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GEVREY CLASS REGULARITY AND APPROXIMATE INERTIAL MANIFOLDS FOR THE NEWTON-BOUSSINESQ EQUATIONS

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Abstract

The authors show the Gevrey class regularity of the solutions for the two-dimensional Newton-Boussinesq Equations. Based on this fact, an approximate inertial manifold for the system is constructed, which attracts all solutions to an exponentially thin neighborhood of it in a finite time.

Keywords Gevrey class regularity, Global attractor, Approximate inertial manifold, Newton-Boussinesq Equation

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§1. Introduction

Approximate inertial manifolds are related to the study of long time behaviour of solutions of dissipative partial differential equations. We recall that an approximate inertial manifold is a finite dimensional smooth manifold such that every solution enters a thin neighborhood of it in a finite time. In particular, the global attractor is contained in this neighborhood. The existence of such manifolds has been obtained for many partial differential equations. In this respect, we refer readers to Foias, Manley and Temam[1]; Temam[2]; Marion[3]; Debussche and Marion[4]; Foias, Manley and Temam[5]; Liu[6] et al.

Our aim of this paper is to deal with approximate inertial manifolds for the following two-dimensional Newton-Boussinesq Equations:

$$\partial_t\xi + u\partial_x\xi + v\partial_y\xi = \triangle\xi - \frac{R_a}{P_r}\partial_x\theta, \tag{1.1}$$

$$\triangle\Psi = \xi, \qquad u = \Psi_y, \qquad v = -\Psi_x, \tag{1.2}$$

$$\partial_t\theta + u\partial_x\theta + v\partial_y\theta = \frac{1}{P_r}\triangle\theta, \tag{1.3}$$

where $\vec{u} = (u, v)$ is the velocity vector, θ is the flow temperature, Ψ is the flow function, ξ is the vortex, $P_r > 0$ is the Prandtl number, and $R_a > 0$ is the Rayleigh number. The equations (1.1)–(1.3) arose in the course of analysing many physical problems such as Benard flow (see [7–9]). In [10], we constructed approximate inertial manifolds for system (1.1)–(1.3), and showed that any orbit enters their very thin (but not exponentially) neighborhoods. In the

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present paper, we first establish the Gevrey class regularity of the solutions for (1.1)–(1.3). And then we show that the approximate inertial manifolds constructed in [10] attract all solutions to their exponentially thin neighborhoods in a finite time, which strengthens the results in [10].

This paper is organized as follows. In the next section, we first show the time analyticity of the solution. And then we obtain the solution and its time derivative are bounded in H^1-norm. In Section 3, we recall the approximate inertial manifold constructed in [10]. By the time analyticity, we obtain the same results as in [10], but weaken the requirement of smoothness for initial data. The Gevrey class regularity of the solution is established in Section 4. As a result, we deduce that the approximate inertial manifold constructed attracts all solutions to an exponentially thin neighborhood of it.

For notational convenience, let $\Omega = (0, L_1) \times (0, L_2)$, and denote by $\|\cdot\|$ the norm of $L^2(\Omega)$ with usual inner product $(\cdot,\cdot)$, by $\|\cdot\|_Y$ the norm of any Banach space Y. When no confusion arising, we also denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product of $H = L^2(\Omega) \times L^2(\Omega)$, respectively. $\forall m \geq 0$, let

$$\begin{aligned} H^m_{per}(\Omega) = \big\{ & u \in H^m(\Omega) : u^{(k)}(x + L_1, y) = u^{(k)}(x, y + L_2) = u^{(k)}(x, y), \\ & \text{for } 0 \leq k \leq m \text{ and } a.e.\ x, y \big\}. \end{aligned}$$

In the sequel, we frequently use the Agmon inequality

$$\|u\|_\infty \leq C\|u\|^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{H^2} \leq C\|u\|_{H^2}, \qquad \forall u \in H^2(\Omega), \tag{1.4}$$

and the Poincare inequality

$$\|u\| \leq C\|\nabla u\|, \qquad \text{where} \int_\Omega u(x)dx = 0, \tag{1.5}$$

and

$$C_1\|u\|_{H^2} \leq \|u\| + \|\triangle u\| \leq C_2\|u\|_{H^2}, \quad \forall u \in H^2(\Omega), \tag{1.6}$$

where C_1, C_2 and C are positive constants.

