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BIFURCATIONS OF LIMIT CYCLES FROM A HETEROCLINIC CYCLE OF HAMILTONIAN SYSTEMS**

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Abstract

This paper is concerned with the bifurcations of limit cycles from a heteroclinic cycle of planar Hamiltonian systems under perturbations. The author obtains a simple condition which guarantees the existence of at most two limit cycles near the heteroclinic cycle.

Keywords Bifurcation, Limit cycle, Heteroclinic cycle
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§1. Normal Forms of Displacement Functions

Consider a planar C^∞ system of the form

$$\dot{x} = f(x) + \lambda f_0(x, \delta, \lambda) \equiv \overline{f}(x, \delta, \lambda), \tag{1.1}$$

where $x \in R^2$, $\lambda \in R$, $\delta \in R^m$, and $\mathrm{tr} Df(x) = 0$. Suppose that for $\lambda = 0$ (1.1) has a heteroclinic cycle L consisting of two hyperbolic saddle points $S_i^0 (i = 1, 2)$ and two separatrice $L_i (1 = 1, 2)$. For definiteness, we assume that L is oriented clockwise and that L_1 starts at S_1^0 and ends at S_2^0. Choose points $A_1 \in L_2$, $A_2 \in L_1$ near S_1^0, and $A_3 \in L_2$, $A_4 \in L_1$ near S_2^0. Let l_i be a cross section through A_i and parallel to the vector $n_i = -\mathrm{grad} H(A_i) / \mid \mathrm{grad} H(A_i) \mid (i = 1, 2, 3, 4)$. We define Poincarè maps $F_1 : l_1 \to l_2$, $F_2 : l_2 \to l_4$ using positive orbits, and $G_1 : l_1 \to l_3$, $G_2 : l_3 \to l_4$ using negative orbits. Let $F_0(u, \delta, \lambda) = F_2 \circ F_1 - G_2 \circ G_1$ which is called a displacement function of (1.1). Our goal in this section is to give a normal form of F_0.

Let $S_i(\delta, \lambda)$ be the saddle point of (1.1) near S_i^0. Set $y = x - S_1(\delta, \lambda)$. We have from (1.1)

$$\dot{y} = \overline{f}(y + S_1(\delta, \lambda), \delta, \lambda). \tag{1.2}$$

Let $\alpha_{11}(\delta, \lambda) > 0$, $-\alpha_{21}(\delta, \lambda) < 0$ be eigenvalues of $B_1(\delta, \lambda) = D_x \overline{f}(S_1, \delta, \lambda)$. Then there exists a reversible matrix $T(\delta, \lambda)$ such that

$$T B_1 T^{-1} = \mathrm{diag}(\alpha_{11}, -\alpha_{21}). \tag{1.3}$$

From [1,2], for any natural number n, there exists a coordinate change:

$$T_{\delta,\lambda} : \ z = T_{\delta,\lambda}(y) = T(\delta, \lambda) y + O(\mid y \mid^2) \tag{1.4}$$

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which carries locally (1.2) into the C^∞ system

$$\dot{z}_1 = \alpha_{11}^0 z_1 + \sum_{i=0}^{n} a_{i1} z_1 (z_1 z_2)^i + z_1 (z_1 z_2)^{n+1} R_1,$$
$$\dot{z}_2 = -\alpha_{11}^0 z_2 + \sum_{i=0}^{n} b_{i1} z_2 (z_1 z_2)^i + z_2 (z_1 z_2)^{n+1} R_2, \tag{1.5}$$

where $\alpha_{11}^0 = \alpha_{11}(\delta, 0),\ a_{01} = \alpha_{11} - \alpha_{11}^0,\ b_{01} = \alpha_{11}^0 - \alpha_{21}$. From (1.5) we have

$$\frac{dz_2}{dz_1} = \frac{z_2}{z_1}\Big[-1 + \sum_{i=1}^{n+1} C_{i1} (z_1 z_2)^{i-1} + (z_1 z_2)^{n+1} R\Big], \tag{1.6}$$

where

$$C_{11} = \frac{1}{\alpha_{11}}(\alpha_{11} - \alpha_{21}). \tag{1.7}$$

Define $\bar{l}_1 = \{z_2 = -\varepsilon,\ 0 \le z_1 \le \varepsilon\}$, $\bar{l}_2 = \{z_1 = \varepsilon, -\varepsilon \le z_2 \le 0\}$ for $\varepsilon > 0$. Let $\overline{Q}_r = (\varepsilon, -q(r, \delta, \lambda))$ be the first intersection point of $\bar{l}_2$ with the positive orbit of (1.5) through $\overline{P}_r = (r, -\varepsilon) \in \bar{l}_1$. This defines the Dulac map q. Letting $z_i = \varepsilon v_i$ in (1.6), from [1,5] we have

