Chin. Ann. of Math.
19B: 2(1998),143-152.

ASYMPTOTIC BEHAVIOR OF SOLUTIONS TO THE SYSTEM OF ONE-DIMENSIONAL NONLINEAR THERMOVISCOELASTICITY***

Hsiao Ling* Jian Huaiyu**

Abstract

The authors study the large-time behaviour of global smooth solutions to initial-boundary value problems for the system of one-dimensional nonlinear thermoviscoelasticity. It is found that the solution may possess phase transition phenomena when the material is not monotone, and the solution may decay to a stable state for the monotone case.

Keywords Asymptotic behavior, Nonlinear thermoviscoelasticity, Initial boundary value problem
1991 MR Subject Classification 35L60
Chinese Library Classification O175.27

§1. Introduction

This paper is devoted to the study of the large-time behaviour of smooth solutions for initial-boundary value problems in one-dimensional nonlinear thermoviscoelasticity. The equations describing the motion of one-dimensional materials with reference density $\rho_0 = 1$ are those of balance of mass, momentum and energy

$$\begin{cases} u_t - v_x = 0, \\ v_t - \sigma_x = 0, \\ (e + v^2/2)_t - (\sigma v)_x + q_x = 0 \end{cases} \tag{1.1}$$

supplemented with the second law of thermodynamics expressed by the Clausius-Duhem inequality

$$\eta_t + \left(\frac{q}{\theta}\right)_x \geq 0, \tag{1.2}$$

where $u, v, e, \sigma, \eta, \theta$ and q denote deformation gradient, velocity, internal energy, stress, specific entropy, temperature and heat flux, respectively.

Consider a body with reference configuration the interval $[0,1]$ and the initial values of u, v and θ, given by

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \theta(x,0) = \theta_0(x), \quad x \in [0,1]. \tag{1.3}$$

When the boundary conditions are imposed as stress-free and thermally insulated, namely

$$\sigma(0,t) = \sigma(1,t) = 0, \quad q(0,t) = q(1,t) = 0, \quad t \geq 0, \tag{1.4}$$

Manuscript received October 9, 1995. Revised February 2, 1997.
*Institute of Mathematics, Academia Sinica, Beijing 100080, China.
**Department of Applied Mathematics, Tsinghua University, Beijing 100084, China.
***Project supported by the National Natural Science Foundation of China.

Dafermos and Hsiao[1] and Dafermos[2] established the global existence of smooth solutions to (1.1), (1.3) and (1.4) for a kind of solid-like materials. Recently, Jiang[3] discussed the following boundary conditions

$$\sigma(1,t) = -\gamma v(1,t), \quad \sigma(0,t) = \gamma v(0,t), \quad \theta(1,t) = \theta(0,t) = T_0, \quad t \geq 0, \tag{1.5}$$

where $\gamma = 0$ or $\gamma = 1$, and $T_0 > 0$ is the reference temperature. He obtained the global existence of classical solutions to (1.1), (1.3) and (1.5) for the same kind of solid-like materials as in [2]. Physically, the case with $\gamma = 1$ represents that the endpoints of the interval $[0,1]$ are connected to some sort of dash pot.

It is well understood that the large-time problem of the system (1.1) is of interest since the pressure function $\hat{p}(u,\theta)$ is not necessarily monotone in u. Unfortunately, this problem has been open (except the investigation for the material of ideal gas, see [8]), no matter whether $\hat{p}(u,\theta)$ is monotone in u or not, untill quite recently. As the first step to solve this problem, Hsiao and Luo[4] considered a kind of material with the following constitutive relations:

$$e = c_0\theta, \quad \sigma = -\hat{p}(u,\theta) + \hat{\mu}(u)v_x, \quad \hat{p}(u,\theta) = f(u)\theta, \quad q = -k(u)\theta_x, \tag{1.6}$$

where $c_0 > 0$ is a constant, $f(u)$ and $k(u)$ are twice continuously differentiable for $u > 0$ such that

$$k(u) > 0 \ \text{ for } \ u > 0, \quad f(u) \geq 0 \ \text{ if } \ 0 < u < \check{u}, \quad f(u) \leq 0 \ \text{ if } \ \check{U} < u < +\infty, \tag{1.7}$$

for some fixed $0 < \check{u} < \check{U} < +\infty$, and

$$\hat{\mu}(u) \geq \mu_0 > 0, \quad 0 < u < +\infty \tag{1.8}$$

for some fixed $\mu_0 > 0$. It is proved in [4] that the solution of (1.1), (1.3) and (1.4) may decay to a unique state, or may possess phase transition phenomena, according as $f(u)$ is monotone or not. For the second case, the large time behaviour of u is described by a Young measure whose support is confined in the set of zero of $f(u)$. The result established in [4] is a good extension of the results obtained by Greenberg and MacCamy in [5], and by Andrews and Ball in [6], respectively for isothermal materials (i.e., $\theta \equiv$ constant in (1.1)).

