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A NOTE ON THE NULLITY OF HOMOTOPY GROUP FOR COMPLETE THREE DIMENSIONAL MANIFOLDS WITH RICCI≥ 0**

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Abstract

This note shows the nullity of homotopy groups for complete three dimensional manifolds with Ricci≥ 0 under some growth condition of the geodesic ball. The author also gives some examples which show the growth condition here is optimal in some sense.

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§1. Introduction

Let M^3 be a complete, noncompact Riemannian manifold, and Ricci_M be its Ricci curvature. Schoen-Yau[1] proved the following

Theorem 1.1. *Let M^3 be a complete, noncompact Riemannian manifold. Assume $\text{Ricci}_M > 0$. Then, M^3 is diffeomorphic to R^3.*

It is natural to ask what can happen, if we only assume $\text{Ricci}_M \geq 0$. It is quite clear that $S^2 \times R$ with standard product metric is a manifold with Ricci≥ 0 and homotopically nontrival.

Zhu[2] used the the growth of the volume for the geodesic ball to control the homotopy group of M^3. He proved

Theorem 1.2. *Let M^3 be a complete, noncompact Riemannian manifold. Let $V(r)$ be the volume of the geodesic ball of radius r. If*

$$V(r) \geq cr^3 \qquad (\textit{for some constant } c > 0),$$

then M^3 is contractible.

In what follows, we assume that M^3 is a complete, noncompact Riemannian manifold with Ricci≥ 0. Let $V(r)$ be the volume of the geodesic ball of radius r. Then, we prove the following:

Proposition 1.1. *If $\limsup_{r\to\infty} \frac{V(r)}{r} = \infty$, then $\pi_2(M) = 0$.*

Proposition 1.2. *If $\limsup_{r\to\infty} \frac{V(r)}{r^2} = \infty$, then $\pi_1(M) = \{e\}$.*

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Main Theorem *Let M^3 be a complete, noncompact Riemannian manifold. Assume* $\mathrm{Ricci}_M \geq 0$, *and*

$$\limsup_{r\to\infty} \frac{V(r)}{r^2} = \infty.$$

Then M^3 is contractible.

We will show this is indeed an extension of Zhu's theorem and is optimal in some sense.

§2. Proof of Main Theorem

Our argument follows closely that of Schoen-Yau and Zhu, with some detailed analysis on the order of $V(r)$.

Proof of Proposition 1.1. If $\pi_2(M) \neq 0$, then the universal covering space $\widetilde{M}$ has similar properties, i.e., $\pi_2(\widetilde{M}) = \pi_2(M) \neq 0$, $\mathrm{Ricci}_{\widetilde{M}} \geq 0$ and

$$\limsup_{r\to\infty} \frac{\widetilde{V}(r)}{r} = \infty.$$

The sphere theorem of three dimensional topology[3] says that there exists an embedded S^2 in $\widetilde{M}$ which is homotopically nontrival.

If $\widetilde{M}\backslash S^2$ were connected, then we could find a loop in $\widetilde{M}$ intersecting S^2 at exactly one point. This loop could not be null homotopic. Hence, $\pi_1(\widetilde{M}) \neq \{e\}$, which is a contradiction. Hence, $\widetilde{M}\backslash S^2$ has two components $\widetilde{M}_1$, $\widetilde{M}_2$.

As $\pi_1(S^2) = \{e\}$, by Van Kampon's theorem, both $\widetilde{M}_1$ and $\widetilde{M}_2$ are simply connected. Assume $\widetilde{M}_1$ were compact, then S^2 is trival in $H_*(\widetilde{M}, S^2)$. S^2 is homotopically trival in $\widetilde{M}_1$, according to Hurewicz theorem. This contradicts the fact that S^2 is homotopically nontrival in $\widetilde{M}$.