$$q(r, \delta, \lambda) = r + C_{11}[r\omega_1 + \cdots] + C_{n1}\varepsilon^{n-1}[r^n \omega_1 + \cdots] + \varphi_{n1}, \tag{1.8}$$

where $\omega_1 = \int_{r/\varepsilon}^{1} t^{-1-C_{11}} dt,\ \varphi_{n1}$ is of C^n for $0 \le r \le \varepsilon$ and n-flat at $r = 0$. Every bracket in (1.8) consists of finite combinations of terms $r^i \omega_1^j (0 \le j \le i \le n)$ with $[r^i\omega_1 + \cdots] = r^i \omega_1 (1 + o(1))$. It is easy to see that the separatrice $L_2^s,\ L_1^u$ of (1.2) near $L_2,\ L_1$ have $T^{-1}(0,1)^T$ and $T^{-1}(1,0)^T$ as their directional vectors respectively. Let $\overline{P}_0 = (0, -\varepsilon),\ \overline{Q}_0 = (\varepsilon, 0)$ satisfy

$$A_1 = T_0^{-1}(\overline{P}_0),\ A_2 = T_0^{-1}(\overline{Q}_0), \tag{1.9}$$

where $T_0 = T_{\delta,0}$. Let $P_u(Q_u)$ be the intersection point of $l_1(l_2)$ with the orbit of (1.2) passing through $T_{\delta,\lambda}^{-1}(\overline{P}_r)$ near $y = 0$. Then we can write

$$P_u = A_1 + u n_1,\ Q_u = A_2 + F_1(u, \delta, \lambda) n_2. \tag{1.10}$$

Using the similar method in proving Lemma 7.11 (see [2]) we have

Lemma 1.1. *It holds that* $u = W_1(r, \delta, \lambda)$ *and* $F_1 = W_2(q, \delta, \lambda)$, *where*

$$W_i = N_i(\delta, \lambda) + M_i(\delta, \lambda) r + O(r^2), \quad N_i(\delta, 0) = 0,$$
$$M_i(\delta, 0) = \beta_i \sin\theta_1 + o(1) \ \textit{for}\ 0 < \varepsilon \ll 1,\ i = 1, 2,$$
$$\beta_1 = | T^{-1}(\delta, 0)(1,0)^T |, \quad \beta_2 = | T^{-1}(\delta, 0)(1,0)^T |,$$

and θ_1 *is the angle between* L_1 *and* L_2 *at* S_1^0.

We have immediately from (1.8) and Lemma 1.1

Lemma 1.2. $F_1(u, \delta, \lambda) = N_2 + M_2(r + C_{11}[r\omega_1 + \cdots] + \cdots + C_{n1}\varepsilon^{n-1}[r^n\omega_1 + \cdots]) + \psi_{n1} + r^2 p_{n1}$, *where* $r \ge 0$ *satisfies* $u = W_1(r, \delta, \lambda)$, ψ_{n1} *is* n*-flat at* $r = 0$, *and* p_{n1} *is a polynomial with respect to* r.

Lemma 1.3. *For* $\varepsilon > 0$ *small we have*

$$| f(A_1) | = \beta_2 \alpha_{11}^0 \varepsilon + O(\varepsilon^2), \quad | f(A_2) | = \beta_1 \alpha_{11}^0 \varepsilon + O(\varepsilon^2).$$

Proof. Since $f(S_1^0)=0$, we have from (1.3), (1.4) and (1.9)

$$\begin{aligned} f(A_1) &= Df(S_1^0)A_1 + O(| A_1 |^2) \\ &= B_1(\delta,0)T^{-1}(\delta,0)\overline{P}_0 + O(| \overline{P}_0 |^2) \\ &= \varepsilon\alpha_{11}^0 T^{-1}(\delta,0)(0,1)^T + O(\varepsilon^2). \end{aligned}$$

Then the first equality follows. The second can be proved similarly.