It is important to investigate the difference in the large-time behavior of solutions caused by different boundary conditions. In the present paper, we develope the analysis in [4] to discuss the case of boundary condition (1.5) under the following contitutive relations:

$$e = c_0\theta, \quad \sigma = -f(u)\theta + \hat{\mu}(u)v_x, \quad q = -k(u,\theta)\theta_x, \tag{1.9}$$

where $c_0 > 0$ is a constant, $f(u)$ and $\hat{\mu}(u)$ satisfy (1.7) and (1.8) respectively, while $k(u,\theta)$ satisfies the condition that for any given positive constants a and A: $0 < a \leq A < +\infty$, there exist positive constants ν and N, possibly depending on a and/or A, such that

$$\nu \leq k(u,\theta) \leq N, \quad |k_u(u,\theta)| \leq N, \quad |k_{uu}(u,\theta)| \leq N, \quad |k_\theta(u,\theta)| \leq N \tag{1.10}$$

for any $(u,\theta) \in [a,A] \times (0,+\infty)$.

For initial data, we assume that $u_0(x), u_0'(x), v_0(x), v_0'(x), v_0''(x), \theta_0(x)$, $\theta_0'(x)$ and $\theta_0''(x)$ are all in $C^\alpha[0,1]$ for some $0 < \alpha < 1$, $u_0(x) > 0$ and $\theta_0(x) > 0$ for $x \in [0,1]$, and the initial data are compatible with the boundary condition (1.5). Normalizing the initial velocity, we may suppose

$$\int_0^1 v_0(x)dx = 0 \ \text{ if } \ \gamma = 0. \tag{1.11}$$

The global existence of smooth solutions to (1.1), (1.3) and (1.5), under (1.7) − (1.11) and the assumptions on initial data, can be established by the approach in [3]. Namely, there exists a unique solution $\{u(x,t), v(x,t), \theta(x,t)\}$ to (1.1), (1.3) and (1.5) on $[0,1] \times [0,+\infty)$ such that for every $T > 0$ the functions $u, u_x, u_t, v, v_x, v_t, v_{xx}, \theta, \theta_x, \theta_t, \theta_{xx}$ are all in $C^{\alpha,\alpha/2}(Q_T)$ and $u_{tt}, v_{xt}, \theta_{xt}$ are in $L^2(Q_T)$, $Q_T = [0,1] \times [0,T]$. Moreover,

$$\theta(x,t) > 0 \quad \text{for} \quad 0 \le x \le 1, \quad t \ge 0, \tag{1.12}$$

$$\bar{u} < u(x,t) < \bar{U} \quad \text{for} \quad 0 \le x \le 1, \quad t \ge 0, \tag{1.13}$$

where $\bar{u}$ and $\bar{U}$ are positive constants, independent of T.

The difference in our boundary condition (1.5) compared to the one in (1.4) causes a lot of new difficulties in establishing the large-time behavior of solutions in the present paper.

Let

$$h(u) = \int_{\bar{u}}^u f(\xi)d\xi, \tag{1.14}$$

$$E_0 = \int_0^1 \Big[c_0(\theta_0 - T_0 \ln \theta_0) + \frac{v_0^2}{2} - T_0 h(u_0)\Big](x)dx, \quad \hat{E}_0 = \int_0^1 \Big(c_0\theta_0 + \frac{v_0^2}{2}\Big)(x)dx. \tag{1.15}$$

The following results will be established in the present paper.

Theorem 1.1. *Let* $\{u(x,t), v(x,t), \theta(x,t)\}$ *be the globally defined smooth solution of* (1.1), (1.3) *and* (1.5), *decribed as above, satisfying* (1.12) *and* (1.13). *Then, it holds that*

(I) *As* $t \to \infty$,

$$\|f(u)(\cdot\,,t)\|_{L^2[0,1]} + \|f(u)\theta(\cdot\,,t)\|_{L^1[0,1]} \to 0, \tag{1.16}$$

$$\|v(\cdot\,,t)\|_{L^2[0,1]} \to 0, \tag{1.17}$$

$$\int_0^t [k(u,T_0)\theta_x(1,\tau) - k(u,T_0)\theta_x(0,\tau)]d\tau - c_0 \int_0^1 \theta(x,t)dx \to I_\gamma - \hat{E}_0. \tag{1.18}$$

There are constants B_1 *and* B_2 *such that*

$$\int_0^1 (\theta - T_0 \ln \theta)(x,t)dx \to B_1 \ \ \textit{and} \ \ \int_0^1 h(u)(x,t)dx \to B_2, \tag{1.19}$$

$$c_0 B_1 - T_0 B_2 = E_0 - I_\gamma - I_0, \tag{1.20}$$

where E_0 *and* $\hat{E}_0$ *are given by* (1.15), I_γ *and* I_0 *are convergent infinite integrals* (*the convergence will be shown in section* 2), *defined by*

$$I_\gamma = \gamma \int_0^\infty [v^2(1,\tau) + v^2(0,\tau)]d\tau,$$
$$I_0 = T_0 \int_0^\infty \int_0^1 \Big[\frac{\hat{\mu}(u){v_x}^2}{\theta} + \frac{k(u,\theta){\theta_x}^2}{\theta^2}\Big](x,\tau)dxd\tau. \tag{1.21}$$