§2. Time Analyticity of the Solutions

We observe that system (1.1)–(1.3) can be rewritten as

$$\frac{\partial}{\partial t}\xi - \triangle\xi + J(\Psi, \xi) + \frac{R_a}{P_r}\frac{\partial \theta}{\partial x} = 0, \tag{2.1}$$

$$\triangle\Psi = \xi, \tag{2.2}$$

$$\frac{\partial}{\partial t}\theta - \frac{1}{P_r}\triangle\theta + J(\Psi, \theta) = 0, \tag{2.3}$$

where $J(u, v) = u_y v_x - u_x v_y$. These equations are supplemented with the initial condition:

$$\xi\Big|_{t=0} = \xi_0(x, y), \qquad \theta\Big|_{t=0} = \theta_0(x, y), \tag{2.4}$$

and the periodic boundary condition:

$$\begin{aligned} &\xi(x + L_1, y) = \xi(x, y + L_2) = \xi(x, y), \\ &\theta(x + L_1, y) = \theta(x, y + L_2) = \theta(x, y). \end{aligned} \tag{2.5}$$

It is known from [11] that for $(\xi_0,\theta_0)\in L^2_{per}(\Omega)\times L^2_{per}(\Omega)$, the problem (2.1)–(2.5) possesses a unique solution (ξ,θ) defined for all $t\geq 0$ such that

$$\xi\in L^\infty(R^+;L^2_{per}(\Omega))\cap L^2(0,T;H^1_{per}(\Omega)),$$
$$\theta\in L^\infty(R^+;L^2_{per}(\Omega))\cap L^2(0,T;H^1_{per}(\Omega)).$$

A particular feature of equation (2.3) is that the average of the solution is conservative for all $t\geq 0$, that is,

$$m(\theta(t))=\frac{1}{|\Omega|}\int\int_\Omega\theta(x,y)dxdy=\frac{1}{|\Omega|}\int\int_\Omega\theta_0(x,y)dxdy=m(\theta_0). \tag{2.6}$$

Thus, there can not exist bounded absorbing sets in the whole space H. To overcome this difficulty, we introduce the subset of H :

$$H_\alpha=\{(\xi,\theta)\in L^2_{per}\times L^2_{per}:\quad |m(\theta)|\leq\alpha\}$$

for some fixed α.

For later purpose, we first establish

Lemma 2.1. *Assume that $(\xi_0,\theta_0)\in H_\alpha,\theta_0\in H^1_{per}$. Then for the solution (ξ,θ) of problem (2.1)–(2.5) we have $\|\xi(t)\|,\ \ \|\theta(t)\|\leq K_1,\ \forall t\geq t_1$, where K_1 denotes a constant depending only on the data $(\alpha,\Omega,P_r,R_\alpha)$, t_1 depends on the data $(\alpha,\Omega,P_r,R_\alpha)$ and R when $\|\xi_0\|\leq R$ and $\|\theta_0\|\leq R$.*

Proof. For convenience, we denote

$$\tilde\theta=\theta-m(\theta), \tag{2.7}$$

where

$$m(\theta)=\frac{1}{|\Omega|}\int\int_\Omega\theta(x,y,t)dxdy, \tag{2.8}$$

and then we have

$$\frac{1}{|\Omega|}\int\int_\Omega\tilde\theta(x,y,t)dxdy=\frac{1}{|\Omega|}\int\int_\Omega\theta(x,y,t)dxdy-\frac{1}{|\Omega|}\int\int_\Omega m(\theta)dxdy=0. \tag{2.9}$$

We easily see that

$$\|\theta\|^2=\|\tilde\theta\|^2+\|m(\theta)\|^2. \tag{2.10}$$

By (2.6), we get

$$\|m(\theta)\|^2=|\Omega||m(\theta)|^2\leq\alpha^2|\Omega|. \tag{2.11}$$

We note that (2.3), (2.6) and (2.7) imply that

$$\frac{\partial}{\partial t}\tilde\theta-\frac{1}{P_r}\triangle\tilde\theta+J(\Psi,\tilde\theta)=0. \tag{2.12}$$

Taking the inner product of (2.12) with $\tilde\theta$ in L^2, we find that

$$\frac{1}{2}\frac{d}{dt}\|\tilde\theta\|^2+\frac{1}{P_r}\|\nabla\tilde\theta\|^2+(J(\Psi,\tilde\theta),\tilde\theta)=0. \tag{2.13}$$

$$\frac{d}{dt}\|\tilde\theta\|^2+\frac{2}{P_r}\|\nabla\tilde\theta\|^2=0. \tag{2.14}$$