Let $\mathrm{diag}(-\alpha_{22},\alpha_{12})$ be Jordan form of $D_x\overline{f}(S_2,\delta,\lambda)$ with $\alpha_{12}>0$. In a similar way to (1.5), we can obtain a normal form of (1.1) at S_2. Then by changing the sign of the time we have the following expression of G_2 similar to that of F_1 in Lemma 1.2:

$$\begin{aligned} G_2(v,\delta,\lambda) = N_4 + M_4(\widetilde{r} + C_{12}[\widetilde{r}\omega_2 + \cdots] + \cdots + C_{n2}\varepsilon^{n-1}[\widetilde{r}^n\omega_1 + \cdots]) \\ + \widetilde{\psi}_{n2}(\widetilde{r},\delta,\lambda) + \widetilde{r}\widetilde{p}_{n2}(\widetilde{r},\delta,\lambda), \end{aligned} \tag{1.11}$$

where $\omega_2 = \int_{\widetilde{r}/\varepsilon}^1 t^{-1-C_{12}}dt$,

$$C_{12} = \frac{1}{\alpha_{22}}(\alpha_{22}-\alpha_{12}),$$

$$v = N_3(\delta,\lambda) + M_3(\delta,\lambda)\widetilde{r} + O(\widetilde{r}^2) \equiv W_3(\widetilde{r_1},\delta,\lambda), \tag{1.12}$$

$$N_i(\delta,0)=0, \quad M_i(\delta,0) = \beta_i\sin\theta_2 + o(1) \quad \text{for} \ \ 0<\varepsilon\ll 1, \ \ i=3,4, \tag{1.13}$$

and θ_2 is the angle between L_1 and L_2 at S_2^0, $\widetilde{\psi}_{n2}$ is n-flat as $\widetilde{r}=0$, $\widetilde{p}_{n2}$ is a polynomial with respect to $\widetilde{r}$.

Also, similar to Lemma 1.3 we have

$$| f(A_3) | = \beta_4\alpha_{22}^0\varepsilon + O(\varepsilon^2), \qquad | f(A_4) | = \beta_3\alpha_{22}^0\varepsilon + O(\varepsilon^2), \tag{1.14}$$

where $\alpha_{22}^0 = \alpha_{22}(\delta,0)$. Let $X_i(t)$ be a representation of L_i and let $t_2<t_4$, $t_1>t_3$ be such that

$$X_1(t_j) = A_j, \ j=2,4, \qquad X_2(t_j)=A_j, \ j=1,3. \tag{1.15}$$

Then it is easy to see that

$$Z(t_2) = -n_2, \qquad Z(t_4) = -n_4, \tag{1.16}$$

where $Z(\theta) = (-V_2(\theta),V_1(\theta))^T$, $(V_1(\theta),V_2(\theta)) = X_1'(\theta)/ \mid X_1'(\theta) \mid$, $t_2\le\theta\le t_4$. By introducing the change of variables

$$x = X_1(\theta) + Z(\theta)\rho, \qquad t_2\le\theta\le t_4, \tag{1.17}$$

to (1.1) we obtain (see [1, 2])

$$\frac{d\rho}{d\theta} = R_0(\theta,\delta)\lambda + A(\theta)\rho + O(| \rho,\lambda |^2), \tag{1.18}$$

where $R_0(\theta,\delta) = Z^T f_0(X_1,\delta,0)$, $A(\theta) = \mathrm{tr}Df(X_1) - (\ln | f(X_1) |)'$. It is direct that

$$| f(X_1) | \, Z^T f_0(X_1,\delta,0) = f(X_1)\wedge f_0(X_1,\delta,0).$$

Let $\rho(\theta,\delta,\lambda)$ be the solution of (1.18) satisfying $\rho(t_2,\delta,\lambda)=-a$. Then from (1.15)–(1.17) we have

$$F_2(a,\delta,\lambda) = -\rho(t_4,\delta,\lambda) = I_1a - I_2(\delta)\lambda + O(| a,\lambda |^2), \tag{1.19}$$

where

$$I_1 = \frac{| f(A_2) |}{| f(A_4) |}, \qquad I_2(\delta) = \frac{1}{| f(A_4) |} \int_{t_2}^{t_4} f(X_1) \wedge f_0(X_1, \delta, 0) d\theta. \tag{1.20}$$

In the same way, we have

$$G_1(a, \delta, \lambda) = J_1 a - J_2(\delta)\lambda + O(| a, \lambda |^2), \tag{1.21}$$

where

$$J_1 = \frac{| f(A_1) |}{| f(A_3) |}, \qquad J_2(\delta) = \frac{1}{| f(A_3) |} \int_{t_1}^{t_3} f(X_2) \wedge f_0(X_2, \delta, 0) d\theta. \tag{1.22}$$