(II) *For the case of* $\gamma = 0$, *there exists a family of probability measure* $\{\nu_x\}_{x\in[0,1]}$ *on* $\Re$ *with* $\mathrm{Supp}\nu_x \subset \{z\colon f(z) = 0\}$ *such that if* $\Phi \in C(R)$ *and* $g_\Phi(x) \overset{\text{def.}}{=} \langle \nu_x, \Phi \rangle$, *a.e., then*

$$\Phi(u(\cdot\,,t)) \overset{*}{\rightharpoonup} g_\Phi(\cdot\,) \ \ \textit{in} \ \ L^\infty[0,1], \quad \textit{as} \quad t \to +\infty. \tag{1.22}$$

Particularly,

$$B_2 = \int_0^1 \int_0^1 h(z) d\nu_x(z) dx. \tag{1.23}$$

Corollary 1.1. *Suppose $\gamma = 0$ and the equation $f(z) = 0$ possesses only one root z_1. Then*

$$u(\cdot\,,t) \to z_1 \quad strongly \quad in \quad L^q[0,1], \quad as \quad t \to +\infty, \tag{1.24}$$

for all $q \geq 1$. Thus

$$B_2 = h(z_1). \tag{1.25}$$

Corollary 1.2. *Suppose $\gamma = 0$ and the equation $f(z) = 0$ has exactly m roots, $z_1, z_2, \cdots,$ z_m, $m > 1$. Then there exist nonnegative functions $\mu_i \in L^\infty[0,1]$, $1 \leq i \leq m$, $\sum\limits_{i=1}^m \mu_i(x) = 1$, a.e. in $[0,1]$, such that for any $\Phi \in C(\Re)$,*

$$\Phi(u(\cdot\,,t)) \overset{*}{\rightharpoonup} \sum_{i=1}^m \Phi(z_i)\mu_i(\cdot) \quad in \quad L^\infty[0,1], \quad as \quad t \to +\infty. \tag{1.26}$$

Particularly,

$$B_2 = \sum_{i=1}^m h(z_i) \int_0^1 \mu_i(x) dx. \tag{1.27}$$

If $f(u)$ is strictly decreasing, namely,

$$-f'(u) > 0, \quad u \in [\bar{u}, \bar{U}], \tag{1.28}$$

then $f(u) = 0$ possesses only one root $u = \hat{z}$. In this case, we have further result.

Theorem 1.2. *In addition to the assumptions of Theorem 1.1, suppose that (1.28) is filfulled. Then as $t \to +\infty$, it holds that*

$$\|v(\cdot\,,t)\|_{H^1[0,1]} + \|\theta(\cdot\,,t) - T_0\|_{H^1[0,1]} + \|u(\cdot\,,t) - \hat{z}\|_{H^1[0,1]} \to 0$$

and $\gamma[v^2(1,t) + v^2(0,t)] \to 0$.

We will prove Theorem 1.1 and the corollaries in section 2 and Theorem 1.2 in section 3, respectively.

§2. Proof of Theorem 1.1

Throughout this section, $\{u(x,t), v(x,t), \theta(x,t)\}$ will denote the solution described in the global existence theorem and Λ will denote a generic constant, independent of t. Denote $Q_t = [0,1] \times [0,t]$ for any $t \geq 0$.

Let

$$\check{\theta} = \theta - T_0 \ln \theta, \quad w(u, \theta, v_x, \theta_x) = \frac{\hat{\mu}(u) {v_x}^2}{\theta} + \frac{k(u,\theta) {\theta_x}^2}{\theta^2}. \tag{2.1}$$

It can be shown , by (1.1), (1.9), (1.12) and (1.14), that

$$\Big[c_0 \check{\theta} + \frac{v^2}{2} - T_0 h(u)\Big]_t + T_0 w = \Big[\sigma v + \frac{(\theta - T_0)}{\theta} k(u,\theta)\theta_x\Big]_x. \tag{2.2}$$