It comes from (2.9) and (1.5) that

$$\frac{d}{dt}\|\tilde\theta\|^2+\frac{2}{C^2P_r}\|\tilde\theta\|^2\leq 0.$$

By Gronwall inequality we infer that

$$\|\tilde{\theta}(t)\|^2 \le \|\tilde{\theta}(0)\|^2 e^{-C_1 t} \le R^2 e^{-C_1 t} \le \alpha^2,$$
$$\forall t \ge t_* = \frac{2}{C_1} \ln \frac{R}{\alpha} \quad \Big(C_1 = \frac{2}{C^2 P_r}\Big). \tag{2.15}$$

By (2.10), (2.11) and (2.15) we have

$$\|\theta\|^2 \le \alpha^2|\Omega| + \alpha^2, \qquad \forall t \ge t_*. \tag{2.16}$$

Taking the inner product of (2.1) with ξ in L^2 we see that

$$\frac{1}{2}\frac{d}{dt}\|\xi\|^2 + \|\nabla\xi\|^2 + (J(\Psi,\xi),\xi) + \frac{R_\alpha}{P_r}\Big(\frac{\partial\theta}{\partial x},\xi\Big) = 0. \tag{2.17}$$

By the arguments similar to the above we can deduce that

$$\|\xi(t)\|^2 \le C, \quad \forall t \ge t_*, \tag{2.18}$$

where t_* depends on the data and R when $\|\xi_0\| \le R$ and $\|\theta_0\| \le R$. (2.16) and (2.18) conclude Lemma 2.1.

Lemma 2.2. *Suppose that the conditions of Lemma* 2.1 *hold, then we have*

$$\|\nabla\xi(t)\|, \quad \|\nabla\theta(t)\| \le K_2, \quad \forall t \ge t_2,$$

where K_2 depends only on the data, t_2 depends on the data and R when $\|\xi_0\| \le R$ and $\|\theta_0\| \le R$.

Proof. Taking the inner product of (2.1) with $\triangle\xi$ in L^2, we find that

$$\frac{d}{dt}\|\nabla\xi\|^2 + \|\triangle\xi\|^2 = (J(\Psi,\xi),\triangle\xi) + \frac{R_a}{P_r}\Big(\frac{\partial\theta}{\partial x},\triangle\xi\Big). \tag{2.19}$$

After simple computations we find that

$$\frac{d}{dt}\|\nabla\xi\|^2 \le 2C_1\|\nabla\xi\|^4 + 2C_3\|\nabla\theta\|^4 + 2C_4. \tag{2.20}$$

Taking the inner product of (2.3) with $\triangle\theta$ in L^2, we find that

$$\frac{d}{dt}\|\nabla\theta\|^2 + \frac{1}{P_r}\|\triangle\theta\|^2 = (J(\Psi,\theta),\triangle\theta). \tag{2.21}$$

From the above we can deduce that

$$\frac{d}{dt}\|\nabla\theta\|^2 \le 2C_5\|\nabla\xi\|^4 + 2C_6\|\nabla\theta\|^4. \tag{2.22}$$

By (2.20) and (2.22) and the uniform Gronwall lemma we can conclude Lemma 2.2.

As an immediate consequence of Lemma 2.1 and Lemma 2.2, we have

$$\|\xi(t)\|_{H^1}, \quad \|\theta(t)\|_{H^1} \le K, \qquad \forall t \ge t_*, \tag{2.23}$$

where K depends only on the data (α,Ω,P_r,R_a), t_* as before.

In order to show the time analyticity of the solution, we first rewrite system (2.1)–(2.3) as an abstract equation. Let

$$u = (\xi,\theta), \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{P_r} \end{pmatrix}.$$

Then by (2.1)–(2.3) we see that

$$\frac{du(t)}{dt} + DAu(t) + R(u(t)) = 0, \tag{2.24}$$

where $A = -\triangle$ is an unbounded self-adjoint operator with domain $D(A) = H^2_{per}(\Omega)$, and

$$A\Psi = -\xi, \qquad R(u) = \begin{pmatrix} J(\Psi,\xi) + \frac{R_a}{P_r}\frac{\partial\theta}{\partial x} \\ J(\Psi,\theta) \end{pmatrix}, \tag{2.25}$$

which is a nonlinear operator from $H^1_{per} \times H^1_{per}$ to $L^2_{per} \times L^2_{per}$. In fact we have