From (1.19) and Lemma 1.2 it follows that

$$\begin{aligned}(F_2 \circ F_1)(u, \delta, \lambda) &= a_1 + a_2 r + a_2 \widetilde{C}_{11}[r\omega + \cdots] + \cdots + a_2 \widetilde{C}_{n1} \varepsilon^{n-1}[r^n \omega_1 + \cdots] \\ &\quad + \widetilde{\psi}_{n1} + r^2 \widetilde{p}_{n1} \equiv \widetilde{F}(r, \delta, \lambda),\end{aligned} \tag{1.23}$$

where

$$\begin{aligned}&u = N_1 + M_1 r + O(r^2), \quad a_1 = I_1 N_2 - I_2 \lambda + O(\lambda^2), \\ &a_2 = I_1 M_2 (1 + O(\lambda)), \qquad \widetilde{C}_{i1} = \widetilde{C}_{i1}(1 + O(\lambda)), \quad i = 1, \cdots, n,\end{aligned} \tag{1.24}$$

and $\widetilde{\psi}_{n1}$ is n-flat at $r = 0, \widetilde{p}_{n1}$ is a polynomial in r. From (1.11) we have

$$\begin{aligned}(G_2 \circ G_1)(u, \delta, \lambda) &= N_4 + M_4 \widetilde{r} + M_4 C_{12}[\widetilde{r}\omega_2 + \cdots] + \cdots + M_4 C_{n2}[\widetilde{r}^n \omega_2 + \cdots] \\ &\quad + \widetilde{\psi}_{n2} + \widetilde{r}^2 \widetilde{\psi}_{n2} \equiv \widetilde{G}(\widetilde{r}, \delta, \lambda),\end{aligned} \tag{1.25}$$

where, from (1.12), $\widetilde{r}$ satisfies $G_1(u, \delta, \lambda) = N_3 + M_3 \widetilde{r} + O(\widetilde{r}^2)$. Hence from (1.21) and (1.24) we obtain

$$\begin{aligned}\widetilde{r} &= \frac{1}{M_3}(G_1(u, \delta, \lambda) - N_3)(1 + O(| G_1 - N_3 |)) \\ &= b_1(\delta, \lambda) + b_2(\delta, \lambda) r + O(r^2),\end{aligned} \tag{1.26}$$

where

$$b_1 = \frac{1}{M_3}(J_1 N_1 - J_2 \lambda - N_3) + O(\lambda^2), \qquad b_2 = J_1 M_1 / M_3. \tag{1.27}$$

Then we can give the following normal form F_0 from (1.23), (1.25) and (1.26):

$$F_0(u, \delta, \lambda) = \overline{F_0}(r, \delta, \lambda) = \widetilde{F}(r, \delta, \lambda) - \widetilde{G}(\widetilde{r}, \delta, \lambda) = \overline{F}(r, \delta, \lambda) - \overline{G}(\widetilde{r}, \delta, \lambda), \tag{1.28}$$

where $u = W_1(r, \delta, \lambda)$, and

$$\begin{aligned}\overline{G}(\widetilde{r}, \delta, \lambda) &= \widetilde{G}(\widetilde{r}, \delta, \lambda) - N_4 - M_4 \widetilde{r} - \widetilde{r}^2 \widetilde{p}_{n2} \\ &= M_4 C_{12}[\widetilde{r}\omega_2 + \cdots] + \cdots + M_4 C_{n2} \varepsilon^{n-1}[\widetilde{r}^n \omega_2 + \cdots] + \widetilde{\psi}_{n2},\end{aligned} \tag{1.29}$$

$$\begin{aligned}\overline{F}(r, \delta, \lambda) &= \widetilde{F}(r, \delta, \lambda) - N_4 - M_4 \widetilde{r} - \widetilde{r}^2 \widetilde{\psi}_{n2} \\ &= \overline{a}_1 + \overline{a}_2 r + a_2 \widetilde{C}_{11}[r\omega_1 + \cdots] + \cdots + a_2 \widetilde{C}_{n1} \varepsilon^{n-1}[r^n \omega_1 + \cdots] \\ &\quad + \widetilde{\psi}_{n1} + r^2 \widetilde{p}_{n1},\end{aligned} \tag{1.30}$$

$$\overline{a}_1 = a_1 - N_4 - M_4 b_1 + O(\lambda^2), \qquad \overline{a}_2 = a_2 - M_4 b_2 + O(\lambda). \tag{1.31}$$