Lemma 2.1. *There exist positive constants Λ, α and β, independent of t, such that for all $t \geq 0$ it holds that*

$$\int_0^t \int_0^1 \Big(\frac{{v_x}^2}{\theta} + \frac{{\theta_x}^2}{\theta^2}\Big)(x,\tau) dx d\tau + \int_0^1 \frac{v^2}{2}(x,t) dx + \gamma \int_0^t [v^2(1,\tau) + v^2(0,\tau)] d\tau \leq \Lambda, \tag{2.3}$$

$$\alpha \le \int_0^1 \theta(x,t)dx \le \beta, \tag{2.4}$$

$$\int_0^t \max_{[0,1]} v^2(\cdot\,,\tau)d\tau \le \Lambda. \tag{2.5}$$

Proof. Integrating (2.2) over Q_t and using (1.3) and (1.5), we have

$$\int_0^1 \Big[c_0\check{\theta} + \frac{v^2}{2} - T_0h(u)\Big](x,t)dx + \int_0^t\int_0^1 T_0w(x,\tau)dxd\tau$$
$$+\gamma\int_0^t [v^2(1,\tau)+v^2(0,\tau)]d\tau = E_0, \tag{2.6}$$

which, combined with the fact $\check{\theta} \ge T_0 - T_0\ln T_0$ and (1.13), implies that

$$\int_0^1 \frac{v^2}{2}dx + \int_0^t\int_0^1 T_0w(x,\tau)dxd\tau + \gamma\int_0^t [v^2(1,\tau)+v^2(0,\tau)]d\tau \le E_1,$$

where

$$E_1 = \int_0^1 \{c_0[(\theta_0((x) - T_0\ln\theta_0(x)) - (T_0 - T_0\ln T_0)] + T_0[\max_{u\in[\bar{u},\bar{U}]} h(u) - h(u_0(x))]\}dx > 0.$$

This, together with (1.8) and (1.10), yields (2.3).

From (2.6), (1.13) and (1.14), it follows that $\int_0^1(\theta - T_0\ln\theta)(x,t)dx \le \Lambda$, for some positive constant Λ, which, applied by Jensen's inequality for the convex function $y = z - \ln z$, implies that $\int_0^1 \theta(x,t)dx$ is between the two positive roots of the equation $z - T_0\ln z = \Lambda$. Thus, (2.4) is obtained.

For the proof of (2.5), we distinguish two cases. If $\gamma = 1$, one has

$$\max_{[0,1]} v^2(\cdot\,,t) \le 2\Big[v^2(0,t) + \int_0^1 \frac{{v_x}^2}{\theta}(x,t)dx\cdot\int_0^1 \theta(x,t)dx\Big], \tag{2.7}$$

which, combined with (2.3) and (2.4), yields (2.5). In the case $\gamma = 0$, integrating $(1.1)_2$ over Q_t, using (1.11), and employing (2.3) and (2.4) then, we are able to get (2.5) in a similar way as in [4] for the proof of (2.10) there.

Lemma 2.2. *For all* $t \ge 0$,

$$\int_0^1 (\theta^2 + v^4)(x,t)dx + \int_0^t\int_0^1 \Big[\check{\theta}_x^2 + v^2\Big({v_x}^2 + \frac{{v_x}^2}{\theta} + \frac{{\theta_x}^2}{\theta^2}\Big)\Big](x,\tau)dxd\tau$$
$$+\gamma\int_0^t [v^4(1,\tau)+v^4(0,\tau)]d\tau \le \Lambda. \tag{2.8}$$

Proof. Multiply $(1.1)_2$ by v^3 and integrate it over Q_t. Using (1.9), (1.3), (1.5) and integration by parts, we obtain, due to (1.13), (1.8), (1.9) and Young's inequality, that

$$\int_0^1 v^4(x,t)dx + \gamma\int_0^t [v^4(1,\tau)+v^4(0,\tau)]d\tau + \int_0^t\int_0^1 v^2{v_x}^2dxd\tau$$
$$\le \Lambda + \Lambda\int_0^t \max_{[0,1]} v^2(\cdot\,,\tau)\cdot\int_0^1 \theta^2(x,\tau)dxd\tau. \tag{2.9}$$

Next, we rewrite (2.2) as $\Big[c_0\check{\theta}+\frac{v^2}{2}\Big]_t + T_0w = [\sigma v + k(u,\theta)\check{\theta}_x]_x + [T_0h(u)]_t$, which, multiplied by $(c_0\check\theta+\frac{v^2}{2})$ and integrated over Q_t, with the help of (1.5), (1.3), (1.9) and (2.3), yields

$$\frac{1}{2}\int_0^1\Big[c_0\check\theta+\frac{v^2}{2}\Big]^2(x,t)dx+\int_0^t\int_0^1\Big[\Big(c_0\check\theta+\frac{v^2}{2}\Big)T_0w+c_0k(u,\theta)\check\theta_x^2+\hat\mu(u)v_x^2v^2\Big]dxd\tau$$
$$\le\Lambda+\int_0^t\int_0^1\{c_0f(u)\theta v\check\theta_x+c_0f(u)\theta v^2v_x-[c_0\hat\mu(u)+k(u,\theta)]vv_x\check\theta_x\}(x,\tau)dxd\tau$$
$$+\int_0^t\int_0^1\frac{v^2}{2}[T_0h(u)]_\tau(x,\tau)dxd\tau+\int_0^t\int_0^1c_0\check\theta[T_0h(u)]_\tau(x,\tau)dxd\tau. \tag{2.10}$$