$$\begin{aligned} \|R(u)\| &\le \|J(\Psi,\xi)\| + \frac{R_a}{P_r}\|\nabla\theta\| + \|J(\Psi,\theta)\| \\ &\le C\|\nabla\triangle\Psi\|\|\nabla\xi\| + \frac{R_a}{P_r}\|\nabla\theta\| + C\|\nabla\triangle\Psi\|\|\nabla\theta\| \\ &\le C\|\nabla\xi\|^2 + \frac{R_a}{P_r}\|\nabla\theta\| + C\|\nabla\xi\|\|\nabla\theta\| \\ &\le C_1\|u\|^2_{H^1} + C_2\|u\|_{H^1}. \end{aligned} \tag{2.26}$$

And then we find that

$$\begin{aligned} |(R(u), Au + u)| &\le \|R(u)\|(\|Au\| + \|u\|) \\ &\le C_1\|u\|^2_{H^1}\|Au\| + C_2\|u\|_{H^1}\|Au\| + C_1\|u\|^3_{H^1} + C_2\|u\|^2_{H^1} \\ &\le \epsilon\|Au\|^2 + C(\epsilon)\|u\|^4_{H^1} + C, \quad \forall\epsilon > 0. \end{aligned} \tag{2.27}$$

By (2.26) And (2.27) we have

Theorem 2.1. *Assume that* $(\xi_0, \theta_0) \in H_\alpha$, $\theta_0 \in H^1_{per}$. *Then there exist* β_0 *and* T_0 *such that the solution* $(\xi(t), \theta(t)$ *of problem* (2.1)–(2.5) *has a* $D(A)$ *valued analytic extension in a complex region of the form*

$$\triangle_1 = \left\{t + se^{i\beta} : t \ge t_1, \quad |\beta| \le \beta_0, 0 \le s \le T_0\right\},$$

where t_1 *as in* (2.23), β_0 *and* T_0 *depend on the data and* $|\beta_0| \le \pi/4$. *In addition, there exists a constant* K *depending on the data such that* $\forall z \in \Delta_2$,

$$\|\xi(z)\|, \quad \|\theta(z)\|, \quad \|A^{1/2}\xi(z)\|, \quad \|A^{1/2}\theta(z)\|, \quad \|A\xi(z)\|, \quad \|A\theta(z)\| \le K, \tag{2.28}$$

where

$$\triangle_2 = \{z : \operatorname{Re} z \ge a, \quad |\operatorname{Im} z| \le b\}.$$

Here a *and* b *are constants depending on the data and* R *when* $\|\xi_0\| \le R$, $\|\theta_0\| \le R$.

Proof. We notice that (2.23) and (2.27) show that the conditions of Theorem 1.1 in [12] hold. Thus, by Theorem 1.1 there we can easily obtain this theorem. The details are omitted here.

§3. Approximate Inertial Manifolds

In this section, we introduce an approximate inertial manifold for the system (2.1)–(2.5). We first note that there exists an orthonormal basis $\{w_j\}_{j=1}^\infty$ of H consisting of eigenvectors of $A = -\triangle$ such that

$$Aw_j = \lambda_j w_j, \quad 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_j \to \infty, \quad \text{as} \quad j \to \infty.$$

Given m, denote by $P = P_m : H \to \operatorname{span}\{w_1, \cdots, w_m\}$ the projector, $Q = Q_m = I - P_m$.

Applying P_m and Q_m to (2.1)–(2.3) we obtain the following coupled system for $\xi_1 = P_m\xi$,

$\xi_2 = Q_m\xi$, $\theta_1 = P_m\theta$, $\theta_2 = Q_m\theta$:

$$\frac{d}{dt}\xi_1 + A\xi_1 + P_m J(\Psi,\xi) + \frac{R_a}{P_r}P_m\frac{\partial\theta}{\partial x} = 0,$$
$$\frac{d}{dt}\xi_2 + A\xi_2 + Q_m J(\Psi,\xi) + \frac{R_a}{P_r}Q_m\frac{\partial\theta}{\partial x} = 0, \tag{3.1}$$

$$\frac{d}{dt}\theta_1 + \frac{1}{P_r}A\theta_1 + P_m J(\Psi,\xi) = 0,$$
$$\frac{d}{dt}\theta_2 + \frac{1}{P_r}A\theta_2 + Q_m J(\Psi,\xi) = 0. \tag{3.2}$$

Lemma 3.1. *Assume that $(\xi_0,\theta_0) \in H_\alpha$, $\theta_0 \in H^1_{per}$. Then there exists a constant K depending only on the data such that*

$$\|A^{1/2}\xi_2(t)\| \le K\lambda_{m+1}^{-1}, \quad \|A^{1/2}\theta_2(t)\| \le K\lambda_{m+1}^{-1}, \quad \forall t \ge t_1,$$

where t_1 depends on the data and R when $\|\xi_0\| \le R$ and $\|\theta_0\| \le R$.