Hence, both $\widetilde{M}_1$, $\widetilde{M}_2$ are noncompact. For any $p_i \in \widetilde{M}_1$ and $q_i \in \widetilde{M}_2$ such that $\mathrm{dist}(p_i, S^2) \to \infty$ and $\mathrm{dist}(q_i, S^2) \to \infty$, there exist minimal geodesics γ_i connecting p_i, q_i and intersecting S^2 at $\gamma_i(0) \in S^2$ (and assume $|\dot{\gamma}_i(0)| = 1$). Then, we can choose a subsequence of γ_i (still denoted by γ_i) such that

$$\lim_{i\to\infty} \gamma_i(0) = p \in S^2, \qquad \lim_{i\to\infty} \dot{\gamma}_i(0) = v \in T_p\widetilde{M}.$$

Then, the geodesic $\gamma(t)$ (such that $\gamma(0) = p$, $\dot{\gamma}(0) = v \in T_p\widetilde{M}$) is a minimal geodesic on $\widetilde{M}^3$. By Cheeger-Gromoll splitting theorem, $\widetilde{M} = \Sigma \times R$ for some complete manifolds Σ (see [4]).

If Σ were an open manifold, $H_2(\Sigma) = 0$. $\pi_1(\Sigma) = \pi_1(\widetilde{M}) = \{e\}$. By Hurewicz theorem, $\pi_2(\Sigma) = H_2(\Sigma) = 0$. This contradicts the fact $\pi_2(\Sigma) = \pi_2(\widetilde{M}) \neq 0$. Hence, Σ is a compact manifold.

Hence, $V(r) \leq 2V(\Sigma)\times r$, where $V(\Sigma)$ is the volume of Σ. This contradicts the assumption $\limsup\limits_{r\to\infty} \frac{V(r)}{r} = \infty$.

Hence, we have $\pi_2(M) = \pi_2(\widetilde{M}) = 0$.

Remark 2.1. We can know from the above that if M^3 is a complete noncompact manifold with Ricci≥ 0 and $\pi_2(M^3) \neq 0$, then $M^3 = S^2 \times R$ (but S^2 may have nonstandard metric).

In fact, we have shown $M^3 = \Sigma \times R$ for some compact manifold Σ. But, $\mathrm{Ricci}_M \geq 0$

shows that the Gaussian curvature K for Σ is nonnegative. The Gauss-Bonnet formula:

$$\int_\Sigma KdA = 4\pi(1-g), \quad \text{where } g \text{ is the genus of } \Sigma,$$

shows that $g \leq 1$. If $g = 1$, $\pi_2(M) = \pi_2(\Sigma) = \pi_2(T^2) = 0$, which is a contradiction. Hence, $g = 0$ or Σ is diffeomophic to S^2.

Proof of Proposition 1.2. If $\limsup\limits_{r\to\infty} \frac{V(r)}{r^2} = \infty$, $\pi_2(M) = 0$ according to Proposition 1.1. Hence, M is a $K(\pi, 1)$ space. $H^i(\pi_1(M)) = H^i(M) = 0$ for $i \geq 2$. Hence $\pi_1(M)$ is torsion free.

Assume $\pi_1(M) \neq \{e\}$. Let $[\sigma] \in \pi_1(M)$ be a nontrival loop, $\pi : \widetilde{M} \to M$ be the universal cover. Let $V(r)$ be the volume of the geodesic ball centered at p. $\pi(\widetilde{p}) = p$. Let F be the fundamental domain of M containing $\widetilde{p}$ in $\widetilde{M}$. Denote by $B_p^M(r)$, $B_{\widetilde{p}}^{\widetilde{M}}(r)$ the geodesic balls of radius r centered at p and $\widetilde{p}$ respectively. Then,

$$\bigcup_{k=1}^N [\sigma]^k(F \cap B_{\widetilde{p}}^{\widetilde{M}}(r)) \subset B_{\widetilde{p}}^{\widetilde{M}}(NL(\sigma) + r).$$

Here, $L(\sigma)$ is the length of σ. $[\sigma]^j(F \cap B_{\widetilde{p}}^{\widetilde{M}}(r)) \cap [\sigma]^k(F \cap B_{\widetilde{p}}^{\widetilde{M}}(r)) = \emptyset$, for $j \neq k$ (because $[\sigma]^k \neq e$ in $\pi_1(\widetilde{M})$ for all $k > 0$). Hence,

$$\begin{aligned} N\text{Vol}(B_p(r)) &= N\text{Vol}(F \cap B_{\widetilde{p}}^{\widetilde{M}}(r)) \\ &= \text{Vol}\Big(\bigcup_{k=1}^N [\sigma]^k\big(F \cap B_{\widetilde{p}}^{\widetilde{M}}(r)\big)\Big) \\ &\leq \text{Vol}(B_{\widetilde{p}}^{\widetilde{M}}(NL(\sigma) + r)) \\ &\leq \frac{4}{3}\pi(NL(\sigma) + r)^3 \quad \text{by volume comparison theorem.} \end{aligned}$$

