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SOLVABILITY TO THE PICARD BOUNDARY VALUE PROBLEM FOR NONLINEAR DIFFERENTIAL SYSTEMS WITH IMPULSES

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Abstract

The author introduces a concept of curvature bound set relative to second order impulsive differential systems and based on this concept discusses the existence of solutions to the Picard boundary value problem of the systems. Compared with the previous works finished by Lakshmikantham and Erbe, the author's results do not require the right-handed function and impulsive functions with special structures such as monotonicity, etc. When the impulsive effects are absent, these results could be viewed as new generalized forms of the two so-called optimal results about second order scalar differential equations derived by Lees and Mawhin respectively.

Keywords Picard boundary value problem, Solvability, Differential system with impulses, Curvature bound set

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§1. Introduction

This paper is devoted to the study of the existence of solutions for the problem

$$x'' = f(t, x, x'), \quad t \neq t_i, \quad t \in (a, b), \tag{1.1}$$

$$\Delta x(t_i) = I_i(x(t_i - 0)), \quad i = 1, 2, \cdots, k, \tag{1.2}$$

$$\Delta x'(t_i) = J_i(x'(t_i - 0)), \quad i = 1, 2, \cdots, k, \tag{1.3}$$

$$x(a) = 0, \quad x(b) = 0, \tag{1.4}$$

where $a = t_0 < t_1 < \cdots < t_{k+1} = b$ are fixed numbers, k is a positive integer; $f(t, x, y) = f_i(t, x, y)$ for $t \in (t_{i-1}, t_i)$, $x, y \in R^n$, and $f_i : [t_{i-1}, t_i] \times R^n \times R^n \to R^n$ are continuous, $i = 1, 2, \cdots, k + 1$;

$$\Delta x(t_i) = x(t_i + 0) - x(t_i - 0), \quad \Delta x'(t_i) = x'(t_i + 0) - x'(t_i - 0),$$

$I_i, J_i : R^n \to R^n$ are continuous for $i = 1, 2, \cdots, k$.

To study the existence of solutions, we will turn the boundary value problem (1.1)–(1.4) into an equivalent operator equation

$$x = Tx, \tag{1.5}$$

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where

$$Tx = \int_a^b G(s,t)f(x,x(s),x'(s))ds + \sum_{t>t_i} I_i(x(t_i)) - \frac{t-a}{b-a}\sum_{i=1}^{k} I_i(x(t_i))$$
$$+\int_a^b \sum_{\tau>t_i} J_i(x'(t_i))d\tau - \frac{t-a}{b-a}\int_a^b \sum_{\tau>t_i} J_i(x'(t_i))d\tau \tag{1.6}$$

and

$$G(s,t) = \frac{(s-a)(t-b)}{b-a}, \quad a \le s < t \le b,$$
$$= \frac{(s-b)(t-a)}{b-a}, \quad a \le t < s \le b. \tag{1.7}$$

The following result is helpful in discussing the operator equation (1.5), which can be found in [1].

Lemma 1.1.[1, Theorem 1] *Let X be a Banach space, $H : X \to X$ be a compact map such that $I - H$ is one to one,and Ω an open bounded subset of X such that $0 \in (I - H)(\Omega)$. Then the compact map $T : \bar{\Omega} \to X$ has a fixed piont in $\bar{\Omega}$ if for any $\lambda \in (0, 1)$, the equation*

$$x = \lambda Tx + (1-\lambda)Hx \tag{1.8}$$

has no solution x on the boundary $\partial\Omega$ of Ω.

This lemma will be used to prove our main existence result, namely, Theorem 2.1 in Section 2. In this section we will also give the definition of curvature bound set relative to (1.1)–(1.3), which is a generalized version of [2, Definition V.2], and use it in the hypotheses of Theorem 2.1. Readers will find that all the results of this paper can be unified in this concept. As applications of our main result, we generalize [1, Theorem 2] and [2, Theorem V.25] to the impulsive case (1.1)–(1.4) and obtain Theorems 3.1 and 3.2. From Theorem 3.1 we derive Corollaries 3.1 and 3.2, which are similar to the main results of [3, 4] obtained by S. Hu and V. Lakshmikantham , L. H. Erbe and X. Liu by use of the upper and lower solution method,but any forms of monotonicity of $f, I_i, J_i,\ i = 1, 2, \cdots, k$, which are necessary in the papers, are not required here. In the last section we will give more concrete conditions, under which the bound function ϕ of Corollary 3.1 will be constructed, and derive Theorems 4.1 and 4.2, which improve two so-called "optimal" results obtained by Gaines and Mawhin[2, Theorem V15] and Lees[5] respectively, even when the impulsive effects are absent, i.e., $I_i(x) \equiv J_i(x) \equiv 0$ for $i = 1, 2, \cdots, k$. As the last results we get from Theorem 4.2 Corollaries 4.1 and 4.2 associated with the solvability of (1.1) and (1.4), in which the hypotheses are the ones breaking some restrictions of [6] for the scalar case $n = 1$.