Proof. The proof of this lemma is standard, so is omitted here.

We observe that Theorem 2.1, Lemma 3.1 and Cauchy formula imply that there exists a constant K_1 depending on the data such that

$$\left\|A^{1/2}\frac{d}{dt}\xi_2(t)\right\|, \quad \left\|A^{1/2}\frac{d}{dt}\theta_2(t)\right\| \le K_1\lambda_{m+1}^{-1}, \quad \forall t \ge t_*. \tag{3.3}$$

We now construct an approximate inertial manifold. To the end, we define a mapping Φ from P_mH to Q_mH such that for $(\xi_1,\theta_1) \in P_mH$, $\Phi(\xi_1,\theta_1) = (f,g)$ is given by

$$Af + Q_m J(\Psi_1,\xi_1) + \frac{R_a}{P_r}Q_m\frac{\partial\theta_1}{\partial x} = 0, \tag{3.4}$$
$$-A\Psi_1 = \xi_1, \tag{3.5}$$
$$\frac{1}{P_r}Ag + Q_m J(\Psi_1,\theta_1) = 0. \tag{3.6}$$

Let Σ = graph (Φ). Then we can show that Σ is an approximate inertial manifold. More precisely, we have

Theorem 3.1. *Suppose $(\xi_0,\theta_0) \in H_\alpha, \theta_0 \in H^1_{per}$. Then there exists a constant K depending only on the data (α,Ω,P_r,R_a) such that any solution $(\xi(t),\theta(t))$ of problem* (2.1)–(2.5) *remains at a distance in H of Σ bounded by $K\lambda_{m+1}^{-2}$ for $t \ge t_*$, where t_* depends on the data and R when $\|\xi_0\| \le R$, $\|\theta_0\| \le R$.*

Proof. The procedure of the proof of this theorem is similar to that of Theorem 4.2 below, so we omit the details here.

We remark that Theorem 3.1 improves the results of [10], where the distance is only bounded by $K\lambda_{m+1}^{-3/2}$ and the initial values are required to be more smooth.

§4. Gevrey Class Regularity

In this section, we show that the solution of (2.1)–(2.5) is analytic with a value not only in $D(A)$, but also in a Gevrey class of functions on Ω, and hence in $C^\infty(\Omega)$. Based on this fact, we strengthen Theorem 3.1.

For simplicity, in the following, we assume that $\Omega = (0,2\pi)^2$ and

$$\int_\Omega u(x,t)dx = 0, \qquad \forall t > 0. \tag{4.1}$$

Lemma 4.1. *Suppose that* $u=(\xi,\theta)$ *and* $w=(w_1,w_2)$ *are both given in* $D(e^{\tau A^{1/2}}A)$, $\tau>0$. *Then we have*

$$\begin{aligned}&|(e^{\tau A^{1/2}}R(u), e^{\tau A^{1/2}}Aw)|\\ &\le C\|e^{\tau A^{1/2}}A^{1/2}u\|^2\|e^{\tau A^{1/2}}Aw\|+C\|e^{\tau A^{1/2}}A^{1/2}u\|\|e^{\tau A^{1/2}}Aw\|,\end{aligned}$$