For $| \lambda |$ small, (1.1) has a periodic orbit near L if and only if $\overline{F}_0$ has a root $r > 0$ with $\widetilde{r} > 0$, and has a homoclinic or heteroclinic loop if and only if $\overline{F}_0$ has a root $r \geq 0$ with

$r\widetilde{r}=0$ such that $r+\widetilde{r}>0$ (homoclinic case) or $r=\widetilde{r}=0$ (heteroclinic case). From [2] we have

$$\begin{aligned} a_1 - N_4 &= -\frac{\overline{M}_1(\delta)}{\mid f(A_4)\mid}\lambda + O(\lambda^2), \\ G_1(N_1,\delta,\lambda) - N_3 &= \frac{\overline{M}_2(\delta)}{\mid f(A_3)\mid}\lambda + O(\lambda^2), \end{aligned} \tag{1.32}$$

where

$$\overline{M}_i(\delta) = \int_{-\infty}^{\infty} f(X_1)\wedge f_0(X_1,\delta,0)dt, \qquad i=1,2. \tag{1.33}$$

From (1.21) and (1.27),

$$M_3 b_1 = \frac{\overline{M}_2(\delta)}{\mid f(A_3)\mid}\lambda + O(\lambda^2). \tag{1.34}$$

§2. Bifurcations Near the Heteroclinic Cycle

Let $A^* = A_4 + (G_2\circ G_1)(u,\delta,\lambda)n_4,\ B^* = A_4 + (F_2\circ F_1)(u,\delta,\lambda)n_4$, where $u = W_1(r,\delta,\lambda)$. We have

$$H(B^*) - H(A^*) = \lambda\oint_{L_h} f\wedge f_0(x,\delta,0)dt + O(\lambda^2),$$

where $h=h(r),\ L_h\subset\{H=h\}$, and $L_h = L$ for $r=0$. On the other hand, the mean value theorem implies that

$$\begin{aligned} H(B^*) - H(A^*) &= DH(A^*)(B^*-A^*) + O(\mid B^*-A^*\mid^2) \\ &= -\mid f(A_4)\mid \overline{F}_0(r,\delta,\lambda)(1+o(1)). \end{aligned}$$

Then we can write

$$\overline{F}_0(r,\delta,\lambda) = \lambda F_0^*(r,\delta,\lambda), \tag{2.1}$$

where

$$F_0^*(0,\delta,0) = -\frac{1}{\mid f(A_4)\mid}\sum_{i=1}^{2}\oint_{L_i} f\wedge f_0(x,\delta,0)dt \equiv -\frac{1}{\mid f(A_4)\mid}[\overline{M}_1(\delta)+\overline{M}_2(\delta)]. \tag{2.2}$$

Since (1.1) is Hamiltonian for $\lambda=0$, we can suppose

$$\begin{aligned} &C_{i2} = \lambda C_{i2}^*(\delta,\lambda), \quad \widetilde{C}_{i1} = \lambda C_{i1}^*(\delta,\lambda), \quad i=1,\cdots,n, \\ &\overline{a}_i = \lambda a_i^*(\delta,\lambda), \qquad \widetilde{\psi}_{ni} = \lambda\psi_{ni}^*, \qquad\quad i=1,2. \end{aligned} \tag{2.3}$$

It follows from (1.29)–(1.30) and (2.1) that

$$F_0^*(r,\delta,\lambda) = F(r,\delta,\lambda) - G(\widetilde{r},\delta,\lambda), \tag{2.4}$$

where

$$G(\widetilde{r},\delta,\lambda) = M_4C_{12}^*[\widetilde{r}\omega_2+\cdots]+\cdots+M_4C_{n2}^*\varepsilon^{n-1}[\widetilde{r}^n\omega_2+\cdots]+\psi_{n2}^*, \tag{2.5}$$

$$F(r,\delta,\lambda) = a_1^* + a_2^*r + a_2C_{11}^*[r\omega_1+\cdots]+\cdots+a_2C_{n1}^*\varepsilon^{n-1}[r^n\omega_1+\cdots]+\psi_{n1}^*+r^2p_{n1}^*. \tag{2.6}$$

It is direct from (1.26), (2.2) and (2.4)–(2.7) that

$$a_1^*(\delta,0) = -\frac{1}{\mid f(A_4)\mid}(\overline{M}_1(\delta)+\overline{M}_2(\delta)). \tag{2.7}$$