Let us denote the ith integral term on the right-hand side of (2.10) by $I_i, i=1,2,3$. Using Young's inequality, (1.13) and (2.9), we can show that for any given $\varepsilon>0$

$$|I_1|\le\Lambda(\varepsilon)\Big[1+\int_0^t\max_{[0,1]}v^2(\cdot,\tau)\int_0^1\theta^2(x,\tau)dxd\tau\Big]+\varepsilon\int_0^t\int_0^1\check\theta_x^2(x,\tau)dxd\tau. \tag{2.11}$$

Similarly, due to (1.14), $(1.1)_1$, (2.5) and (2.9), it holds that

$$|I_2|\le\Lambda+\Lambda\int_0^t\max_{[0,1]}v^2(\cdot,\tau)\int_0^1\theta^2(x,\tau)dxd\tau. \tag{2.12}$$

In order to bound I_3, we combine $(1.1)_3$ and (2.2) to get

$$[T_0h(u)]_t=T_0\Big[\Big(\frac{k(u,\theta)\theta_x}{\theta}\Big)_x-(c_0\ln\theta)_t+w\Big], \tag{2.13}$$

by which I_3 can be expressed as

$$I_3=\int_0^t\int_0^1c_0\check\theta T_0w(x,\tau)dxd\tau+c_0T_0\int_0^t\int_0^1\Big[\frac{k(u,\theta)\theta_x}{\theta}\Big]_x\check\theta(x,\tau)dxd\tau$$
$$-c_0^2T_0\int_0^t\int_0^1[\ln\theta]_\tau\check\theta(x,\tau)dxd\tau. \tag{2.14}$$

We denote the second and the third integral terms on the right-hand side of (2.14) by I_4 and I_5 respectively. With the help of integration by parts, (1.5), (2.3) and Young's inequality, it can be proved that for any given $\varepsilon>0$,

$$I_4\le\Lambda(\varepsilon)+\varepsilon\int_0^t\int_0^1\check\theta_x^2dxd\tau+\Lambda\Big|\int_0^t[k(u,\theta)\theta_x(1,\tau)-k(u,\theta)\theta_x(0,\tau)]d\tau\Big|.$$

Furthermore, by integrating $(1.1)_3$ over Q_t and using (1.5), (1.9), (2.3) and (2.4), it turns

$$\Big|\int_0^t[k(u,\theta)\theta_x(1,\tau)-k(u,\theta)\theta_x(0,\tau)]d\tau\Big|\le\Lambda. \tag{2.15}$$

Therefore, for any given $\varepsilon>0$,

$$I_4\le\Lambda(\varepsilon)+\varepsilon\int_0^t\int_0^1\check\theta_x^2(x,\tau)dxd\tau. \tag{2.16}$$

Since $(\ln\theta)_\tau\check\theta=\theta_\tau-\frac{T_0}{2}(\ln^2\theta)_\tau$, it follows easily from (1.3) and (2.4) that

$$I_5\le\Lambda+\frac{c_0^2T_0^2}{2}\int_0^1\ln^2\theta(x,t)dx. \tag{2.17}$$

Due to (2.10)–(2.12), (2.14), (2.16), (2.17) and the smallness of ε, we arrive at

$$\frac{1}{2}\int_0^1\overline{W}(\theta,v)(x,t)+\frac{T_0}{2}\int_0^t\int_0^1v^2w(x,\tau)dxd\tau+\int_0^t\int_0^1\Big[\frac{c_0\nu}{2}\check\theta_x^2+\hat\mu(u)v^2v_x^2\Big]dxd\tau$$
$$\le\Lambda[1+\int_0^t\max_{[0,1]}v^2(\cdot,\tau)\int_0^1\theta^2(x,\tau)dxd\tau], \tag{2.18}$$

where $\overline{W}(\theta,v)=c_0^2\theta^2+\frac{v^4}{4}-2c_0^2T_0\theta\ln\theta-c_0T_0v^2\ln\theta+c_0v^2\theta$.

From the fact that $\lim\limits_{\eta\to+\infty}\frac{\ln\eta}{\eta}=0$ and $\ln\eta<0$ as $0<\eta<1$, it follows that there exists a positive constant M such that

$$\overline{W}(\theta,v)\ge\frac{c_0^2\theta^2}{2}-M(v^4+1)\quad\text{for any}\quad(\theta,v)\in(0,+\infty)\times(-\infty,+\infty). \tag{2.19}$$

(2.19), (2.18) and (2.9) yield

$$\int_0^1(\theta^2+v^4)(x,t)dx+\int_0^t\int_0^1\Big[\check{\theta}_x^2+v^2\Big(v_x^2+\frac{v_x^2}{\theta}+\frac{\theta_x^2}{\theta^2}\Big)\Big](x,\tau)dxd\tau$$
$$\le\Lambda\Big[1+\int_0^t\max_{[0,1]}v^2(\cdot\,,\tau)\int_0^1\theta^2(x,\tau)dxd\tau\Big].$$

This and (2.5), with the help of Gronwall's inequality, imply (2.8).