where C *is an appropriate constant.*

Proof. We set

$$\Psi=\sum_{k,l\in Z}\Psi_{k,l}e^{i(kx+ly)},\tag{4.2}$$

$$\Psi^*=e^{\tau A^{1/2}}\Psi=\sum_{k,l\in Z}\Psi^*_{k,l}e^{i(kx+ly)},$$

$$\Psi^*_{k,l}=e^{\tau|(k,l)|}\Psi_{k,l};\tag{4.3}$$

$$w_1=\sum_{p,q\in Z}w_{p,q}e^{i(px+qy)},\tag{4.4}$$

$$w_1^*=e^{\tau A^{1/2}}w_1=\sum_{p,q}w^*_{p,q}e^{i(px+qy)},$$

$$w^*_{p,q}=e^{\tau|(p,q)|}w_{p,q}.\tag{4.5}$$

And then we have

$$\xi=\triangle\Psi=-\sum_{k,l}\Psi_{k,l}(k^2+l^2)e^{i(kx+ly)},\tag{4.6}$$

$$\xi^*=\triangle\Psi^*=-\sum_{k,l}\Psi^*_{k,l}(k^2+l^2)e^{i(kx+ly)}.\tag{4.7}$$

By (2.25) we have

$$\begin{aligned}&(e^{\tau A^{1/2}}R(u), e^{\tau A^{1/2}}Aw)\\ &=(e^{\tau A^{1/2}}J(\Psi,\xi), e^{\tau A^{1/2}}Aw_1)+\frac{R_a}{P_r}\Big(e^{\tau A^{1/2}}\frac{\partial\theta}{\partial x}, e^{\tau A^{1/2}}Aw_1\Big)\\ &\quad+(e^{\tau A^{1/2}}J(\Psi,\theta), e^{\tau A^{1/2}}Aw_2).\end{aligned}\tag{4.8}$$

We now majorize every term in the right-hand side of (4.8) as follows. By (3.7) we have

$$(J(\Psi,\xi),w_1)=\int\int_\Omega\Psi_y\xi_x\overline{w}_1dxdy-\int\int_\Omega\Psi_x\xi_y\overline{w}_1dxdy.\tag{4.9}$$

Since

$$\begin{aligned}&\int\int_\Omega\Psi_y\xi_x\overline{w}_1dxdy\\ &=-\int\int_\Omega\sum_{k,l}\Psi_{k,l}ile^{i(kx+ly)}\sum_{r,s}\Psi_{r,s}(r^2+s^2)ire^{i(rx+sy)}\sum_{p,q}\overline{w}_{p,q}e^{-i(px+qy)}dxdy\\ &=4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi_{k,l}\cdot l)\Psi_{r,s}(r^2+s^2)r\overline{w}_{p,q},\end{aligned}\tag{4.10}$$

and similarly,

$$\int\int_\Omega\Psi_x\xi_y\overline{w}_1dxdy=4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi_{k,l}\cdot k)\Psi_{r,s}s(r^2+s^2)\overline{w}_{p,q},\tag{4.11}$$

we see that

$$
\begin{aligned}
&(e^{\tau A^{1/2}}J(\Psi,\xi),e^{\tau A^{1/2}}Aw_1)\\
=&(J(\Psi,\xi),e^{2\tau A^{1/2}}Aw_1)\\
=&4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi_{k,l}\cdot l)\Psi_{r,s}r(r^2+s^2)e^{2\tau|(p,q)|}(p^2+q^2)\overline{w}_{p,q}\\
&-4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi_{k,l}\cdot k)\Psi_{r,s}s(r^2+s^2)e^{2\tau|(p,q)|}(p^2+q^2)\overline{w}_{p,q}\\
=&4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi^*_{k,l}\cdot l)\Psi^*_{r,s}r(r^2+s^2)\overline{w}^*_{p,q}(p^2+q^2)e^{\tau(|(p,q)|-|(k,l)|-|(r,s)|)}\\
&-4\pi^2\sum_{k+r=p,\ l+s=q}(\Psi^*_{k,l}\cdot k)\Psi^*_{r,s}s(r^2+s^2)\overline{w}^*_{p,q}(p^2+q^2)e^{\tau(|(p,q)|-|(k,l)|-|(r,s)|)}.
\end{aligned}
\tag{4.12}
$$

Due to $p=k+r,\ q=l+s$, we see that $|(p,q)|\le|(k,l)|+|(r,s)|$, and then

$$e^{\tau(|(p,q)|-|(k,l)|-|(r,s)|)}\le 1. \tag{4.13}$$

And thus by (4.12) and (4.13) we find that

$$
\begin{aligned}
&|(e^{\tau A^{1/2}}J(\Psi,\xi),e^{\tau A^{1/2}}Aw_1)|\\
\le&4\pi^2\sum_{k+r=p,\ l+s=q}|\Psi^*_{k,l}\cdot l|\cdot|\Psi^*_{r,s}r(r^2+s^2)|\cdot|\overline{w}^*_{p,q}(p^2+q^2)|\\
&+4\pi^2\sum_{k+r=p,\ l+s=q}|\Psi^*_{k,l}\cdot k|\cdot|\Psi^*_{r,s}s(r^2+s^2)|\cdot|\overline{w}^*_{p,q}(p^2+q^2)|.
\end{aligned}
\tag{4.14}
$$

