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INFINITELY MANY HOMOCLINIC ORBITS FOR A CLASS OF HAMILTONIAN SYSTEMS WITH SYMMETRY

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Abstract

This paper deals via variational methods with the existence of infinitely many homoclinic orbits for a class of first order time dependent Hamiltonian systems

$$\dot{z} = JH_z(t, z)$$

without any periodicity assumption on H, providing that $H(t, z)$ is even with respect to $z \in \boldsymbol{R}^{2N}$, superquadratic or subquadratic as $|z| \to \infty$, and satisfies some additional assumptions.

Keywords Variational method, Homoclinic orbits, Hamiltonian systems
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§1. Introduction and Main Results

This paper is an extension of the work [8].

We consider the existence of infinitely many homoclinic orbits for the first order time dependent Hamiltonian systems

$$\dot{z} = JH_z(t, z), \tag{HS}$$

where $z = (p, q) \in \boldsymbol{R}^{2N}$, $H \in C^1(\boldsymbol{R} \times \boldsymbol{R}^{2N}, \boldsymbol{R})$, $H(t, 0) \equiv 0$, and J is the standard symplectic structure on $\boldsymbol{R}^{2N}$,

$$J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$$

with I_N being the $N \times N$ identity matrix. By a homoclinic orbit we mean a solution $z \in C^1(\boldsymbol{R}, \boldsymbol{R}^{2N})$ of (HS) which satisfies $z(t) \not\equiv 0$ and the asymptotic condition $z(t) \to 0$ as $|t| \to \infty$.

The existence of homoclinic orbits of systems like (HS) is a very classical problem. Up to about 1990, apart from a few isolated results, the only method for dealing with such a problem was the small perturbation techniques of Melinkov. In very recent years this kind of problem has been deeply investigated via variational methods pioneered by Rabinowitz, Coti-Zelati, Ekeland, Séré, Hofer, Wysocki and others (see, for example, [2,4–7, 10–11, 13–17]).

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These papers for the first order systems (HS) considered the Hamiltonian of the form

$$H(t,z) = \frac{1}{2}Az\cdot z + R(t,z),$$

where A is a $2N\times 2N$ symmetric and constant matrix such that each eigenvalue of JA has a non-zero real part, and $R(t,z)$ is periodic in t and globally superquadratic in z. They showed that (HS) has at least one homoclinic orbit. The existence of infinitely many homoclinic orbits of (HS) was also established in [15,16] if, in addition, $R(t,z)$ is convex in z.

Recall that, for a particular case of second order systems of the type

$$-\ddot{q} = -L(t)q + W_q(t,q),$$

where $L\in C(\boldsymbol{R},\boldsymbol{R}^{N^2})$ is a symmetric matrix valued function, the works [14] (among other results) and [6,11] obtained some existence results of homoclinic orbits without any periodicity assumption on the Hamiltonian

$$H(t,p,q) = \frac{1}{2}|p|^2 - \frac{1}{2}L(t)q\cdot q + W(t,q), \qquad (p,q)\in \boldsymbol{R}^{2N},$$

providing instead that the smallest eigenvalue of $L(t)\to\infty$ as $|t|\to\infty$, and $W(t,q)$ satisfies some growth assumptions.

Motivated by the works of [6,11,14], we studied in [8] the following Hamiltonian

$$H(t,z) = -\frac{1}{2}M(t)z\cdot z + R(t,z), \tag{1.1}$$

where

$$M(t) = \begin{pmatrix} 0 & L(t) \\ L(t) & 0 \end{pmatrix}$$

with L being an $N\times N$ symmetric matrix valued function. We proved that (HS) has at least one homoclinic orbit under the assumptions:

(L_1) the smallest eigenvalue of $L(t)\to\infty$ as $|t|\to\infty$, i.e.,

$$l(t)\equiv \inf_{\xi\in\boldsymbol{R}^N,|\xi|=1} L(t)\xi\cdot\xi \to \infty \quad\text{as}\quad |t|\to\infty;$$

(L_2) $L\in C^1(\boldsymbol{R},\boldsymbol{R}^{N^2})$ and there exists $T_0>0$ such that $2L(t)\pm\frac{d}{dt}L(t)$ are nonnegative definite for all $|t|\ge T_0$;

(R_1) $R\in C^1(\boldsymbol{R}\times\boldsymbol{R}^{2N},\boldsymbol{R})$ and there exists $\mu>2$ such that

$$0<\mu R(t,z)\le R_z(t,z)z, \quad \forall t\in\boldsymbol{R} \text{ and } z\ne 0;$$

(R_2) $0<\underline{b}\equiv \inf_{t\in\boldsymbol{R},|z|=1} R(t,z)$;

(R_3) $|R_z(t,z)| = o(|z|)$ as $z\to 0$ uniformly in t;

(R_4) there exists $0\le a_1(t)\in L^1(\boldsymbol{R})\cap C(\boldsymbol{R})$, $\gamma>1$ and $a_2>0$ such that

$$|R_z(t,z)|^\gamma \le a_1(t) + a_2R_z(t,z)z, \quad \forall (t,z).$$

Moreover, in [8] the case that $R(t,z)$ is subquadratic growth as $|z|\to\infty$ is also considered.

The purpose of this paper is to show that (HS) possesses infinitely many homoclinic orbits if $H(t,z)$ is even in z and satisfies the above assumptions or others.

Our first result reads as follows.