Lemma 2.3. *For all $t\ge0$,*

$$\int_0^t\int_0^1v_x^2(x,\tau)dxd\tau\le\Lambda. \tag{2.20}$$

Proof. We multiply $(1.1)_2$ by v and integrate it then over Q_t. By use of integration by parts, (1.5) and (1.9) yield

$$\int_0^1\frac{v^2}{2}(x,t)dx+\gamma\int_0^t[v^2(1,\tau)+v^2(0,\tau)]d\tau+\int_0^t\int_0^1\hat{\mu}(u)v_x^2dxd\tau$$
$$=\int_0^1\frac{v^2}{2}(x,0)dx+\int_0^t\int_0^1f(u)\theta v_xdxd\tau. \tag{2.21}$$

To control the last term, we use an idea from [4] to estimate $I_6\overset{\text{def.}}{=}\int_0^t\int_0^1f(u)\check{\theta}v_xdxd\tau$ first. Using (1.1), integration by parts, mean's value theorem and Young's inequality, with the help of (1.13), (2.3), (2.5) and (2.8), we can obtain, as in [4], that

$$|I_6|\le\Lambda(\varepsilon)+2\varepsilon\int_0^t\int_0^1v_x^2dxd\tau. \tag{2.22}$$

Next, we estimate $I_7\overset{\text{def.}}{=}\int_0^t\int_0^1f(u)v_xT_0\ln\theta dxd\tau$, which, due to $(1.1)_1$ and (2.13), can be written as

$$I_7=T_0\int_0^t\int_0^1w\ln\theta dxd\tau+T_0\int_0^t\int_0^1\Big[\frac{k(u,\theta)\theta_x}{\theta}\Big]_x\ln\theta dxd\tau-c_0T_0\int_0^t\int_0^1(\ln\theta)(\ln\theta)_\tau dxd\tau$$
$$\overset{\text{def.}}{=}I_8+I_9-I_{10},$$

where I_8,I_9 and I_{10} denote the above three integral terms successively.

In view of (2.1), (2.3) and (2.8), it is known that

$$\int_0^t\int_0^1\theta_x^2dxd\tau\le\Lambda. \tag{2.23}$$

As $\lim\limits_{\eta\to+\infty}\ln\eta/\sqrt{\eta}=0$, there exists a constant $\widehat{M}>0$ such that $\ln\eta\le\widehat{M}\sqrt{\eta}\le\widehat{M}\eta$ for all $\eta\ge1$. Thus, it follows from (2.3) and (2.23) that

$$I_8\le\widehat{M}T_0\Big\{\int_0^t\int_0^1\Big[\frac{\hat{\mu}(u)v_x^2}{\sqrt{\theta}}+\frac{k(u,\theta)\theta_x^2}{\theta}\Big]dxd\tau\Big\}\le\Lambda(\varepsilon)+\varepsilon\int_0^t\int_0^1v_x^2dxd\tau.$$

By virtue of integration by parts, (1.5) and (2.3), we have

$$I_9\le\ln T_0\int_0^t[k(u,\theta)\theta_x(1,\tau)-k(u,\theta)\theta_x(0,\tau)]d\tau+\Lambda,$$

which is bounded due to (2.15).

It is obvious to get the boundedness for $-I_{10}$.

Therefore, it follows that

$$I_7 \leq \varepsilon \int_0^t \int_0^1 v_x^2 dx d\tau + \Lambda(\varepsilon). \tag{2.24}$$

This, together with (2.22), shows that for any $\varepsilon > 0$,

$$\int_0^t \int_0^1 f(u)\theta v_x dx d\tau = I_6 + I_7 \leq \Lambda(\varepsilon) + 3\varepsilon \int_0^t \int_0^1 v_x^2 dx d\tau,$$

which together with (2.21) implies (2.20).

The following two lemmas can be proved according to a similar arguement in [4]. In order to make this paper not too lengthy, we have to omit their proofs.