Throughout this paper, each set we introduce is always meant to be nonempty. And a map $x : [a, b] \to R^n$ is said to be a solution of (1.1)–(1.4) if $x(t)$ is twice differentiable for $t \ne t_i$, $x'(t_i+0)$ and $x'(t_i-0)$ exist for $i = 1, 2, \cdots, k$, and $x(t)$ satisfies relations (1.1)–(1.4). At this time we always denote $x(t_i) = x(t_i - 0), x'(t_i) = x'(t_i - 0)$ for $i = 1, 2, \cdots, k$.

§2. Curvature Bound Set and Main Result

In this section we will first enlarge the definition of curvature bound set relative to (1.1) to the impulsive equations (1.1)–(1.3) and then make use of it to discuss the solvability of (1.1)–(1.4).

Definition 2.1. *If $G \subset [a,b] \times R^n$ is open in the relative topology on $[a,b] \times R^n$ and bounded, we will call G a curvature bound set relative to (1.1)–(1.3) if for any $(t_0, x_0) \in \partial G$ with $t_0 \in (a,b)$, there exists a continuous function $V(t,x) = V(t_0, x_0; t, x)$ such that*

(i) *$V(t,x)$ is twice differentiable for $t \neq t_i, i = 1, 2, \cdots, k,\quad x \in R^n$.*

(ii) *$G \subset \{(t,x) : V(t,x) < 0\}$ and $V(t_0, x_0) = 0$.*

(iii) *We denote $\bar{t}_o = t_0$ as $t_0 \neq t_i, i = 1, 2, \cdots, k$ and $\bar{t}_0 \in \{t_0, t_0 + 0\}$ as $t_0 \in \{t_i\}_{i=1}^k$. Then*

$$H(\bar{t}_0, x_0)\begin{pmatrix}1\\ y\end{pmatrix} \cdot \begin{pmatrix}1\\ y\end{pmatrix} + \mathrm{grad}V(\bar{t}_0, x_0) \cdot \begin{pmatrix}0\\ f(\bar{t}_0, x_0, y)\end{pmatrix} > 0$$

for all y such that

$$\mathrm{grad}V(\bar{t}_0, x_0) \cdot \begin{pmatrix}1\\ y\end{pmatrix} = 0,$$

where $H(t,x)$ denotes the Hessian matrix of V at (t,x) and $\mathrm{grad}V(t,x)$ *denotes the gradient of V at (t,x). Moreover,*

$$H(t_i, x) = H(t_i - 0, x), \quad \mathrm{grad}V(t_i, x) = \mathrm{grad}V(t_i - 0, x), \quad i = 1, 2, \cdots, k.$$

(iv) *If $t_0 = t_{i_0}$ for some $i_0 \in \{1, 2, \cdots, k\}$, then*

$$V(t_{i_0}, x_0 + I_{i_0}(x_0)) \geq 0, \quad \text{and} \quad \mathrm{grad}V(t_{i_0}, x_0 + I_{i_0}(x_0)) \cdot \begin{pmatrix}1\\ y + J_{i_0}(y)\end{pmatrix} > 0$$

$$\text{if} \quad \mathrm{grad}V(t_0, x_0) \cdot \begin{pmatrix}1\\ y\end{pmatrix} > 0;$$

$$V(t_{i_0}, \bar{x}_0) \geq 0, \quad \mathrm{grad}V(t_{i_0}, \bar{x}_0) \cdot \begin{pmatrix}1\\ y\end{pmatrix} < 0$$

$$\text{if } \bar{x}_0 + I_{i_0}(\bar{x}_0) = x_0, \quad \bar{y} + J_{i_0}(\bar{y}) = y \quad \text{and} \quad \mathrm{grad}V(t_{i_0}, x_0) \cdot \begin{pmatrix}1\\ y\end{pmatrix} < 0.$$

Lemma 2.1. *Let G be a curvature bound set relative to (1.1)–(1.3). If $x(t)$ is a solution to (1.1)–(1.3) on $[a,b]$ with $(a, x(a)), (b, x(b)) \in G$ and $(t, x(t)) \in \bar{G}$ for $t \in [a,b]$, then $(t, x(t)) \in G$ for $t \in [a,b]$ and $(t_i, x(t_i + 0)) \in G$ for $i = 1, 2, \cdots, k$.*

Proof. If not, there are two cases:

(a) $(\bar{t}, x(\bar{t})) \in \partial G$ for some $\bar{t} \in (a,b)$.

(b) $(t_{i_o}, x(t_{i_0} + 0)) \in \partial G$ for some $i_0 \in \{1, 2, \cdots, k\}$.