Theorem 1.1. *Let H be of the form* (1.1) *with L satisfying* (L_1)–(L_2) *and R satisfying* (R_1)–(R_4). *Suppose, in addition, that $R(t,z)=R(t,-z)$ for all $(t,z)\in\boldsymbol{R}\times\boldsymbol{R}^{2N}$. Then* (HS) *possesses infinitely many homoclinic orbits* $\{z_k\}$ *such that*

$$\int_{\boldsymbol{R}}\Big[-\frac{1}{2}J\dot{z}_k\cdot z_k-H(t,z_k)\Big]dt\to\infty\quad as\quad k\to\infty.$$

Our second result handles the case that $R(t,z)$ *is subquadratic in* z. *We need the following assumptions:*

(L_3) *there exists* $\alpha<2$ *such that* $l(t)|t|^{\alpha-2}\to\infty$ *as* $|t|\to\infty$;

(R_5) $R\in C^1(\boldsymbol{R}\times\boldsymbol{R}^{2N},\boldsymbol{R})$ *and there is* $1<\beta\in(\frac{2}{3-\alpha},2)$ *such that*

$$0\le R_z(t,z)\cdot z\le\beta R(t,z),\quad\forall(t,z);$$

(R_6) *there exists* $a_3>0$ *such that*

$$a_3|z|^\beta\le R(t,z),\quad\forall(t,z);$$

(R_7) *there exists* $a_4>0$ *such that*

$$|R_z(t,z)|\le a_4|z|^{\beta-1},\quad\forall t\in\boldsymbol{R}\ \ and\ \ |z|\le1;$$

(R_8) $|R_z(t,z)|\in L^\infty(\boldsymbol{R}\times B_R)$ *for any* $R>0$, *where* $B_R=\{z\in\boldsymbol{R}^{2N};|z|\le R\}$, *and* $|z|^{-1}|R_z(t,z)|\to0$ *as* $|z|\to\infty$ *uniformly in* $t\in\boldsymbol{R}$.

Theorem 1.2. *Let* H *be of the form* (1.1) *with* L *satisfying* (L_2)–(L_3) *and* R *satisfying* (R_5)–(R_8), *and suppose, in addition, that* $R(t,z)=R(t,-z)$ *for all* $(t,z)\in\boldsymbol{R}\times\boldsymbol{R}^{2N}$. *Then* (HS) *possesses infinitely many homoclinic orbits* $\{z_k\}$ *such that*

$$0<\int_{\boldsymbol{R}}\Big[\frac{1}{2}J\dot{z}_k\cdot z_k+H(t,z_k)\Big]dt\to0\quad as\quad k\to\infty.$$

Typical examples fitting in with our situation are $L(t)=|t|^\theta I_N$ with $\theta>1$ and $R(t,z)=b(t)W(z)$, where $b\in C(\boldsymbol{R},\boldsymbol{R})$ satisfying $\underline{b}\le b(t)\le\bar{b}$ for some positive constants $\underline{b}\le\bar{b}$ and all $t\in\boldsymbol{R}$, and $W(z)=\sum_{i=1}^m c_i|z|^{\mu_i}$, for some integer $m>0$, with $c_i>0 (1\le i\le m)$ and $1<\mu_1\le\mu_2\le\cdots\le\mu_m$. Clearly, $R(t,z)$ is even in z, and satisfies (R_1)–(R_4) if $\mu_1>2$, (R_5)–(R_8) if $m=1$ and $\mu_1\in(\frac{2}{3-\alpha},2)$.

§2. Preliminary Results

The following two critical point propositions will be used for proving the previous Theorem 1.1 and Theorem1.2.

Let E be a real Hilbert space with the norm $\|\cdot\|$. Suppose that E has an orthogonal decomposition $E=E_1\oplus E_2$ with both E_1 and E_2 being infinite dimensional. Suppose $\{v_n\}$ (resp. $\{w_n\}$) is an orthonormal basis for E_1 (resp. E_2), and set

$$X_n=\operatorname{span}\{v_1,\cdots,v_n\}\oplus E_2,\quad X^m=E_1\oplus\operatorname{span}\{w_1,\cdots,w_m\}.$$

For a functional $I\in C^1(E,\boldsymbol{R})$ we denote by $I_n=I|_{X_n}$ the restriction of I on X_n. Recall that we say I satisfies $(PS)^*$ conditon if any sequence $\{u_n\}$ with $u_n\in X_n$ for which $0\le I(u_n)\le$ const. and $I_n'(u_n)\equiv\nabla I_n(u_n)\to0$ as $n\to\infty$ possesses a convergent subsequence. We also say that I satisfies $(PS)^{**}$ condition if for each $n\in\boldsymbol{N}$, I_n satisfies the Palais-Smale condition, i.e., any sequence $\{u_k\}\subset X_n$ for which $I(u_k)$ is bounded and $I_n'(u_k)\to0$ as $k\to\infty$ possesses a convergent subsequence.

Proposition 2.1. *Let* E *be as above and let* $I\in C^1(E,\boldsymbol{R})$ *be even, satisfy* $(PS)^*$ *and* $(PS)^{**}$, *and* $I(0)=0$. *Suppose moreover that* I *satisfies, for any* $m\in\boldsymbol{N}$,

(I_1) *there is* $R_m > 0$ *such that* $I(u) \le 0, \quad \forall u \in X^m \quad$ *with* $\|u\| \ge R_m$;

(I_2) *there are* $r_m > 0, a_m > 0$ *with* $a_m \to \infty$ *as* $m \to \infty$ *such that*

$$I(u) \ge a_m, \quad \forall u \in (X^{m-1})^\perp \quad \text{with} \quad \|u\| = r_m;$$

(I_3) I *is bounded from above on bounded sets of* X^m.

Then I *has a positive critical value sequence* $\{c_k\}$ *satisfying* $c_k \to \infty$ *as* $k \to \infty$.