Hence we know from (2.1), (2.4)–(2.7) that a necessary condition for L to generate a cycle is $\overline{M}_1(\delta_0)+\overline{M}_2(\delta_0)=0$ for some $\delta_0\in R^m$. Hence, we make the following assumption

$$\overline{M}_1(\delta_0)+\overline{M}_2(\delta_0)=0,\quad C^*_{1j}(\delta_0,0)=(C_{1j})'_\lambda(\delta_0,0)\neq 0,\quad j=1,2. \tag{2.8}$$

From (1.26), there is a unique function given by

$$r=-\frac{b_1}{b_2}(1+O(b_1))\equiv r_0(\delta,\lambda) \tag{2.9}$$

such that

$$\widetilde{r}\geq 0 \text{ if and only if } r\geq r_0(\delta,\lambda). \tag{2.10}$$

Notice that from (2.4), (2.6) and (2.10)

$$F^*_0\mid_{\widetilde{r}=0}=F(r_0(\delta,\lambda),\delta,\lambda),\quad F^*_0\mid_{r=0}=a^*-G(b_1,\delta,\lambda).$$

If $b_1<0$, the homoclinic loop at S_1 is formed if and only if

$$F(r_0(\delta,\lambda),\delta,\varepsilon)=0. \tag{2.11}$$

If $b_1>0$, the homoclinic loop at S_2 is formed if and only if $a^*_1-G(b,\delta,\lambda)=0$. And from (1.31), (2.3), (1.32) and (1.34), the heteroclinic cycle is formed if and only if

$$a_1-N_4=0,\quad M_3b_1=0. \tag{2.12}$$

Consider the following two curves

$$K_1: y=F(r,\delta,\lambda) \text{ and } K_2: y=G(\widetilde{r},\delta,\lambda). \tag{2.13}$$

It is easy to see that the curves K_1 and K_2 has at most one intersection point if $C^*_{11}C^*_{12}<0$, which implies the uniquenss of limit cycles near L (see [3]). Hence, we may suppose

$$C^*_{11}C^*_{12}>0,\quad C_{11}>0,\quad \gamma_0=\frac{C^*_{11}(\delta_0,0)}{C^*_{12}(\delta_0,0)}. \tag{2.14}$$

From (2.5), (1.36) and [4,5], we have

$$\begin{aligned} F'_r&=a_2C^*_{11}\omega_1(1+(o(1)), & F''_r&=-a_2C^*_{11}\varepsilon^{C_{11}}r^{-1-C_{11}}(1+o(1)),\\ G'_r&=M_4b_2C^*_{12}\omega_2(1+o(1)), & G''_r&=-M_4b_2^2C^*_{12}\varepsilon^{C_{12}}\widetilde{r}^{-1-C_{12}}(1+o(1)). \end{aligned} \tag{2.15}$$

Lemma 2.1. *Let* (2.8) *and* (2.14) *hold. If* $\gamma_0\neq 1$, *for any small number* $\varepsilon_1>0$, *there is* $0<\varepsilon_0<\varepsilon$ *such that*

$$C^*_{11}(1-\gamma_0)(F-G)<0 \quad \textit{for } \varepsilon_0\leq r\leq\varepsilon_1,\quad |\delta-\delta_0|\leq\varepsilon_0,\quad |\lambda|\leq\varepsilon_0.$$

Proof. It suffices to prove that $C^*_{11}(1-\gamma_0)(F-G)<0$ for $\delta=\delta_0,\ \lambda=0$ and $0<r\ll 1$. In fact, from (1.31), (2.3), (2.5)-(2.8), we have

$$a_2=M_4b_2,\ \frac{F}{G}=\gamma_0(1+o(1))\qquad \text{for } \lambda=0 \text{ and } |\delta-\delta_0|\ll 1. \tag{2.16}$$

Then the conclusion follows.

Let

$$u(\delta,\lambda)=\begin{cases} F(r_0,\delta,\lambda), & \text{for } b_1\leq 0,\\ a^*_1-G(b_1,\delta,\lambda), & \text{for } b_1\geq 0.\end{cases} \tag{2.17}$$

Theorem 2.1. *Let* (2.8) *and* (2.14) *hold, and let* $\gamma_0\neq 1$ *and* $b_1(1-\gamma_0)\leq 0$. *Then* (1.1) *has at most one limit cycle near L for* $|\delta-\delta_0|$ *and* $|\lambda|>0$ *small. More precisely,* (1.1) *has a unique limit cycle if and only if* $C^*_{11}(1-\gamma_0)u(\delta,\lambda)>0$, *and has a homoclinic loop* (*heteroclinic loop*) *if and only if* $u(\delta,\lambda)=0,\ b_1\neq 0,\ u(\delta,\lambda)=b_1=0$.