For the first case, let $\phi(t) = V(t, x(t))$, where V is as in Definition 2.1. Obviously, we have $\phi(t) \leq 0$ for $t \in [a,b]$ and $\phi(\bar{t}) = 0$. Thus

$$0 \leq \phi'(\bar{t}) = \mathrm{grad}V(\bar{t}, x(\bar{t})) \cdot \begin{pmatrix}1\\ x(\bar{t})\end{pmatrix}.$$

If $\phi'(\bar{t}) = 0$, then $\phi(t)$ reaches the local maximum value 0 at $t = \bar{t}$ on a left neighbourhood of $\bar{t}$, and hence

$$\begin{aligned} 0 \geq \phi''(\bar{t} - 0) = {} & H(\bar{t}, x(\bar{t}))\begin{pmatrix}1\\ x'(\bar{t})\end{pmatrix} \cdot \begin{pmatrix}1\\ x'(\bar{t})\end{pmatrix} \\ & + \mathrm{grad}V(\bar{t}, x(\bar{t})) \cdot \begin{pmatrix}0\\ f(\bar{t}, x(\bar{t}), x'(\bar{t}))\end{pmatrix}, \end{aligned}$$

which contradicts condition (iii) of Definition 2.1. If $0 < \phi'(\bar{t})$, then $\bar{t} = t_{i_0}$ for some

$i_0 \in \{1, 2, \cdots, k\}$, and from condition (iv) of Definition 2.1, on the one hand we have

$$\mathrm{grad}V(\bar{t}, x(\bar{t}) + I_{i_0}(x(\bar{t}))) \cdot \begin{pmatrix} 1 \\ x'(\bar{t}) + J_{i_0}(x'(\bar{t})) \end{pmatrix} > 0,$$

on the other hand it follows that $\phi(\bar{t}+0) = 0$, and

$$0 \geq \phi'(\bar{t}+0) = \mathrm{grad}V(\bar{t}, x(\bar{t}) + I_{i_0}(x(\bar{t}))) \cdot \begin{pmatrix} 1 \\ x'(\bar{t}) + J_{i_o}(x'(\bar{t})) \end{pmatrix},$$

a contradiction. Thus case (a) can not occur.

Similarly we can show that case (b) will also lead to a contradiction. Hence, the proof is finished.

Definition 2.2. *For a bounded subset $G \subseteq [a, b] \times R^n$, we will say that $f : [a, b] \times R^n \times R^n \to R^n$ satisfies a Nagumo's condition with respect to G if*

$$| f_j(t, x, y) | \leq h(| y_i |), \quad j = 1, 2, \cdots, n, \tag{2.1}$$

for $(t, x) \in \overline{G}$, $y \in R^n$, where h is a positive continuous function on $[0, +\infty)$ and

$$\int_0^{+\infty} \frac{sds}{h(s)} = +\infty, \tag{2.2}$$

or if

$$| f(t, x, y) | \leq \psi(| y |)$$

for $(t, x) \in \overline{G}$, $y \in R^n$, where ψ is a positive continuous and nondecreasing function on $[0, +\infty)$ such that

$$\lim_{s \to +\infty} \frac{s^2}{\psi(s)} = +\infty.$$

Lemma 2.2.[7, Lemma 4] *Suppose that $G \subset [a, b] \times R^n$ is bounded and $f(t, x, y) = f_i(t, x, y)$ for $t \in (t_{i-1}, t_i)$, $x, y \in R^n$, $f_i : [t_{i-1}, t_i] \times R^n \times R^n \to R^n$ are continuous, $i = 1, 2, \cdots, k+1$, f satisfies a Nagumo's condition with respect to G. Then there exists a positive number M dependent only on G and h or ψ such that $| x'(t) | \leq M$ for $t \in [a, b]$ if $x(t)$ is a solution of (1.1)–(1.3) with $(t, x(t)) \in \overline{G}$ for $t \in [a, b]$.*

We now can state our main result.

Theorem 2.1. *Suppose*

(a) *G is a curvature bound set relative to $(2.5)_\lambda$–$(2.7)_\lambda$ for $\lambda \in (0, 1)$ with $(t, 0) \in G$ for $t \in [a, b]$ and $c \geq 0$ is fixed, where*

$$x'' = \lambda f(t, x, x') + (1 - \lambda)cx, \quad t \neq t_i, \quad t \in (a, b), \tag{2.5$_\lambda$}$$

$$\Delta x(t_i) = \lambda I_i(x(t_i - 0)), \quad i = 1, 2, \cdots, k, \tag{2.6$_\lambda$}$$

$$\Delta x'(t_i) = \lambda J_i(x'(t_i - 0)), \quad i = 1, 2, \cdots, k. \tag{2.7$_\lambda$}$$

(b) *f satisfies a Nagumo's condition with respect to G.*

Then the problem (1.1)–(1.4) has at least one solution such that $(t, x(t)) \in \overline{G}$ for $t \in [a, b]$.