Proof. This proposition is a special case of [3, Theorem 3.1]. However, in the present form its proof is simple and we sketch it here for the reader's convenience. Set $X_n^m = X_n \cap X^m = \mathrm{span}\{v_1, \cdots, v_n, w_1, \cdots, w_m\}$ and

$$Q_n^m = \{u \in X_n^m; \|u\| \le R_m\}, \quad S_m = \{u \in (X^{m-1})^\perp; \|u\| = r_m\},$$

$$\Gamma_n^m = \{\gamma \in C(Q_n^m, X_n); \gamma \text{ is odd and } \gamma|_{\partial Q_n^m} = id\}.$$

Then $\gamma(Q_n^m) \cap S^m \ne \emptyset$ for all $\gamma \in \Gamma_n^m$. In fact, let $V = \{u \in Q_n^m;\ \|\gamma(u)\| < r_m\}$. Since γ is odd, $0 \in V$. By (I_1)–(I_2), $R_m > r_m$ and so $V \cap \partial Q_n^m = \emptyset$. Thus V is open in X_n^m. Let $P : X_n \to X_n^{m-1}$ be the projector. Then $P \circ \gamma|_V : V \to X_n^{m-1}$ is odd and continuous. Hence there is $u \in \partial V$ such that $P \circ \gamma(u) = 0$, i.e., $\|\gamma(u)\| = r_m$ and $\gamma(u) \in (X^{m-1})^\perp$. Now define

$$c_n^m = \inf_{\gamma \in \Gamma_n^m} \max_{u \in Q_n^m} I(\gamma(u)).$$

Then we have, by (I_2) and (I_3),

$$a_m \le c_n^m \le b_m, \tag{2.1}$$

where $b_m = \sup\limits_{u \in X^m, \|u\| \le R_m} I(u) < \infty$. By (PS)**, a standard argument (see [1, 12]) shows that c_n^m is a critical value of I_n. Letting $n \to \infty$ in (2.1) yields $a_m \le c^m \le b_m$ and c^m is a positive critical value of I by (PS)*. Since $a_m \to \infty$, the proposition follows immediately.

Proposition 2.2. *Let* E *be as above and let* $I \in C^1(E, \boldsymbol{R})$ *be even, satisfy* (PS)* *and* (PS)**, *and* $I(0) = 0$. *Suppose moreover that* I *satisfies, for each* $m \in \boldsymbol{N}$,

(I_4) *there exist* $r_m > 0, a_m > 0$ *such that* $a_m \le I(u)$, $\forall u \in X^m$ *with* $\|u\| = r_m$;

(I_5) *there is* $b_m > 0$ *with* $b_m \to 0$ *as* $m \to \infty$ *such that* $I(u) \le b_m, \qquad \forall u \in (X^{m-1})^\perp$.

Then I *possesses a positive critical value sequence* $\{c_k\}$ *such that* $c_k \to 0$ *as* $k \to \infty$.

Proof. Let Σ denote the class of closed (in E) subsets of $E \setminus \{0\}$ symmetric with respect to the origin, and let $\gamma : \Sigma \to \boldsymbol{N} \cup \{0, \infty\}$ be the genus map. Set

$$\Sigma_n^m = \{A \in \Sigma; A \subset X_n \text{ and } \gamma(A) \ge n + m\}.$$

Define

$$c_n^m = \sup_{A \in \Sigma_n^m} \inf_{u \in A} I(u).$$

Since for each $A \in \Sigma_n^m$, $A \subset X_n$ and $\gamma(A) \ge n + m$, $A \cap (X^{m-1})^\perp \ne \emptyset$. Thus

$$\inf_{u \in A} I(u) \le \sup_{u \in (X^{m-1})^\perp} I(u) \le b_m \tag{2.2}$$

by (I_5). Since $\gamma(\partial B_{r_m} \cap X_n^m) = n + m$ where $B_{r_m} = \{u \in E; \|u\| \le r_m\}$, $\partial B_{r_m} \cap X_n^m \in \Sigma_n^m$ and by (I_4)

$$\inf_{\partial B_{r_m} \cap X_n^m} I(u) \ge a_m. \tag{2.3}$$

Combining (2.2) and (2.3) shows

$$a_m \le c_n^m \le b_m. \tag{2.4}$$

Since I satisfies (PS)**, by using the genus theory and a positive rather than a negative gradient flow, a standard argument[1,12] shows that c_n^m is a critical value of I_n. By (2.4), letting $n \to \infty$, we see that $c_n^m \to c^m$ and, by (PS)*, c^m is a critical value of I satisfying $a_m \le c^m \le b_m$, which, together with (I_5), gives the conclusion of the proposition.

The proof is complete.

We now consider the symmetric matrix valued functions $M \in C(\boldsymbol{R}, \boldsymbol{R}^{2N\times 2N})$ of the form

$$M(t) = \begin{pmatrix} 0 & L(t) \\ L(t) & 0 \end{pmatrix}.$$

Suppose that L satisfies (L_1) and (L_2). Let $A \equiv -J\frac{d}{dt} + M$ be the selfadjoint operator with the domain $D(A) \subset L^2 \equiv L^2(\boldsymbol{R}, \boldsymbol{R}^{2N})$, defined as a sum of quadratic forms. Let $\{E(\lambda); -\infty < \lambda < \infty\}$ be the resolution of A, and $U = I - E(0) - E(-0)$. Then U commutes with A, $|A|$ and $|A|^{1/2}$, and $A = |A|U$ is the polar decomposition of A (see [9]). $D(A) = D(|A|) = D(I + |A|)$ is a Hilbert space equipped with the norm

$$\|z\|_1 = \|(I + |A|)z\|_{L^2}, \quad \forall z \in D(A),$$

where $\|\cdot\|_{L^2}$ is the norm of L^2. It is easy to check that $D(A)$ is continuously embedded in $W^{1,2} \equiv W^{1,2}(\boldsymbol{R}, \boldsymbol{R}^{2N})$ (see [8]). Moreover we have

Lemma 2.3. *Let L satisfy* (L_1) *and* (L_2). *Then $D(A)$ is compactly embedded in L^2.*

Proof. See [8, Lemma 2.1].

Remark 2.4. In virtue of Lemma 2.3, $(I+|A|)^{-1} : L^2 \to L^2$ is a compact linear operator. Therefore a standard argument shows that $\sigma(A)$, the spectrum of A, consists of eigenvalues numbered by (counted in their multiplicities):

$$\cdots \le \lambda_{-2} \le \lambda_{-1} \le 0 < \lambda_1 \le \lambda_2 \le \cdots$$

with $\lambda_{\pm k} \to \pm\infty$ as $k \to \infty$, and a corresponding system of eigenfunctions $\{e_k\}$ of A forms an orthonormal basis in L^2.