Proof. We suppose $b_1 \le 0, \gamma_0 < 1$. It follows that $\widetilde{r} \le b_2 r + O(r^2)$. Then from (2.15),

$$\frac{F_r^{"}}{G_r^{"}} \le \frac{a_2 C_{11}^*}{M_4 b_2^2 C_{12}^*}(b_2 + O(r))^{1+C_{12}} r^{C_{12}-C_{11}}.$$

Note that $r^{C_{12}-C_{11}} < 1$ from (2.14). We have

$$\frac{F_r^{"}}{G_r^{"}} \le \gamma_0(1 + o(1)) < 1 \quad \text{for } 0 < r \ll 1. \tag{2.18}$$

Therefore, the curves K_1 and K_2 in (2.13) have at most two intersection points for $r \ge 0$. In the case of $b_1 < 0$, we have

$$G = 0, \ 0 < C_{11}^* F_r' < \infty, \quad C_{11}^* G_r' = +\infty \quad \text{for } \ r = r_0.$$

It implies from (2.18) and Lemma 2.1 that K_1 and K_2 have no point (a unique point (r^*, y^*)) in common for $r \ge r_0$ when $uC_{11}^* < 0 (\ge 0)$. Moreover, $r^* > (=) r_0$ if $u \ne (=) 0$. Hence, noting (2.11), the conclusion follows. In the case of $b_1 = 0$, we can prove that $F_r'/G_r' \to 0$ as $r \to 0$, and the conclusion follows in the same way.

Lemma 2.2. *Let* (2.8) *and* (2.14) *hold. If* $\gamma_0 \ne 1$, *then the function* $(F - G)_r^{"}$ *has at most one root with respect to* r *for* $| \delta - \delta_0 |, | \lambda |, r > 0$ *and* $\widetilde{r} > 0$ *all small.*

Proof. From (2.5), (2.6) and [4,5], we have

$$\frac{F_r^{"}}{G_r^{"}} = \varepsilon^{C_{12}-C_{11}} \frac{a_2 C_{11}^* r^{-1-C_{11}}(A(\lambda) + g_1(r))}{M_4 b_2^2 C_{12}^* \widetilde{r}^{-1-C_{12}}(B(\lambda) + g_2(\widetilde{r}))},$$

where

$$A(0) = B(0) = 1, \quad g_1(0) = g_2(0) = 0, \quad r g_1'(r) = o(1), \quad \widetilde{r} g_2'(\widetilde{r}) = o(1). \tag{2.19}$$

Hence, $F_r^{"} - G_r^{"} = 0$ if and only if

$$g(r) = \Big(\frac{a_2 C_{11}^*}{M_4 b_2^2 C_{12}^*} \varepsilon^{C_{12}-C11}\Big)^{\frac{1}{1+C_{11}}} \widetilde{r}^{\frac{1+C_{12}}{1+C_{11}}} (B + g_2)^{-\frac{1}{1+C_{11}}} - r(A + g_1)^{-\frac{1}{1+C_{11}}} = 0.$$

Notice that $\widetilde{r}_r' = b_2 + O(r)$. By (2.19), we have

$$g'(r) = \gamma_0 \widetilde{r}^{\frac{C_{12}-C_{11}}{1+C_{11}}}(1 + \widetilde{g}_2(\lambda, \widetilde{r})) - (1 + \widetilde{g}_1(\lambda, r)),$$

where $\widetilde{g}_i = o(1), \ i = 1, 2$. It gives that $g'(r) < 0 (> 0)$ if $\gamma_0 < 1 (> 1)$ since $\widetilde{r}^{\frac{C_{12}-C_{11}}{1+C_{11}}} < 1 (> 1)$ if $\gamma_0 < 1 (> 1)$.

This finishes the proof.

Theorem 2.2. *Let* (2.8) *and* (2.14) *hold,* $\gamma_0 \ne 1, b_1(1 - \gamma_0) > 0$. *Then there exists* $u^*(\delta, \lambda)$ *satisfying* $C_{11}^*(1 - \gamma_0)(u^* - u(\delta, \lambda)) < 0$ *such that for* $| \delta - \delta_0 |$ *and* $| \lambda | > 0$ *small and in a neighbourhood of* L (1.1) *has*

(i) *a unique limit cycle if* $C_{11}^*(1 - \gamma_0)u > 0$*;*

(ii) *a unique homoclinic loop and a unique limit cycle if* $u = 0$*;*

(iii) *exactly two limit cycles if* $C_{11}^*(1 - \gamma_0)u^* < C_{11}^*(1 - \gamma_0)u < 0$*;*

(iv) *a unique semistable limit cycle if* $u^* = u$*;*

(v) *no cycles if* $C_{11}^*(1 - \gamma_0)u < C_{11}^*(1 - \gamma_0)u$.