Proof. Denote

$$X = C^1[a, b; t_1, t_2, \cdots, t_k] = \{x : [a, b] \to R^n \mid x'(t) \text{ exists for } t \neq t_1, t_2, \cdots, t_k, \\ x'(t_i + 0), x'(t_i - 0) \text{ exist for } i = 1, 2, \cdots, k\}$$

with the norm

$$\| x \| = \max\Big\{ \sup_{t\in[a,b]} | x'(t) |, \quad \sup_{t\in[a,b]} | x(t) | \Big\}.$$

Then X is a Banach space. Let $H : X \to X$ be defined by

$$Hx = \int_a^b G(s,t)cx(s)ds \quad \text{for } x \in X,$$

and $T : X \to X$ by (1.6), where $G(s,t)$ is as (1.7). Then it is easily seen that both H and T are compact, $I - H$ is one to one because the unique solution of

$$\begin{aligned} x'' &= cx, \quad t \neq t_i, \quad i = 1,2,\cdots,k, \\ \Delta x(t_i) &= \Delta x'(t_i) = 0, \quad i = 1,2,\cdots,k, \\ x(a) &= x(b) = 0 \end{aligned}$$

is $x(t) \equiv 0$.

Since G is bounded, there exists $M_1 > 0$ such that if $(t,x) \in G$, then $| x | \leq M_1$ and $| x_i | \leq M_1$ for $i = 1,2,\cdots,n$. Let $\overline{h}(s) = h(s) + cM_1$, and $\overline{\psi}(s) = \psi(t) + cM_1$. Then for every $\lambda \in (0,1)$, $\lambda f(t,x,y) + (1-\lambda)cx$ satisfies a Nagumo's condition with respect to G with $\overline{h}$ and $\overline{\psi}$ instead of h and ψ in (2.1)–(2.4). By Lemma 2.2, there exists $\overline{M} > 0$ dependent only on G and $\overline{h}$ or $\overline{\psi}$ such that $| x'(t) | \leq \overline{M}$ for $t \in [a,b]$ if $x(t)$ is a solution of $(2.5)_\lambda$–$(2.7)_\lambda$ with $(t,x(t)) \in \overline{G}\, for\, t \in [a,b]$.

Let

$$\begin{aligned} \Omega = \{x \in X \mid (t,x(t)) \in G, \ | x'(t) | < \overline{M} + 1 \ \text{ for } \ t \in [a,b] \ \text{ and} \\ (t_i, x(t_i + 0)) \in G \ \text{ for } \ i = 1,2,\cdots,k\}. \end{aligned}$$

From condition (a) and (2.8) we have $0 \in (I - H)(\Omega)$. Suppose that $x(t)$ is a solution of $(2.5)_\lambda$–$(2.7)_\lambda$ and (1.4) with $x \in \overline{\Omega}$. Then $(t,x(t)) \in \overline{G}$ for $t \in [a,b]$, and it follows that $| x'(t) | \leq \overline{M}$ for $t \in [a,b]$. By Lemma 2.1, to finish the proof, it is enough to show that $(t,x(t)) \in G$, $(t_i, x(t_i + 0)) \in G$ for $t \in [a,b]$, $i = 1,2,\cdots,k$. This follows from Lemma 2.2 because G is a curvature bound set relative to $(2.5)_\lambda$–$(2.7)_\lambda$. The proof is complete.

Remark. Condition (b) can be replaced by

(b)$'$ There exist numbers $\alpha \in [0,1)$ and $\beta \geq 0$ such that, for any (t,x,y) with $(t,x) \in \overline{G}, y \in R^n$, one has

$$-x \cdot f(t,x,y) \leq \alpha \mid y|^2 + \beta, \tag{2.9}$$

$$| y \cdot f(t,x,y) | \leq h(| y |) \mid y |, \tag{2.10}$$

where $h : [0,+\infty) \to (0,\infty)$ is increasing, continuous and satisfies

$$\int_0^{+\infty} \frac{s^2 ds}{h(s)} = +\infty$$

just as (3.5) and (3.6) in [1] if we assume that for any solution $x(t)$ of $(2.5)_\lambda$–$(2.7)_\lambda$ and (1.4) with $(t,x(t)) \in \overline{G}$ for $t \in [a,b]$, one has

$$\Big| \sum_{i=1}^{k} [x(t_i + 0) \cdot x'(t_i + 0) - x(t_i) \cdot x'(t_i)] \Big| \leq M,$$

where $M > 0$ is a fixed number dependent only on α, β, h and G. In fact, as Lemma 2.2 we can derive the a priori bound as follows : Let $x(t)$ be a solution of $(2.5)_\lambda - (2.7)_\lambda$ and (1.4) with $(t, x(t)) \in \overline{G}$ for $t \in [a, b]$. Then

$$x(t) \cdot x''(t) = \lambda x(t) \cdot f(t, x(t), x'(t)) + (1-\lambda) c |x(t)|^2.$$