Now we set $E = D(|A|^{1/2}) = D((I + |A|)^{1/2})$. E is a Hilbert space under the following inner product

$$(z_1, z_2)_0 = (|A|^{1/2}z_1, |A|^{1/2}z_2)_{L^2} + (z_1, z_2)_{L^2}$$

and norm

$$\|z\|_0 = (z, z)_0^{1/2} = \|(I + |A|)^{1/2}z\|_{L^2},$$

where $(\cdot, \cdot,)_{L^2}$ denotes the L^2 inner product.

Let $E^\circ = KerA$, $E^+ = Cl_E(\text{span}\{e_1, e_2, \cdots, \})$, and $E^- = (E^\circ \oplus E^+)^{\perp_E}$, where $Cl_E S$ stands for the closure of S in E and $S^{\perp_E}$ the orthogonal complementary subspace of S in E. Then

$$E = E^- \oplus E^\circ \oplus E^+. \tag{2.5}$$

Since, by Lemma 2.3, 0 is at most an isolated eigenvalue of A, for the later convenience, we introduce on E the following inner product

$$(z_1, z_2) = (|A|^{1/2}z_1, |A|^{1/2}z_2)_{L^2} + (z_1^0, z_2^0)_{L^2}$$

for all $z_i = z_i^- + z_i^0 + z_i^+ \in E^- \oplus E^0 \oplus E^+ (i = 1, 2)$, and norm

$$\|z\| = (z, z)^{1/2} \tag{2.6}$$

for all $z \in E$. Clearly, $\|\cdot\|$ is equivalent to $\|\cdot\|_0$. Moreover, E is continuously embedded in $H^{1/2}(\boldsymbol{R}, \boldsymbol{R}^{2N})$, the Sobolev space of fractional order (see [8]).

Lemma 2.5. *Let L satisfy* (L_1) *and* (L_2). *Then E is compactly embedded in L^p for all $p \in [2, \infty)$.*

Proof. See [8, Lemma 2.2].

Recall that the assumption (L_3) is stronger than (L_1). By (L_3), since $\alpha < 2$, we see $\frac{2}{3-\alpha} < 2$. Corresponding to this case we have

Lemma 2.6. *Let L satisfy* (L_2) *and* (L_3). *Then E is compactly embedded in L^p for all $1 \le p \in (\frac{2}{3-\alpha}, \infty)$.*

Proof. See [8, Lemma 2.3].

Finally we introduce

$$a(z,x) = (|A|^{1/2}Uz, |A|^{1/2}x)_{L^2} \tag{2.7}$$

for all $z, x \in E$. $a(\cdot,\cdot)$ is the quadratic form associated with A. Clearly, for $z \in D(A)$ and $x \in E$ we have

$$a(z,x) = (Az,x)_{L^2} = \int_{\boldsymbol{R}} (-J\dot{z} + M(t)z)x. \tag{2.8}$$

Plainly, E^-, E^0 and E^+ are orthogonal to each other with respect to $a(\cdot,\cdot)$, and moreover

$$\begin{aligned} a(z,x) &= ((P^+ - P^-)z, x), \quad \forall z, x \in E, \\ a(z,z) &= \|z^+\|^2 - \|z^-\|^2, \quad \forall z \in E, \end{aligned} \tag{2.9}$$

where $P^\pm : E \to E^\pm$ are the orthogonal projectors and $z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+$.

§3. Proof of Theorem 1.1

Throughout this section, let the assumptions of Theorem 1.1 be satisfied. Let $E = D(|A|^{1/2})$ with the norm (2.6). By (R_4) one has

$$|R_z(t,z)| \le C(1 + |z|^{\gamma'-1}), \quad \forall (t,z), \tag{3.1}$$

where $\gamma' = \frac{\gamma}{\gamma-1}$, which, jointly with (R_3), yields that for any $\epsilon > 0$ there is $C_\epsilon > 0$ such that

$$|R_z(t,z)| \le \epsilon|z| + C_\epsilon|z|^{\gamma'-1}, \quad \forall (t,z), \tag{3.2}$$

and

$$|R(t,z)| \le \epsilon|z|^2 + C_\epsilon|z|^{\gamma'}, \quad \forall (t,z). \tag{3.3}$$

Here (and after) C (or C_i) stands for generic positive constants not depending on t and z. By (R_1) and (R_2) one also has

$$R(t,z) \ge \underline{b}|z|^\mu, \quad \forall t \in \boldsymbol{R} \text{ and } |z| \ge 1. \tag{3.4}$$

Note that (3.1) and (3.4) imply $\gamma' \ge \mu > 2$.

Let

$$\varphi(z) = \int_{\boldsymbol{R}} R(t,z), \quad \forall z \in E.$$

(3.1)–(3.4) imply that φ is well defined, $\varphi \in C^1(E, \boldsymbol{R})$ and

$$\varphi'(z)x = \int_{\boldsymbol{R}} R_z(t,z)x, \quad \forall x, z \in E \tag{3.5}$$

by Lemma 2.5. In addition, φ' is a compact map. To see this, let $z_n \to z$ weakly in E. By Lemma 2.5 one can assume that $z_n \to z$ strongly in L^p for $p \in [2, \infty)$. By (3.5)

$$\|\varphi'(z_n) - \varphi'(z)\| = \sup_{\|x\|=1} \Big| \int_{\boldsymbol{R}} (R_z(t, z_n) - R_z(t, z))x \Big|.$$

By (3.2) and the Hölder inequality, for any $R > 0$,

$$\begin{aligned} &\Big| \int_{|t|\geq R} (R_z(t, z_n) - R_z(t, z))x \Big| \\ &\leq C \int_{|t|\geq R} (|z_n| + |z| + |z_n|^{\gamma'-1} + |z|^{\gamma'-1})|x| \\ &\leq C\Big[\|x\|_{L^2}\Big(\int_{|t|\geq R} |z_n|^2 + |z|^2\Big)^{1/2} + \|x\|_{L^{\gamma'}}\Big(\int_{|t|\geq R} |z_n|^{\gamma'} + |z|^{\gamma'}\Big)^{(\gamma'-1)/\gamma'}\Big]. \end{aligned} \tag{3.6}$$