Proof. Without loss of generality, we suppose that $\gamma_0 < 1, \ C_{11}^* > 0$ and $b_1 > 0$. Then from (2.17), if $u > 0$, we have $F > G$ for $r = 0$. Notice that $C_{11}^* F_r' = +\infty, \ 0 < C_{11}^* G_r' < \infty$ at $r = 0$. By Lemmas 2.1 and 2.2, the curves K_1 and K_2 have a unique intersection point for $r > 0$. When $u = 0$, the end point $(r, y) = (0, 0)$ of K_1 and K_2 is also a intersection point.

For $u < 0$, let $h(r, a_1^*, b_1, \delta, \lambda) = F - G$. We have $\frac{\partial h}{\partial a_1^*} = 1$. From the above discussion, when $a_1^* = G(b_1, \delta, \lambda)$, K_1 and K_2 have exactly two intersection points (including the end point) for $r \geq 0$. By Lemma 2.2 we can prove that when $0 < -h(0, a_1^*, b_1, \delta, \lambda) \ll 1$, K_1 and K_2 have exactly two intersection points $P_i(r_i, y_i)$ with $r_2 > r_1$. Obviously

$$(-1)^i (F - G)'_r \mid_{r=r_i} < 0, \qquad i = 1, 2.$$

It follows that there exists $A^*(b_1, \delta, \lambda)$ such that $P_1 \neq P_2 (P_1 = P_2 \equiv P)$ if $a_1^* > A^*(a_1^* = A^*)$, and P disappears if $a_1^* < A^*$. Then the theorem follows by letting $u^* = A^*(b_1, \delta, \lambda) - G(b_1, \delta, \lambda)$. The proof is completed.

Remark 2.1. Suppose that $\delta \in r$, $\overline{M}_1(\delta_0) \neq 0$, and $(\overline{M}_1 + \overline{M}_2)'(\delta_0) \neq 0$. From (2.7) and (2.17), we can solve $\delta = \delta_1^*(\lambda) = \delta_0 + o(1)$ from $u(\delta, \lambda) = 0$, and $\delta = \delta_2^*(\lambda) = \delta_0 + o(1)$ from $u^*(\delta, \lambda) = u(\delta, \lambda)$. They correspond to homoclinic loop and semistable cycle bifurations. If $\delta \in R^2$, $\overline{M}_i(\delta_0) = 0$, $i = 1, 2$, and $\det \frac{\partial}{\partial \delta}(\overline{M}_1, \overline{M}_2) \neq 0$ at $\delta = \delta_0$, then equation (2.12) has a unique solution $\delta = \widetilde{\delta}(\lambda) = \delta_0 + O(\lambda)$ which gives the heteroclinic loop bifurcation. We can also give bifurcation surfaces for two homoclinic loops and the semistable cycle.

The following is immediate from Theorems 2.1 and 2.2.

Theorem 2.3. *Let* (2.8) *and* (2.14) *hold. If* $\gamma_0 \neq 1$, (1.1) *has at most two limit cycles near* L *for* $| \delta - \delta_0 |$ *and* $| \lambda | > 0$ *small.*

Remark 2.2. Recently Han Maoan and Zhng Zhifen prove that Theorem 2.3 is true if $C_{11}^*(\delta_0, 0) \neq C_{12}^*(\delta_0, 0)$.

References

[1] Han Maoan, On the number of limit cycles bifurcated from a homoclinic or heteroclinc loop, *Science in China* (*Series A*), **2**(1993), 113-122.

[2] Han Maoan & Zhu Deming, Bifurcation theory of differential equations, Coal Mine Industry Publishing House, Beijing, 1994.

[3] Han Maoan, Luo Dingjun & Zhu Deming, The uniqueness of limit cycles bifurcated from a separatrix cycle (III), *Acta Math. Sinica,* **5**(1992), 673–684.

[4] Dumortier, F., Roussarie, R. & Rousseau, C., Elementary graphics of cyclicity 1 and 2, *Nonlinearity,* **3**(1994), 1001–1044.

[5] Roussarie, R., On the number of limit cycles which appear by perturbation of separatrix loop of planar vector fields, *Bol. Soc. Brasil Mat.,* **17**(1986), 67–101.