Hence,

$$V(r) \leq \frac{4}{3}\pi \frac{(NL(\sigma) + r)^3}{N}, \quad \text{for all } N > 0 \text{ and } r \geq 1.$$

Choosing $N = [r] + 1$, we have

$$V(r) \leq cr^2, \quad \text{for some constant } c > 0.$$

This contradicts the assumption $\limsup\limits_{r\to\infty} \frac{V(r)}{r^2} = \infty$.

Hence, M is simply connected.

Remark 2.2. A detail analysis shows that if $\pi_1(M) \neq \{e\}$, then

$$\limsup_{r\to\infty} \frac{V(r)}{r^2} \leq 9\pi L.$$

Here, $L = \inf\{L(\sigma) | [\sigma] \in \pi_1(M) \text{ and } [\sigma] \neq e \text{ in } \pi_1(M)\}$.

Proof of Main Theorem. From Propositions 1.1 and 1.2, we have $\pi_1(M) = \{e\}$, $\pi_2(M) = 0$. But, M is an open manifold, $H_3(M) = 0$. Hence, according to Hurewicz theorem, $\pi_k(M) = 0$ for all $k \geq 3$. According to Whitehead theorem, M is contractible.

§3. Examples

Example 3.1. Let $M^3 = S^2 \times R$ with standard product metric. Then, $\text{Ricci}_M \geq 0$ and

$\limsup_{r\to\infty} \frac{V(r)}{r} = 2\mathrm{Vol}(S^2) \leq \infty.$

$$\pi_2(M^3) = Z \neq 0.$$

This example shows that the volume growth condition for Proposition 1.1 is optimal.

Example 3.2. Let $M^3 = S^1 \times R^2$ with standard product metric. Then, $\mathrm{Ricci}_M = 0$, and $\limsup_{r\to\infty} \frac{V(r)}{r^2} = 4\pi\mathrm{diam}(S^1)$.

$$\pi_2(M^3) = 0, \quad \pi_1(M^3) = Z \neq \{e\}.$$

This example shows that the volume growth condition for Proposition 1.2 is optimal.

Example 3.3. Let $M^3 = R^3$ with metric $ds^2 = dr^2 + f^2(r)d\theta^2_{S^2}$, where $d\theta^2_{S^2}$ is the standard metric of S^2 and (r, θ) is the polar coordinate of R^3 (see [5]). Then, $f(0) = 0$, $f'(0) = 1$ and

$$f''(r) + K(r)f(r) = 0 \quad \text{for all } r \geq 0,$$

where $K(r)$ is the radicial curvature of M^3. If $K(r) \geq 0$, then $\mathrm{Ricci}_{M^3} \geq 0$ and

$$V(r) = \mathrm{Vol}(S^2)\int_0^r f^2(s)ds.$$

Let $f(r) = cr^\alpha$ for $r \geq r_0$ large enough. Then,

$$K(r) = \frac{(1-\alpha)\alpha}{r^2} \quad \text{for } r \geq r_0.$$

If $0 \leq \alpha \leq 1$, $K(r) \geq 0$ for $r \geq r_0$. For α and r_0 fixed, one can find $c > 0$ and $f(r)$ in $[0, r_0)$, such that $K(r) \geq 0$.

$$\begin{aligned} V(r) \sim & \mathrm{Vol}(S^2)c^2\int_{r_0}^r s^{2\alpha}ds \\ = & \frac{c^2 r^{2\alpha+1}}{2\alpha+1}\mathrm{Vol}(S^2) \qquad \text{as } r \to \infty. \end{aligned}$$

Hence, M^3 satisfies our condition, but not the condition of Zhu[2] if $\frac{1}{2} < \alpha < 1$.

This example shows that our theorem is indeed an extension of Zhu's theorem.

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