For any $\epsilon > 0$, by (3.6) one can take R_0 large such that

$$\Big| \int_{|t|\geq R_0} (R_z(t, z_n) - R_z(t, z))x \Big| < \frac{\epsilon}{2} \tag{3.7}$$

for all $\|x\| = 1$ and $n \in \boldsymbol{N}$. On the other hand, it is well-known (see [12]) that since $z_n \to z$ strongly in L^2,

$$\|R_z(\cdot, z_n) - R_z(\cdot, z)\|_{L^2(B_{R_0})} \to 0$$

as $n \to \infty$ where $B_{R_0} = (-R_0, R_0)$. Therefore there is $n_0 \in \boldsymbol{N}$ such that

$$\Big| \int_{|t|\leq R_0} (R_z(t, z_n) - R_z(t, z))x \Big| < \frac{\epsilon}{2} \tag{3.8}$$

for all $\|x\| = 1$ and $n \geq n_0$. Combining (3.7) and (3.8) yields

$$\|\varphi'(z_n) - \varphi'(z)\| < \epsilon, \quad \forall n \geq n_0.$$

Hence φ' is compact.

Let $a(\cdot, \cdot)$ be the quadratic form given by (2.7), and define

$$I(z) = \frac{1}{2} a(z, z) - \varphi(z), \quad \forall z \in E.$$

By (2.9)

$$I(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \varphi(z), \quad \forall z \in E$$

for all $z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+$. Then $I \in C^1(E, \boldsymbol{R})$. Noting that (2.8) holds, by a standard argument we show that the nontrivial critical points of I on E are homoclinic orbits of (HS).

Let $E_1 = E^- \oplus E^0$ and $E_2 = E^+$ with $\{v_n = e_{-n}\}_{n=1}^\infty$ and $\{w_m = e_m\}_{m=1}^\infty$ respectively, where $\{e_n\}_{n=-\infty}^\infty$ is the system of eigenfunctions of A (see Remark 2.4). Set also $X_n = \operatorname{span}\{v_1, \cdots, v_n\} \oplus E_2$, $X^m = E_1 \oplus \operatorname{span}\{w_1, \cdots, w_m\}$, and $I_n = I|_{X_n}$. We will verify that I satisfies the assumptions of Proposition 2.1.

Lemma 3.1. *I satisfies the* (PS)* *and* (PS)** *conditions.*

Proof. The verification procedure for (PS)* and (PS)** are the same, and so we only check the (PS)*. Suppose $z_n \in X_n$ such that $0 \leq I(z_n) \leq C$ and $\epsilon_n = \|I_n'(z_n)\| \to 0$. By definition and ($\mathrm{R}_1$),

$$\begin{aligned} I(z_n) - \frac{1}{2}I'(z_n)z_n &= \int_{\boldsymbol{R}} \Big(\frac{1}{2}R_z(t,z_n)z_n - R(t,z_n)\Big) \\ &\geq \Big(\frac{1}{2} - \frac{1}{\mu}\Big) \int_{\boldsymbol{R}} R_z(t,z_n)z_n \geq \Big(\frac{\mu}{2} - 1\Big) \int_{\boldsymbol{R}} R(t,z_n). \end{aligned} \tag{3.9}$$

(3.9) and (R_4) yield $\|R_z(t,z_n)\|_{L^\gamma}^\gamma \leq C(1+\|z_n\|)$, and hence by Lemma 2.5,

$$\|z_n^+\|^2 = I'(z_n)z_n^+ + \int_{\boldsymbol{R}} R_z(t,z_n)z_n^+ \leq C\|z_n^+\|(1+\|R_z(t,z)\|_{L^\gamma}),$$

or

$$\|z_n^+\| \leq C(1+\|z_n\|^{1/\gamma}). \tag{3.10}$$

Similarly one gets

$$\|z_n^-\| \leq C(1+\|z_n\|^{1/\gamma}). \tag{3.11}$$

If $E^0 = \{0\}$, (3.10)–(3.11) imply $\|z_n\| \leq$ const. Suppose $E^0 \neq \{0\}$. For $z \in E$, let

$$z^1(t) = \begin{cases} z(t) & \text{if } |z(t)| < 1, \\ 0 & \text{if } |z(t)| \geq 1, \end{cases} \quad z^2(t) = \begin{cases} 0 & \text{if } |z(t)| < 1, \\ z(t) & \text{if } |z(t)| \geq 1. \end{cases} \tag{3.12}$$

Since by Lemma 2.5

$$\int_{\boldsymbol{R}} |z_n^1|^\mu \leq \int_{\boldsymbol{R}} |z_n^1|^2 \leq \int_{\boldsymbol{R}} |z_n|^2 \leq C\|z_n\|^2,$$

one has

$$\|z_n^1\|_{L^\mu} \leq C\|z_n\|^{2/\mu}. \tag{3.13}$$

By (3.4) and (3.9)

$$\|z_n^2\|_{L^\mu} \leq C(1+\|z_n\|^{1/\mu}). \tag{3.14}$$

By L^2 orthogonality and Hölder inequality with $\mu' = \frac{\mu}{\mu-1}$,

$$\|z_n^0\|_{L^2}^2 = (z_n^0, z_n)_{L^2} \leq \|z_n^0\|_{L^{\mu'}}(\|z_n^1\|_{L^\mu} + \|z_n^2\|_{L^\mu}).$$

Hence, since $\dim E^0 < \infty$ and (3.13)–(3.14) hold, one sees

$$\|z_n^0\| \leq C(\|z_n\|^{2/\mu} + \|z_n\|^{1/\mu}). \tag{3.15}$$

The combination of (3.10)–(3.11) and (3.15) shows that again $\|z_n\| \leq$ const. Finally, since φ' is compact, a standard argument shows that $\{z_n\}$ has a convergent subsequence, proving the $(\mathrm{PS})^*$.

Lemma 3.2. *I satisfies* (I_1).