Let

$$\alpha(x,y)=\sum_{k,l}|\Psi^*_{k,l}|\cdot|l|e^{i(kx+ly)}, \tag{4.15}$$

$$\beta(x,y)=\sum_{r,s}|\Psi^*_{r,s}|(r^2+s^2)|r|e^{i(rx+sy)}, \tag{4.16}$$

$$\eta(x,y)=\sum_{p,q}|\Psi^*_{p,q}|(p^2+q^2)e^{i(px+qy)}. \tag{4.17}$$

Then we see that

$$
\begin{aligned}
&4\pi^2\sum_{k+r=p,\ l+s=q}|\Psi^*_{k,l}\cdot l|\cdot|\Psi^*_{r,s}r(r^2+s^2)|\cdot|\overline{w}^*_{p,q}(p^2+q^2)|\\
=&\int\int_\Omega\alpha(x,y)\beta(x,y)\eta(x,y)dxdy\\
\le&\|\alpha(x,y)\|_\infty\|\beta(x,y)\|\|\eta(x,y)\|\\
\le&C\|\alpha(x,y)\|_{H^2}\|\beta(x,y)\|\|\eta(x,y)\|\\
\le&C(\|\alpha(x,y)\|+\|\triangle\alpha(x,y)\|)\|\beta(x,y)\|\|\eta(x,y)\|\\
\le&C\|\nabla\triangle\Psi^*\|^2\|\triangle w_1^*\|=C\|\nabla\xi^*\|^2\|\triangle w_1^*\|.
\end{aligned}
\tag{4.18}
$$

Similarly, the second term in the right-hand side of (4.14) can also be bounded by (4.18). And then it follows from (4.14) and (4.18) that

$$
\begin{aligned}
\left|(e^{\tau A^{1/2}}J(\Psi,\xi),e^{\tau A^{1/2}}Aw_1)\right|&\le C\|e^{\tau A^{1/2}}A^{1/2}\xi\|^2\|e^{\tau A^{1/2}}Aw_1\|\\
&\le C\|e^{\tau A^{1/2}}A^{1/2}u\|^2\|e^{\tau A^{1/2}}Aw\|.
\end{aligned}
\tag{4.19}
$$

Similarly, we can also deduce that

$$\Big|\frac{R_a}{P_r}\Big(e^{\tau A^{1/2}}\frac{\partial\theta}{\partial x}, e^{\tau A^{1/2}}Aw_1\Big)\Big| \le C_1\|e^{\tau A^{1/2}}A^{1/2}u\|\|e^{\tau A^{1/2}}Aw\|, \tag{4.20}$$

$$\Big|\Big(e^{\tau A^{1/2}}J(\Psi,\theta), e^{\tau A^{1/2}}Aw_2\Big)\Big| \le C_2\|e^{\tau A^{1/2}}A^{1/2}u\|^2\|e^{\tau A^{1/2}}Aw\|. \tag{4.21}$$

By (4.8) and (4.19)–(4.21) we conclude Lemma 4.1.

We note that if $u(t) = (\xi(t), \theta(t))$ is a solution of problem (2.1)–(2.5), then Lemma 4.1 implies that

$$\begin{aligned}
&|(e^{\tau A^{1/2}}R(u), e^{\tau A^{1/2}}Au)| \\
&\le C\|e^{\tau A^{1/2}}A^{1/2}u\|^2\|e^{\tau A^{1/2}}Au\| + C\|e^{\tau A^{1/2}}A^{1/2}u\|\|e^{\tau A^{1/2}}Au\| \\
&\le \varepsilon\|e^{\tau A^{1/2}}Au\|^2 + C(\varepsilon)\|e^{\tau A^{1/2}}A^{1/2}u\|^4 + C_1, \quad \forall\varepsilon > 0.
\end{aligned} \tag{4.22}$$

This inequality shows Lemma 3.1 in [12], and thus by Theorem 3.1 in that paper we obtain

Theorem 4.1. *Assume that $(\xi_0, \theta_0) \in H_\alpha$, $\theta_0 \in H^1_{per}$. Then there exists a constant σ depending on the data $(\alpha, \Omega, P_r, R_a)$ such that any solution $(\xi(t), \theta(t))$ of problem (2.1)–(2.5) has a $D(A^{1/2}\exp(\sigma A^{1/2}))$ valued analytic extension in a complex region of the form*

$$\triangle = \{t + se^{i\beta} : t \ge t_*, \quad |\beta| \le \beta_0, \quad 0 \le s \le T_0\}. \tag{4.23}$$