Lemma 2.4. *For all $t \geq 0$,*

$$\int_0^t \int_0^1 [f(u)\theta]^2(x,\tau) dx d\tau \leq \Lambda. \tag{2.25}$$

Lemma 2.5. *For all $t \geq 0$,*

$$\int_0^t \int_0^1 f^2(u)(x,\tau) dx d\tau \leq \Lambda. \tag{2.26}$$

Moreover, as $t \to \infty$,

$$\int_0^1 v^2(x,t) dx \to 0, \tag{2.27}$$

$$\int_0^1 f^2(u)(x,t) dx \to 0, \tag{2.28}$$

$$\int_0^1 |f(u)\theta|(x,t) dx \to 0. \tag{2.29}$$

Now, we are ready to complete the proof of Theorem 1.1 and the corollaries. Obviously, (1.16) and (1.17) follow directly from Lemma 2.5. Integrating $(1.1)_3$ over Q_t and using (2.27), we obtain (1.18). Due to (2.26) and (2.20),

$$\int_0^t \Big| \frac{d}{d\tau} \int_0^1 h(u)(x,\tau) dx \Big| d\tau \leq 2 \int_0^t \int_0^1 [f^2(u) + v_x^2] dx d\tau \leq \Lambda,$$

which yields the existence of $\lim\limits_{t \to +\infty} \int_0^1 h(u)(x,t)dx$. Thus, (2.6) leads to (1.19) and (1.20), where the convergence of I_γ and I_0 can be guaranteed by (2.3).

Using the results obtained above and employing an idea in [6] and the same arguments as those used in [4] (see the approach from (2.28) to Lemma 3.7 in [4]), we can prove the second part of Theorem 1.1. Due to the results of Theorem 1.1, one is able to prove Corollaries 1.1 and 1.2 by applying the same methods as those used in [6] to show Corollaries 4.2 and 4.3 there.

§3. Proof of Theorem 1.2

Due to the assumptions of Theorem 1.2, all of the results obtained in section 2 can be used in this section. Furthermore, it follows from (1.28) that there exists a positive constant λ_0 such that

$$-f'(u) \geq \lambda_0 > 0 \quad \text{for} \quad u \in [\bar{u}, \bar{U}]. \tag{3.1}$$

Define $H(u)=\int_{\bar u}^{u}\hat\mu(\xi)d\xi$. Then, $(1.1)_2$ may be written as

$$v_t+[f(u)\theta]_x=[H(u)_t]_x \quad (\equiv [H(u)_x]_t). \tag{3.2}$$

Multiply (3.2) by $H(u)_x=\hat\mu(u)u_x$ and integrate it over Q_t. Applying Young's inequality, with the help of integration by parts, (1.13), (2.3), (2.23), (3.2), (2.5), and (2.8) and (3.1), we obtain

$$\int_0^1 u_x^2(x,t)dx+\int_0^t\int_0^1 \theta u_x^2(x,\tau)dxd\tau\le\Lambda,\ \text{for all}\ \ t\ge 0. \tag{3.3}$$

By the same argument as used in [4] (see (3.6) to (3.7) in [4]), (3.3) can imply that

$$\int_0^t\int_0^1 u_x^2dxd\tau\le\Lambda\Big[\int_0^t\int_0^1\theta u_x^2dxd\tau+\Big(\max_{\tau\ge 0}\int_0^1 u_x^2(x,\tau)dx\Big)\cdot\int_0^t\int_0^1\frac{\theta_x^2}{\theta^2}dxd\tau\Big]\le\Lambda. \tag{3.4}$$

Using $(1.1)_2$, (1.13), (1.9) and Young's inequality, we can show that for any $T>0$ and $t>T$,

$$\frac12\int_T^t\int_0^1 v_{xx}^2dxd\tau\le\Lambda\int_T^t\int_0^1(u_x^2v_x^2+u_x^2\theta^2+\theta_x^2)dxd\tau+\Lambda\int_T^t\int_0^1 v_\tau\sigma_xdxd\tau. \tag{3.5}$$

Let us denote the two terms on the right-hand side of (3.5) by $I^{(1)}$ and $I^{(2)}$ respectively. To control these integral terms, we need the following well-known result: for every $X\in W^{1,1}([0,1]\times[T,t])$ and any $\varepsilon>0$, it holds that

$$\max_{[0,1]}X^2(\cdot,\tau)\le\Lambda(\varepsilon)\int_0^1 X^2(x,\tau)dx+\varepsilon\int_0^1 X_x^2(x,\tau)dx,\quad\text{for all}\ \tau\in[T,t]. \tag{3.6}$$

Applying (3.6) to v_x and θ respectively, and then using (3.3) and (2.8) to control

$$\max_{t\ge0}\int_0^1 u_x^2(x,t)dx\quad\text{and}\quad\max_{t\ge0}\int_0^1\theta^2(x,t)dx,$$

we can show that

$$I^{(1)}\le\frac18\int_T^t\int_0^1 v_{xx}^2(x,\tau)dxd\tau+\Lambda\int_T^t\int_0^1(v_x^2+u_x^2+\theta_x^2)(x,\tau)dxd\tau. \tag{3.7}$$

On the other hand, by virtue of (1.5) and integration by parts, it turns

$$\begin{aligned}\int_T^t\int_0^1 v_\tau\sigma_xdxd\tau=&\frac{\gamma}{2}\{[v^2(1,T)+v^2(0,T)]-[v^2(1,t)+v^2(0,t)]\}\\&-\frac12\int_0^1\hat\mu(u)v_x^2(x,t)dx+\frac12\int_0^1\hat\mu(u)v_x^2(x,T)dx\\&+\frac12\int_T^t\int_0^1\hat\mu'(u)v_x^3(x,\tau)dxd\tau+\int_T^t\int_0^1 f(u)\theta v_{x\tau}dxd\tau.\end{aligned} \tag{3.8}$$