Proof. By (3.4), (R_1) and noting that $|z|^\mu \leq |z|^2$ for $|z| \leq 1$, one has for any $0 < \epsilon \leq \underline{b}$,

$$R(t,z) \geq \epsilon(|z|^\mu - |z|^2), \quad \forall (t,z). \tag{3.16}$$

Let $d > 0$ be such that $\|z\|_{L^2}^2 \leq d\|z\|^2$ for all $z \in E$ (by Lemma 2.5) and take $\epsilon = \min\{\frac{1}{4d}, \underline{b}\}$. Then for $z = z^- + z^0 + z^+ \in X^m$ we have

$$\begin{aligned} I(z) &= \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \int_{\boldsymbol{R}} R(t,z) \\ &\leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 + \epsilon\|z\|_{L^2}^2 - \epsilon\|z\|_{L^\mu}^\mu \\ &\leq \|z^+\|^2 - \frac{1}{4}\|z^-\|^2 + \frac{1}{4}\|z^0\|^2 - \epsilon\|z\|_{L^\mu}^\mu. \end{aligned} \tag{3.17}$$

Since $\dim(E^0 \oplus \mathrm{span}\{w_1, \cdots, w_m\}) < \infty$, we have

$$\|z^0+z^+\|_{L^2}^2 = (z^0+z^+, z)_{L^2} \le \|z^0+z^+\|_{L^{\mu'}}\|z\|_{L^\mu} \le C(m)\|z^0+z^+\|_{L^2}\|z\|_{L^\mu},$$

and so $\|z^0+z^+\| \le C'(m)\|z\|_{L^\mu}$, or

$$C''(m)\|z^0+z^+\|^\mu \le \|z\|_{L^\mu}^\mu, \tag{3.18}$$

where $C(m), C'(m)$ and $C''(m) > 0$ depending on m but not on $z \in X^m$. (3.17) and (3.18) imply

$$I(z) \le \|z^0+z^+\|^2 - \frac{1}{4}\|z^-\|^2 - \epsilon C''(m)\|z^0+z^+\|^\mu \tag{3.19}$$

for all $z \in X^m$. Since $\mu > 2$, (3.19) implies that there is $R_m > 0$ such that

$$I(z) \le 0, \quad \forall z \in X^m \ \text{ with } \ \|z\| \ge R_m,$$

proving (I_1).

Lemma 3.3. *I satisfies* (I$_2$).

Proof. Define

$$\eta_m = \sup_{z \in (X^m)^\perp \setminus \{0\}} \frac{\|z\|_{L^{\gamma'}}}{\|z\|}.$$

Clearly, $\eta_m \ge \eta_{m+1} > 0$. We claim that

$$\eta_m \to 0 \ \text{ as } \ m \to \infty. \tag{3.20}$$

Arguing indirectly, assume $\eta_m \to \eta > 0$. Then there is a sequence $z_m \in (X^m)^\perp$ with $\|z_m\| = 1$ and $\|z_m\|_{L^{\gamma'}} \ge \frac{\eta}{2}$. Since $(z_m, w_k) \to 0$ as $m \to \infty$ for each w_k, one see that $z_m \to 0$ weakly in E, and so by Lemma 2.5, $\|z_m\|_{L^{\gamma'}} \to 0$, a contradiction. (3.20) is proved.

By (3.3) with $\epsilon = \frac{1}{4d}$ and $C = C_\epsilon$ one has, for $z \in (X^{m-1})^\perp$,

$$I(z) = \frac{1}{2}\|z\|^2 - \int_{\boldsymbol{R}} R(t,z) \ge \frac{1}{4}\|z\|^2 - C\|z\|_{L^{\gamma'}}^{\gamma'} \ge \frac{1}{4}\|z\|^2 - C\eta_{m-1}^{\gamma'}\|z\|^{\gamma'}.$$

Taking $r_m = (2\gamma' C\eta_{m-1}^{\gamma'})^{\frac{-1}{\gamma'-2}}$ and $a_m = (\frac{1}{4} - \frac{1}{2\gamma'})r_m^2$ one obtains

$$I(z) \ge a_m, \quad \forall z \in (X^{m-1})^\perp \ \text{ with } \ \|z\| = r_m.$$

Since $\gamma' > 2$, (3.20) shows that $a_m \to \infty$ as $m \to \infty$. (I_2) follows.

Lemma 3.4. *I satisfies* (I$_3$).

Proof. (I$_3$) follows directly from (3.19).

Now we are in a position to give the following

Proof of Theorem 1.1. Clearly $I(0) = 0$ and I is even since $R(t,z)$ is even with respect to $z \in \boldsymbol{R}^{2N}$. Lemma 3.1–Lemma 3.4 show that I satisfies all the assumptions of Proposition 2.1. Hence I has a positive critical value sequence c_k with $c_k \to \infty$. Let z_k be the critical point of I such that $I(z_k) = c_k$. Then z_k are homoclinic orbits of (HS) and

$$\int_{\boldsymbol{R}} \Big(-\frac{1}{2}J\dot{z}_k \cdot z_k - H(t,z_k)\Big)dt = I(z_k) = c_k \to \infty$$

as $k \to \infty$. The proof is complete.

§4. Proof of Theorem 1.2

In this section, let the assumptions of Theorem 1.2 be satisfied. We prove Theorem 1.2 via Proposition 2.2. Let $E = D(|A|^{1/2})$ be as before.

By (R_5)–(R_8) there are $\bar{a} \geq a_3$ such that

$$a_3|z|^\beta \leq R(t,z) \leq \bar{a}|z|^\beta, \quad \forall (t,z) \in \boldsymbol{R} \times \boldsymbol{R}^{2N}. \tag{4.1}$$

Moreover, one has

$$|R_z(t,z)| \leq C(|z|^{\beta-1} + |z|), \quad \forall (t,z). \tag{4.2}$$

Define again

$$\varphi(z) = \int_{\boldsymbol{R}} R(t,z), \quad \forall z \in E.$$

(4.1)–(4.2) imply that φ is well-defined, $\varphi \in C^1(E,\boldsymbol{R})$ and

$$\varphi'(z)x = \int_{\boldsymbol{R}} R_z(t,z)x, \quad \forall z, x \in E. \tag{4.3}$$