Integrating by parts with (1.4) and (2.9) we have

$$-\sum_{i=1}^{k} [x(t_i+0) \cdot x'(t_i+0) - x(t_i) \cdot x'(t_i)] - \int_a^b | x'(t)|^2 dt$$
$$\le -\lambda\beta(b-a) + \alpha\lambda \int_a^b | x'(t)|^2 dt + (1-\lambda) \int_a^b c \mid x(t)|^2 dt.$$

It follows that

$$\int_{t_i}^{t_i+1} | x'(t)|^2 dt \le \int_a^b | x'(t)|^2 dt \le [\beta(b-a) + M]/(1-\alpha) \overset{\triangle}{=} M_1 \quad \text{for} \quad i = 1, 2, \cdots, k,$$

Using [1, Lemma 1] we get

$$| x'(t) | \le \overline{M} \quad \text{for} \quad t \in [a, b],$$

where $\overline{M} > 0$ depends only on a, b, h and M_1.

§3 Extensions of A Theorem of Fabry-Habets and A Theorem of Gaines-Mawhin

In this section as applications of Theorem 2.1 we will enlarge [1, Theorem 1] and [2, Theorem V.25] to the impulsive case.

Theorem 3.1. *Assume that there exists a continuous function* $\phi : [a, b] \to (0, +\infty)$ *verifying*

(i) $\phi(t)$ *is twice differentiable for* $t \ne t_i$ *and* $\phi''(t_i+0)$, $\phi''(t_i-0)$ *exist for* $i = 1, 2, \cdots, k$.

(ii) *For any* (t, x, y) *such that* $t \in [a, b] \cup \{t_i+0\}_{i=1}^k$, $| x |= \phi(t)$, $x \cdot y =| x | \phi'(t)$, *one has*

$$-x \cdot f(t, x, y) \le -\phi(t)\phi''(t) + | y|^2 - \phi'(t)^2.$$

(iii) $| x + \lambda I_i(x) | -\phi(t_i) \ge 0$,

$$(x + \lambda I_i(x)) \cdot (y + \lambda J_i(y)) - \phi'(t_i+0)\phi(t_i) > 0 \quad \textit{for } \lambda \in (0, 1),$$
$$\textit{if} \quad | x | -\phi(t_i) = 0, \ \ x \cdot y - \phi(t_i)\phi'(t_i) > 0;$$
$$| \overline{x} | -\phi(t_i) \ge 0, \quad \overline{x} \cdot \overline{y} - \phi'(t_i)\phi(t_i) < 0,$$
$$\textit{if} \quad | x | -\phi(t_i) = 0, \quad x \cdot y - \phi(t_i)\phi'(t_i+0) < 0$$
$$\textit{and} \ \ \overline{x} + \lambda I_i(\overline{x}) = x, \quad \overline{y} + \lambda J_i(\overline{y}) = y \quad \textit{for some} \ \ \lambda \in (0, 1).$$

Assume moreover that

(iv) f *satisfies a Nagumo's condition with respect to* $\{(t, x) : t \in [a, b], | x |< \phi(t)\}$.

Then the problem (1.1)–(1.4) *has at least one solution* $x(t)$ *such that* $| x(t) | \le \phi(t)$ *for* $t \in [a, b]$.

Proof. Let $G = \{(t, x) : | x |< \phi(t)\}$. From Theorem 2.1, it is enough to show that for sufficiently large $c > 0$, G is a curvature bound set relative to $(2.5)_\lambda$–$(2.7)_\lambda$ for $\lambda \in (0, 1)$. To this end, for any $(t_0, x_0) \in \partial G$ with $t_0 \in (a, b)$, let $V(t, x) = \frac{1}{2} x \cdot x - \frac{1}{2}\phi(t)^2$; then conditions (i) and (ii) of Definition 2.1 are satisfied immediately. From assumption (iii) we have $V(t_i, x_0 + \lambda I_i(x_0)) \geq 0$, and

$$\mathrm{grad}V(t_i, x_0 + \lambda I_i(x_0)) \cdot \begin{pmatrix} 1 \\ y + J_i(y)\lambda \end{pmatrix} > 0 \quad \text{for } \lambda \in (0,1)$$

$$\text{if } \ V(t_i, x_0) = 0,\ \mathrm{grad}V(t_i, x_0) \cdot \begin{pmatrix} 1 \\ y \end{pmatrix} > 0;$$

$$V(t_i, \overline{x}_0) \geq 0, \quad \mathrm{grad}V(t_i, \overline{x}_0)\begin{pmatrix} 1 \\ \overline{y} \end{pmatrix} < 0 \ \text{ if } \ \overline{x}_0 + \lambda I_i(\overline{x}_0) = x_0, \quad \overline{y} + \lambda I_i(\overline{y}) = y$$

for some $\lambda \in (0,1)$ and $V(t_i, x) = 0$, $\mathrm{grad}V(t_i, x_0) \cdot \binom{1}{y} < 0$.

