1}_{2k+1} \Big] \Big\} \varphi_j(x-\ell h_j)$$
$$+ \sum_{\ell=0}^{K_j-1} \Big\{ \sum_{k=0}^{K_j-1} \Big[b^j_{\ell-k} c^{j+1}_{2k} + q^j_{\ell-k} c^{j+1}_{2k+1} \Big] \Big\} L_j(x-\ell h_j),$$

which induces the following Decomposition Formulas

$$c^j_\ell = \sum_{k=0}^{K_j-1} \Big[a^j_{\ell-k} c^{j+1}_{2k} + p^j_{\ell-k} c^{j+1}_{2k+1} \Big], \quad d^j_\ell = \sum_{k=0}^{K_j-1} \Big[b^j_{\ell-k} c^{j+1}_{2k} + q^j_{\ell-k} c^{j+1}_{2k+1} \Big].$$

Since

$$\varphi_j(x) = \sum_{\ell=0}^{K_{j+1}-1} \varphi_j(\ell h_{j+1}) \varphi_{j+1}(x-\ell h_{j+1})$$
$$L_j(x) = \sum_{\ell=0}^{K_{j+1}-1} L_j(\ell h_{j+1}) \varphi_{j+1}(x-\ell h_{j+1}),$$

a simple calculation gives

$$\sum_{k=0}^{K_j-1} c^j_k \varphi_j(x-kh_j) + \sum_{k=0}^{K_j-1} d^j_k L_j(x-kh_j)$$
$$= \sum_{k=0}^{K_j-1} c^j_k \sum_{\ell=0}^{K_{j+1}-1} \varphi_j(\ell h_{j+1}) \varphi_{j+1}(x-\ell h_{j+1}-2kh_{j+1})$$
$$+ \sum_{k=0}^{K_j-1} d^j_k \sum_{\ell=0}^{K_{j+1}-1} L_j(\ell h_{j+1}) \varphi_{j+1}(x-\ell h_{j+1}-2kh_{j+1})$$
$$= \sum_{\ell=0}^{K_{j+1}} \Big\{ \sum_{k=0}^{K_j-1} c^j_k \varphi_j((\ell-2k)h_{j+1}) + \sum_{k=0}^{K_j-1} d^j_k L_j((\ell-2k)h_{j+1}) \Big\} \varphi_{j+1}(x-\ell h_{j+1}).$$

Therefore, we obtain the Reconstruction Formulas

$$c^{j+1}_\ell = \sum_{k=0}^{K_j-1} \{ c^j_k \varphi_j((\ell-2k)h_{j+1}) + d^j_k L_j((\ell-2k)h_{j+1}) \},$$

or

$$c^{j+1}_{2\ell} = c^j_\ell + \sum_{k=0}^{K_j-1} d^j_k L_j((\ell-k)h_j) \quad c^{j+1}_{2\ell+1} = \sum_{k=0}^{K_j-1} c^j_k \varphi_j((\ell-k)h_j + h_{j+1}) + d^j_\ell.$$

Now, we are going to compute $a^j_\ell, b^j_\ell, p^j_\ell$, and q^j_ℓ.

From

$$\varphi_{j+1}(x) = \sum_{\ell=0}^{K_j-1} \{ a^j_\ell \varphi_j(x-\ell h_j) + b^j_\ell L_j(x-\ell h_j) \},$$

and the duality of $\varphi_j(x)$ and $\tilde{\varphi_j(x)}$, we have $a^j_\ell = \langle \tilde{\varphi}_j(x-\ell h_j), \varphi_{j+1}(x) \rangle$. Recall that

$$\tilde{\varphi}_j(x) = \sum_{k=0}^{K_{j+1}-1} \tilde{\varphi}_j(kh_{j+1}) \varphi_{j+1}(x-kh_{j+1}) \in V_{j+1},$$

thus,

$$a_\ell^j = \sum_{k=0}^{K_{j+1}-1} \tilde{\varphi}_j(kh_{j+1})\langle \varphi_{j+1}(x-kh_{j+1}-\ell h_j), \varphi_{j+1}(x)\rangle$$
$$= \sum_{k=0}^{K_{j+1}-1} \tilde{\varphi}_j((k-2\ell)h_{j+1})\langle \varphi_{j+1}(x-kh_{j+1}), \varphi_{j+1}(x)\rangle,$$

analogously, we have,

$$b_\ell^j = \sum_{k=0}^{K_{j+1}-1} \tilde{L}_j((k-2\ell)h_{j+1})\langle \varphi_{j+1}(x-kh_{j+1}), \varphi_{j+1}(x)\rangle$$
$$p_\ell^j = \sum_{k=0}^{K_{j+1}-1} \tilde{\varphi}_j((k-2\ell+1)h_{j+1})\langle \varphi_{j+1}(x-kh_{j+1}), \varphi_{j+1}(x)\rangle$$
$$q_\ell^j = \sum_{k=0}^{K_{j+1}-1} \tilde{L}_j((k-2\ell+1)h_{j+1})\langle \varphi_{j+1}(x-kh_{j+1}), \varphi_{j+1}(x)\rangle.$$

§7. Final Remarks

In this paper, we construct interpolatory wavelets and their corresponding dual wavelets from a periodic function. These wavelets are symmetric, but are not orthogonal. Examples show that they share some localization property, but we do not prove that now.

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