By Lemma 2.6, in addition, φ' is compact. To show this let $z_n \in E$ be such that $z_n \to z$ weakly in E. By Lemma 2.6, one can assume $z_n \to z$ strongly in L^p for $1 \leq p \in (\frac{2}{3-\alpha},\infty)$ without loss of generality. By (4.3)

$$\|\varphi'(z_n) - \varphi'(z)\| = \sup_{\|x\|=1} \Big| \int_{\boldsymbol{R}} (R_z(t,z_n) - R_z(t,z))x \Big|. \tag{4.4}$$

For any $R > 0$, by (4.2)

$$\begin{aligned}
&\Big| \int_{|t|\geq R} (R_z(t,z_n) - R_z(t,z))x \Big| \\
&\leq C \int_{|t|\geq R} (|z_n|^{\beta-1} + |z|^{\beta-1} + |z_n| + |z|)|x| \\
&\leq C\Big[\|x\|_{L^\beta}\Big(\int_{|t|\geq R} |z_n|^\beta + |z|^\beta\Big)^{(\beta-1)/\beta} + \|x\|_{L^2}\Big(\int_{|t|\geq R} |z_n|^2 + |z|^2\Big)^{1/2}\Big].
\end{aligned}$$

Hence by Lemma 2.6 for any $\epsilon > 0$ one can take R_0 large such that

$$\Big| \int_{|t|\geq R_0} (R_z(t,z_n) - R_z(t,z))x \Big| < \frac{\epsilon}{2} \tag{4.5}$$

for all $\|x\| = 1$ and $n \in \boldsymbol{N}$. On the other hand, it is easy to see that

$$\|R_z(\cdot,z_n) - R_z(\cdot,z)\|_{L^2(B_{R_0})} \to 0 \;\text{ as }\; n \to \infty$$

(see [12]). Hence there exists n_0 such that

$$\Big| \int_{|t|<R_0} (R_z(t,z_n) - R_z(t,z))x \Big| < \frac{\epsilon}{2}$$

for all $\|x\| = 1$ and $n \geq n_0$, which, together with (4.4) and (4.5), shows

$$\|\varphi'(z_n) - \varphi'(z)\| < \epsilon, \quad \forall n \geq n_0.$$

Therefore φ' is compact.

Now we define the following functional

$$I(z) = \varphi(z) - \frac{1}{2}a(z,z) = \varphi(z) - \frac{1}{2}\|z^+\|^2 + \frac{1}{2}\|z^-\|^2$$

for all $z = z^- + z^0 + z^+ \in E^- \oplus E^0 \oplus E^+ = E$. The above argument shows that $I \in C^1(E,\boldsymbol{R})$. Again by (2.8) a standard argument shows that the nontrivial critical points of I on E give rise to homoclinic orbits of (HS).

Let E_1, E_2, X_n and X^m be as in Section 3, and set $I_n = I|_{X_n}$. We verify that I satisfies the assumptions of Proposition 2.2.

Lemma 4.1. *I satisfies* (PS)* *and* (PS)**.

Proof. The proof has essentially been shown in [8]. For reader's convenience we present it here. Since the reasonings for (PS)* and (PS)** are the same, we only show (PS)*. Let $z_n \in X_n$ be such that $0 \le I(z_n) \le$ const. and $I_n'(z_n) \to \infty$. By (R$_5$)

$$I(z_n) - \frac{1}{2}I_n'(z_n)z_n = \int_{\boldsymbol{R}} \Big(R(t,z_n) - \frac{1}{2}R_z(t,z_n)z_n\Big) \ge \Big(1-\frac{\beta}{2}\Big)\int_{\boldsymbol{R}} R(t,z_n).$$

It then follows from (R$_6$) that

$$C(1+\|z_n\|) \ge a_3\|z\|_{L^\beta}^\beta. \tag{4.6}$$

Since dim $E^0 < \infty$, we have $\|z_n^0\|_{L^2}^2 = (z_n^0, z_n)_{L^2} \le C\|z_n^0\|_{L^2}\|z_n\|_{L^\beta}$, which, together with (4.6), shows

$$\|z_n^0\| \le C(1+\|z_n\|^{1/\beta}). \tag{4.7}$$

By (R$_6$) and (R$_8$), for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that $|R_z(t,z)| \le \epsilon|z| + C_\epsilon|z|^{\beta-1}$, $\forall (t,z)$. Hence one gets

$$\begin{aligned}\|z_n^+\|^2 &= \int_{\boldsymbol{R}} R_z(t,z_n)z_n^+ - I'(z_n)z_n^+ \\ &\le \epsilon\|z_n\|_{L^2}\|z_n^+\|_{L^2} + C_\epsilon\|z_n\|_{L^\beta}^{\beta-1}\|z_n^+\|_{L^\beta} + C\|z_n^+\| \\ &\le \epsilon d\|z_n\|\|z_n^+\| + C_\epsilon C(1+\|z_n\|^{\beta-1})\|z_n^+\|.\end{aligned} \tag{4.8}$$

Similarly we have

$$\|z_n^-\| \le \epsilon d\|z_n\| + C_\epsilon C(1+\|z_n\|^{\beta-1}). \tag{4.9}$$

It then follows from (4.7)–(4.9), letting $\epsilon > 0$ small enough, that $\|z_n\|$ is bounded. Now since φ' is compact, the form of I shows that $\{z_n\}$ has a convergent subsequence, proving the (PS)*.

Lemma 4.2. *I satisfies* (I$_4$).