Letting $| x |= \phi(t_0)$, $x_0 \cdot y =| x | \phi'(t_0)$ and taking into account assumption (ii) we have

$$\begin{aligned} & H(t_0, x_0)\begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y \end{pmatrix} + \mathrm{grad}V(t_0, x_0) \cdot \begin{pmatrix} 0 \\ \lambda f(t_0, x_0, y) + (1-\lambda) c x_0 \cdot x_0 \end{pmatrix} \\ =& | y|^2 - \phi'(t_0)^2 - \phi(t_0)\phi''(t_0) + \lambda x_0 \cdot f(t_0, x_0, y) + (1-\lambda) c x_0 \cdot x_0 \\ \geq& (1-\lambda)[c x_0 \cdot x_0 - \phi'(t_0)^2 - \phi(t_0)\phi''(t_0)] > 0 \end{aligned}$$

if $c > 0$ is large enough. Thus conditions (iii) and (iv) are also valid. The proof is complete.

Corollary 3.1. *Assume that comditions* (i), (ii) *and* (iv) *are satisfied, and*

(iii)$'$

$$\begin{aligned} & I_i(x) = 0 \quad \text{if} \quad | x |= \phi(t_i); \\ & | x + I_i(x) |\leq \phi(t_i) \quad \text{if} \quad | x |< \phi(t_i); \\ & x \cdot (y + J_i(y)) > \phi'(t_i + 0) \quad \text{if} \ \ x \cdot y > \phi'(t_i) \end{aligned}$$

with $| x |= 1$ *and* $\phi'(t_i) \geq \phi'(t_i + 0)$.

Then the problem (1.1)–(1.4) *has at least one solution* $x(t)$ *such that* $| x(t) |\leq \phi(t)$ *for* $t \in [a, b]$.

Proof. It is easily seen that condition (iii) of Theorem 3.1 follows from assumption (iii) of Corollary 3.1. Thus the proof is finished.

Corollary 3.2. *Let* $n = 1$. *Assume that there exist functions* $\alpha(t)$, $\beta(t)$ *verifying*

(a) $\alpha, \beta : [a, b] \to R$ *are continuous and twice differentiable for* $t \neq t_i$; $\alpha''(t_i+0)$, $\alpha''(t_i - 0), \beta''(t_i + 0)$ *and* $\beta''(t_i - 0)$ *exist for* $i = 1, 2, \cdots, k$.

(b) $\alpha(a) < 0 < \beta(a)$, $\alpha(b) < 0 < \beta(b)$, $\alpha(a) + \beta(a) = \alpha(b) + \beta(b) = 0$, *and* $\alpha(t) < \beta(t)$ *for* $t \in [a, b]$.

(c) $\beta''(t) - f(t, \beta, \beta') \leq 0 \leq \alpha''(t) - f(t, \alpha, \alpha')$ *for* $t \neq t_i$, $i = 1, 2, \cdots, k$.

(d) *For* $i = 1, 2, \cdots, k$, *one has*

$$\begin{aligned} & I_i(\beta(t_i)) = I_i(\alpha(t_i)) = 0, \quad \alpha(t_i) \leq x + I_i(x) \leq \beta(t_i) \ \ \text{if}\ \alpha(t_i) < x < \beta(t_i); \\ & \beta'(t_i + 0) \leq \beta'(t_i), \quad \alpha'(t_i + 0) \geq \alpha'(t_i), \quad \text{and}\ y + J_i(y) > \beta'(t_i + 0) \ \ \text{if} \ \ y > \beta'(t_i); \\ & J_i(y) + y < \alpha'(t_i + 0) \ \text{if}\ y < \alpha'(t_i). \end{aligned}$$

(e) f *satisfies a Nagumo's condition with respect to* $G = \{(t, x) : \alpha(t) \leq x \leq \beta(t),\ t \in [a, b]\}$.

Then the problem (1.1)–(1.4) *has at least one solution* $x(t)$ *such that* $\alpha(t) \leq x(t) \leq \beta(t)$ *for* $t \in [a, b]$.

Proof. Making the variable change $x = u + \frac{\beta+\alpha}{2}$ we have that $x(t)$ is a solution of (1.1)–(1.4) if and only if u satisfies the following relations:

$$\begin{aligned} u'' &= \overline{f}(t,u,u') \\ &\triangleq f\left(t, u + \frac{\beta(t)+\alpha(t)}{2}, u' + \frac{\beta'(t)+\alpha'(t)}{2}\right) - \frac{\beta''(t)+\alpha''(t)}{2}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} \Delta u(t_i) &= \overline{I}_i(u(t_i-0)) \\ &\triangleq I_i\left(u(t_i-0) + \frac{\beta(t_i)+\alpha(t_i)}{2}\right) - \frac{\Delta\beta(t_i)+\Delta\alpha(t_i)}{2}, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \Delta u'(t_i) &= \overline{J}_i(u'(t_i-0)) \\ &\triangleq J_i\left(u'(t_i-0) + \frac{\beta'(t_i)+\alpha'(t_i)}{2}\right) - \frac{\Delta\beta'(t_i)+\Delta\alpha'(t_i)}{2}, \end{aligned} \tag{3.3}$$

$$u(a) = u(b) = 0. \tag{3.4}$$

Let $\phi(t) = \frac{\beta(t)-\alpha(t)}{2}$. Then all the conditions of Corollary 3.1 can be verified one by one for the problem (3.1)–(3.4). Hence, the proof is complete.