We use (3.6) to deal with the last two terms on the right-hand side of (3.8) as the way in [4] and can finally obtain

$$\begin{aligned}&G(t)+\int_T^t\int_0^1(v_{xx}^2+\theta_{xx}^2)(x,\tau)dxd\tau\\&\le\Lambda G(T)+\Lambda\int_0^1[|f(u)\theta|(x,t)+f^2(u)\theta^2(x,T)]dx\\&\quad+\Lambda\Big[\int_0^t|f(u)\theta|(x,t)dx\Big]^2+\Lambda\int_T^t\int_0^1(\theta_x^2+v_x^2+u_x^2)(x,\tau)dxd\tau\\&\quad+\Lambda\sup_{\tau\in[T,t]}\int_0^1(v_x^2+\theta_x^2)(x,\tau)dx\cdot\int_T^t\int_0^1(\theta_{xx}^2+v_{xx}^2+v_x^2+\theta_x^2)dxd\tau,\end{aligned} \tag{3.9}$$

where $G(t) = \gamma[v^2(1,t) + v^2(0,t)] + \int_0^1 (v_x^2 + \theta_x^2)(x,t)dx$.

Now we can apply the arguement in section of [4] to (3.9) to obtain

Lemma 3.1.

$$\lim_{t\to+\infty}\{\gamma[v^2(1,t)+v^2(0,t)]+\int_0^1(v_x^2+\theta_x^2)(x,t)dx\}=0, \tag{3.10}$$

$$\int_0^t\int_0^1(v_{xx}^2+\theta_{xx}^2)dxd\tau\le\Lambda,\quad for\ \ all\ \ t\ge0. \tag{3.11}$$

We are ready to finish the proof of Theorem 1.2 now.

It is obvious from (3.10) and (2.27) that

$$\lim_{t\to+\infty}\|v(\cdot\,,t)\|_{H^1[0,1]}=0.$$

Moreover, due to (3.10), Poincare's inequality and (1.5), it follows that

$$\lim_{t\to+\infty}\|\theta(\cdot\,,t)-T_0\|_{H^1[0,1]}=0.$$

Since (3.1) and $f(\hat{z})=0$,

$$\int_0^1(u-\hat{z})^2(x,t)dx\le\Lambda\int_0^1 f^2(u)(x,t)dx,$$

which, together with (2.28), implies

$$\lim_{t\to+\infty}\int_0^1(u-\hat{z})^2(x,t)dx=0. \tag{3.12}$$

On the other hand , $(1.1)_1$, (3.4) and (3.11) lead to

$$\int_0^{+\infty}\Big|\frac{d}{dt}\int_0^1u_x^2(x,t)dx\Big|dt\le\Lambda.$$

This, combined with (3.4), yields

$$\lim_{t\to+\infty}\int_0^1u_x^2(x,t)dx=0,$$

from which and (3.12) we arrive at

$$\lim_{t\to+\infty}\|u(\cdot\,,t)-\hat{z}\|_{H^1[0,1]}=0.$$

So far, we have completed the proof of Theorem 1.2.

References

[1] Dafermos, C. M. & Hsiao, L., Global smooth thermomechanical processes in nonlinear thermoviscoelasticity, *Nonlinear Anal. T. M. A.,* **6** (1982), 435–454.

[2] Dafermos, C. M., Global solutions to the initial boundary value problem for the equations of one-dimensional nonlinear thermoviscoelasticity, *SIAM. J. Math. Anal.,* **13**(1982), 397–408.

[3] Jiang, S., Global large solutions to initial boundary value problems in one-dimensional nonlinear thermoviscoelasticity, *Quart. Appl. Math.,* **51**(1993), 731–744.

[4] Hsiao, L. & Luo, T., Large-time behavior of solutions to the equations of one-dimensional nonlinear thermoviscoelasticity (to appear in *Quart. Appl. Math.*).

[5] Greenberg, J. M. & MacCamy, R. C., On the exponential stabilities of solutions of $E(u_x)u_{xx}+\lambda u_{xtx}=\rho u_{tt}$, *J. Math. Appl.,* **31**(1970), 406–417.

[6] Andrews, G. & Ball, J. M., Asymptotic behavior and changes of phase in one-dimensional nonlinear viscoelasticity, *J. Diff. Eqn.,* **44**(1982), 306–341.

[7] Okada, M. & Kawashima, S., On the equation of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.,* **23**(1983), 55–71.

[8] Nagasawa, T., On the one-dimensional motion of polytropic ideal gas non-fixed on the boundary, *J. Diff. Eqn.,* **65**(1986), 49–67.