Proof. For any $z \in X^m$, we have by (R$_6$)

$$I(z) = \int_{\boldsymbol{R}} R(t,z) - \frac{1}{2}\|z^+\|^2 + \frac{1}{2}\|z^-\|^2 \ge a_3\|z\|_{L^\beta}^\beta - \frac{1}{2}\|z^+\|^2 + \frac{1}{2}\|z^-\|^2. \tag{4.10}$$

Again since dim$\,(E^0 \oplus \text{span}\,\{w_1, \cdots, w_m\}) < \infty$, one has ($\beta' = \frac{\beta}{\beta-1} > 2$)

$$\|z^0+z^+\|_{L^2}^2 = (z^0+z^+, z)_{L^2} \le \|z^0+z^+\|_{L^{\beta'}}\|z\|_{L^\beta} \le C(m)\|z^0+z^+\|_{L^2}\|z\|_{L^\beta}$$

and so there holds $C'(m)\|z^0+z^+\|^\beta \le a_3\|z\|_{L^\beta}^\beta$ which, together with (4.10), yields $I(z) \ge C'(m)\|z^0+z^+\|^\beta - \frac{1}{2}\|z^0+z^+\|^2 + \frac{1}{2}\|z^-\|^2$ for all $z = z^- + z^0 + z^+ \in X^m$. Therefore, since $\beta < 2$, there are $r_m > 0$ and $a_m > 0$ such that $I(z) \ge a_m$, $\forall z \in X^m$ with $\|z\| = r_m$, i.e., I satisfies (I$_4$).

Lemma 4.3. *I satisfies* (I$_5$).

Proof. Let $z \in (X^{m-1})^\perp$. By (4.1) we have

$$I(z) = \int_{\boldsymbol{R}} R(t,z) - \frac{1}{2}\|z\|^2 \le \bar{a}\|z\|_{L^\beta}^\beta - \frac{1}{2}\|z\|^2. \tag{4.11}$$

Let ζ_m be defined by

$$\zeta_m = \sup_{z \in (X^m)^\perp \setminus \{0\}} \frac{\|z\|_{L^\beta}}{\|z\|}.$$

Similarly to the proof of Lemma 3.3, one has

$$0 < \zeta_m \to 0 \quad \text{as} \quad m \to \infty. \tag{4.12}$$

Now by (4.11), for $z \in (X^{m-1})^\perp$, there holds

$$I(z) \leq \bar{a}\zeta_{m-1}^\beta \|z\|^\beta - \frac{1}{2}\|z\|^2. \tag{4.13}$$

Let $b_m = \left(1 - \frac{\beta}{2}\right)\bar{a}\zeta_{m-1}^\beta(\bar{a}\beta\zeta_{m-1}^\beta)^{\beta/(2-\beta)}$. Then by (4.12), $b_m \to 0$ as $m \to \infty$, and by (4.13) $I(z) \leq b_m, \quad \forall z \in (X^{m-1})^\perp$, i.e., I satisfies (I_5).

Now we can give the following

Proof of Theorem 1.2. Clearly, $I(0) = 0$, and since $R(t,z)$ is even with respect to $z \in \boldsymbol{R}^{2N}$, I is even. Lemma 4.1–Lemma 4.3 show that I satisfies all the assumptions of Proposition 2.2. Therefore I possesses a sequence of positive critical values, $\{c_k\}$, satisfying $c_k \to 0$ as $k \to \infty$. Let z_k be the critical points of I associated with c_k, i.e., $I'(z_k) = 0$ and $I(z_k) = c_k$. Then z_k are homoclinic orbits of (HS) such that

$$0 < \int_{\boldsymbol{R}} \left[\frac{1}{2}J\dot{z}_k \cdot z_k + H(t, z_k)\right] dt = I(z_k) = c_k \to 0$$

as $k \to \infty$. The proof is complete.

References

[1] Ambrosetti, A. & Rabinowitz, P. H., Dual variational methods in critical point theory and applications, *J. Funct. Anal.,* **14**(1973), 349–381.

[2] Ambrosetti, A. & Coti-Zelati, V., Multiple homoclinic orbits for a class of conservative systems, Preprint, Sucola Normale Superiore, 1992.

[3] Bartsch, T. & Willem, M., Periodic solutions of non-autonomous Hamiltonian systems with symmetries, Preprint.

[4] Coti-Zelati, V., Ekeland, I. & Séré, E., A variational approach to homoclinic orbits in Hamiltonian systems, *Math. Ann.,* **288**(1990), 133–160.

[5] Coti-Zelati, V. & Rabinowitz, P. H., Homoclinic orbits for a second order Hamiltonian systems possessing superquadratic potentials, *Jour. Am. Math. Soc.,* **4**(1991), 693–727.

[6] Ding, Y., Existence and multiplicity results for homoclinic solutions to a class of Hamiltonian systems, *Nonlinear Analysis T.M.A.,* **25**(1995), 1095–1113.

[7] Ding, Y. & Girardi, M., Periodic and homoclinic solutions to a class of Hamiltonian systems with the potentials changing sign, *Dyn. Sys. and Appl.,* **2**(1993), 131–145.

[8] Ding, Y. & Li, S., Homoclinic orbits for first order Hamiltonian systems, *J. Math. Anal and Appl.,* **189**(1995), 585–601.

[9] Edmunds, D. E. & Evans, W. D., Spectral theory and differential operators, Clarendon Press, Oxford, 1987.

[10] Hofer, H. & Wysocki, K., First order elliptic systems and the existence of homoclinic orbits in Hamiltonian systems, *Math. Ann.,* **288**(1990), 483–503.

[11] Omana, W. & Willem, M., Homoclinic orbits for a class of Hamiltonian systems, *Diff. and Int. Eq.,* **5**(1992), 1115–1120.

[12] Rabinowitz, P. H., Minimax methods in critical point theory with applications to differential equations, CBMS Reg. Conf. Ser. in Math., 65, A.M.S. Providence, 1986.

[13] Rabinowitz, P. H., Homoclinic orbits for a class of Hamiltonian systems, *Proceed, Royal. Soc. Edinburgh,* **114A**(1990), 33–38.

[14] Rabinowitz, P. H. & Tanaka, K., Some results on connecting orbits for a class of Hamiltonian systems, *Math. Z.,* **206**(1991), 473–499.

[15] Séré, E., Existence of infinitely many homoclinic orbits in Hamiltonian systems, *Math. Z.,* **209**(1992), 27–42.

[16] Séré, E., Looking for the Bernoulli shift, Preprint, CEREMADE, 1992.

[17] Tanaka, K., Homoclinic orbits in a first order superquadratic Hamiltonian system: convergence of subharmonic orbits, *J. Diff. Eq.,* **94**(1991), 315–339.