Remark 3.1. Conditions (i)–(iii) of Theorem 3.1 are generalized versions of conditions (i)–(ii) of [1, Theorem 2] in the impulsive case.

Remark 3.2. Corollaries 3.1 and 3.2 are similar to the main results in [3,4] obtained by Hu and Lakshmikantham, Erbe and Liu with the upper and lower solution method. But f, I_i, J_i do not display the special structure necessary in the latter. For example, in [3] $f(t,x,y)$ is quasinondecreasing,

$$I_i(x) = (I_{i1}(x_1), \cdots, I_{in}(x_n)),\ J_i(y) = (J_{i1}(y_1), \cdots, J_{in}(y_n))$$

and I_{ij}, J_{ij} are nondecreasing, whereas in Corollaries 3.1 and 3.2 those assumptions do not appear. Hence, our results can be applied to some new problems.

Remark 3.3. Condition $\alpha(a)+\beta(a) = \alpha(b)+\beta(b) = 0$ of Corollary 3.2 can be removed. In fact, if we consider $x(a) = \xi$, $x(b) = \eta$ with $\xi, \eta \in R^n$ fixed instead of the ones in (1.1) and assume moreover that the unique solution $\overline{x}(t)$ of $x'' = cx$ for $t \in (a,b)$, $x(a) = \xi$, $x(b) = \eta$ satisfies $(t, \overline{x}(t)) \in G$ for $t \in [a,b]$, then Theorem 2.1 is also valid. And hence, Theorem 3.1 holds under the more assumption $|\ \xi\ | < \phi(a)$, $|\ \eta\ | < \phi(b)$ because for $t \in (a,b)$ we have $\overline{x}(t) \to 0$ as $c \to +\infty$.

Example 3.1. Assume that there exist $\underline{M}$, $\overline{M} \in R$ with $\underline{M} < 0 < \overline{M}$ such that

(1) $f : [0,1] \times R \times R \to R$ is continuous and satisfies

$$f(t, \overline{M}, 0) \geq 0 \geq f(t, \underline{M}, 0) \quad \text{for} \quad t \in [0,1],$$

(2) For every $i = 1, 2, \cdots, k$,

$$I_i(\overline{M}) = I_i(\underline{M}) = 0 \text{ and } \underline{M} \leq x + I_i(\underline{M}) \leq \overline{M}$$

for $\underline{M} < x < \overline{M}$ and $y(J_i(y)+y) \geq 0$ for $y \in R$.

(3) f satisfies a Nagumo's condition with respect to

$$G = \{x :\ \underline{M} \leq x \leq \overline{M},\ t \in [0,1]\}.$$

Then problem (1.1)–(1.4) $(n = 1, a = 0, b = 1)$ has at least one solution by Corollary 3.2 and Remark 3.3.

Theorem 3.2. *Suppose that there exists a positive continuous function* $\beta(t)$ *satisfying condition* (a) *of Corollary* 3.2 *and a real valued function* $W(x) \in C^2(R^n)$ *such that*

(a) $\beta''(t) < 0\, for\, t \in [a,b] \cup \{t_i + 0\}_{i=1}^k$; *and*

$$W_{xx}(x)y \cdot y + W_x(x) \cdot f(t,x,y) > \beta''(t)$$

for (t,x,y) *such that* $t \in [a,b] \cup \{t_i\}_{i=1}^k$, $W_x(x) \cdot y = \beta'(t)$.

(b) $W(x + \lambda I_i(x)) - \beta(t_i) \geq 0$,

$$\begin{aligned} &W_x(x + I_i(x)) \cdot (y + \lambda J_i(y)) - \beta'(t_i + 0) > 0 \quad for \quad \lambda \in (0,1) \\ &if \;\; W(x) - \beta(t_i) = 0, \quad W_x(x) \cdot y - \beta'(t_i) > 0; \\ &W(\overline{x}) - \beta(t_i) \geq 0, \quad \overline{x} \cdot \overline{y} - \beta'(t_i) < 0 \\ &if \;\; W(x) - \beta(t_i) = 0, \;\; W_x(x) \cdot y - \beta'(t_i + 0) < 0 \\ &and \;\; \overline{x} + \lambda I_i(\overline{x}) = x, \overline{y} + \lambda J_i(\overline{y}) = y \quad for\ some \quad \lambda \in (0,1). \end{aligned}$$

(c) $G = \{(t,x) : W(x) - \beta(t) < 0\}$ *is bounded and* $(t,0) \in G$ *for* $t \in [a,b]$, $W_{xx}(x)$ *is positive semidefinite for* $x \in R^n$.