Moreover,

$$\|e^{\sigma A^{1/2}}A^{1/2}\xi(z)\| \le K, \quad \|e^{\sigma A^{1/2}}A^{1/2}\theta(z)\| \le K, \qquad \text{for} \quad z \in \triangle, \tag{4.24}$$

where, β_0, T_0, and K depend on the data, $|\theta_0| \le \frac{\pi}{4}$, t_* depends on the data and R when

$$\|\xi_0\| \le R, \quad \|\theta_0\| \le R.$$

Proof. (2.23) and (4.22) verify the conditions of Theorem 3.1 in [12], and hence this theorem follows from that result.

Lemma 4.2. *Assume $(\xi_0, \theta_0) \in H_\alpha, \theta_0 \in H^1_{per}$. Then there exists a constant K depending only on the data such that*

$$\|A^{1/2}\xi_2(t)\|, \quad \|A^{1/2}\theta_2(t)\| \le K\lambda_{m+1}^{-1/2}e^{-\sigma\lambda_{m+1}^{1/2}}, \quad \forall t \ge t_0,$$

where t_0 depends on the data and R when

$$\|\xi_0\| \le R, \quad \|\theta_0\| \le R.$$

Proof. The proof of this lemma is standard now, and hence is omitted here.

Based on Lemma 4.2, we can easily deduce

Theorem 4.2. *Assume that $(\xi_0, \theta_0) \in H_\alpha, \theta_0 \in H^1_{per}$. Then there exists a constant K depending on the data such that any solution $(\xi(t), \theta(t))$ of problem (2.1)–(2.5) remains at a distance in H of $\sum$ bounded by $K\lambda_{m+1}^{-3/2}\ e^{-\sigma\lambda_{m+1}^{1/2}}$ when $t \ge t_*$, where t_* depends on the data and R when*

$$\|\xi_0\| \le R, \quad \|\theta_0\| \le R.$$

Proof. The proof of this theorem is omitted.

We remark that Theorem 4.2 states that Σ attracts any solution to an exponentially thin neighborhood of it. And therefore Theorem 4.2 is stronger than Theorem 3.1.

References

[1] Foias, C., Manley, O. & Temam, R., Sur'l interaction des petites et grands tourbillon's dan's les ecoulements turbulents, *C. R. Acad. Sci. Paris Ser. I. Math.,* **305**(1987), 497–500.

[2] Temam, R., Attractors for the Navier-Stokes equations : localization and approximation, *J. Fac. Sci. Univ. Tokyo. Sec. IA,* **36**(1989), 629–647.

[3] Marion, M., Approximate inertial manifolds for reaction-diffusion equations in high space dimension, *J. Dynamics Diff. Equ.,* **1**(1989), 245–267.

[4] Debussche, A. & Marion, M., On the construction of families of approximate inertial manifolds, *J. Diff. Equ.,* **100**(1992), 173–201.

[5] Foias, C., Manley, O. & Temam, R., Approximate inertial manifolds and effective viscosity in turbulent flows, *Phys. of Fluids A,* **3**(1991), 898–911.

[6] Liu, X., Gevrey class regularity and approximate inertial manifolds for the Kuramoto-Sivashinsky equation, *Physica D,* **50**(1990), 135–151.

[7] Guo, B. L., Spectral method for solving two dimensional Newton-Boussinesq equations, *Acta. Math. Appl.,* **5**(1989), 208–218.

[8] Chen, S., Symmetry analysis of convect on patterns, *Comm. Theor. Phys.,* **1**(1982), 413-426.

[9] Feigenbaum, M. J., The onset spectrum of turbulence, *Phys. Lett. A,* **74**(1979), 375–378.

[10] Guo, B. L. & Wang, B. X., Approximate inertial manifold for the two-dimensional Newton-Boussinesq equations, *J. Partial Diff. Equ.,* **9:**6(1996), 237–250.

[11] Guo, B. L., Nonlinear Galerkin method for solving two-dimensional Newton-Boussinesq Equations, *Chin. Ann. of Math.,* **16B:**3(1995), 375–390.

[12] Promislow, K., Time analyticity and Gevrey regularity for solutions of a class of dissipative partial differential equations, *Nonl. Anal. TMA,* **16**(1991), 959–980.