(d) f *satisfies a Nagumo's condition with respect to* G.

Then the problem (1.1)–(1.4) *has at least one solution* $x(t)$ *such that* $(t, x(t)) \in \overline{G}$ *for* $t \in [a,b]$.

Proof. It is enough to show that G is a curvature bound set relative to $(2.5)_\lambda$–$(2.7)_\lambda$ with $c = 0$ for $\lambda \in (0,1)$ from Theorem 2.1. To this end, let $V(t,x) = W(x) - \beta(t)$. Noticing that

$$\begin{aligned} &\text{grad}V(t,x) = \begin{pmatrix} -\beta'(t) \\ W_x(x) \end{pmatrix}, \\ &H(t,x)\begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ y \end{pmatrix} + \text{grad}V(t,x) \cdot \begin{pmatrix} 0 \\ \lambda f(t,x,y) \end{pmatrix} \\ &= W_{xx} y \cdot y - \beta'' + \lambda W_x(x) \cdot f(t,x,y), \end{aligned}$$

we see that the curvature bound set of G follows from conditions (a)–(c), which ends the proof.

Remark 3.4. When $I_i(x) \equiv J_i(x) \equiv 0$, $i = 1, 2, \cdots, k$, this theorem reduces to [2, Theorem V.25].

§4. Construction of the Bound Function ϕ

In this section we will discuss the existence of solutions of (1.1)–(1.4) by constructing the bound function $\phi(t)$ of Corollary 3.1.

Theorem 4.1. *Assume that*

(i) *There exist positive numbers* A_i,B_i *and* $C_i, i = 0, 1$ *such that*

$$-x \cdot f(t,x,y) \leq A_i \mid x \mid^2 + B_i \mid x \cdot y \mid + C_i \mid x \mid \quad for\ t \in (t_{i-1}, t_i), \mid x \mid \geq M_1 > 0$$

and $\Gamma(A_i, B_i) > t_i - t_{i-1}, i = 1, 2,$ *where*

$$\begin{aligned} \Gamma(A,B) &= 2\sigma^{-1/2}\text{tan}h^{-1}(\sqrt{\sigma}/B)\ for\ \sigma = B^2 - 4A > 0 \\ &= 2(-\sigma)^{-1/2}\text{tan}(\sqrt{-\sigma}/B)\ for\ \sigma < 0 \\ &= 2/B\ for\ \sigma = 0. \end{aligned}$$

(ii) $I_1(x) = 0$ *for* $|x| = R^*$; $|I_1(x)| \leq R^* - |x|$ *for* $|x| < R^*$, *where* R^* *is sufficiently large.*

(iii) *There exist two positive numbers* k_1, k_2 *such that for every* x *with* $|x| = 1$ *if* $x \cdot y \geq k_1$ *then* $x \cdot (y + J_1(y)) \geq -k_2$.

(iv) *For every* $R > 0$, f *satisfies a Nagumo's condition with respect to* $\{(t, x) : t \in [a, b], |x| < R\}$.

Then the problem (1.1)–(1.4) *has at least one solution.*

Proof. Let $\phi_1(t)$ and $\phi_2(t)$ be the unique solutions of the following problems:

$$x'' = -(A_1 |x| + B_1 |x'| + C_1), \ t \in [a, t_1], \quad x(t_1) = M, \quad x'(t_1) = k_1$$

and

$$\begin{aligned} x'' &= -(A_2 |x| + B_2 |x'| + C_2), \, t \in [t_1, b], \\ x(t_1) &= M, \, x'(t_1) = -k_2 \end{aligned}$$

respectively, where $M > 0$. Let $\phi(t) = \phi_1(t)$ for $t \in [a, t_1]$, $\phi(t) = \phi_2(t)$ for $t \in (t_1, b]$. From the proof of [8, Theorem 2] it follows that there exists a sufficiently large $M^* > 0$ such that $\phi(t) > M_1$ for $t \in [a, b]$. We claim that $\phi(t)$ satisfies all the conditions of Corollary 3.1.

In fact, at first conditions (i),(iii)$'$ and (iv) of Corollary 3.1 are valid obviously. Secondly, from assumption (i) we have

$$\begin{aligned} -x \cdot f(t, x, y) &\leq A_i |x|^2 + B_i |x \cdot y| + C_i |x| \\ &= \phi(t)(A_i \phi(t) + B_i |\phi'(t)| + C_i) \\ &= -\phi(t)\phi''(t) \\ &\leq -\phi(t)\phi'' + |y|^2 - \phi'(t)^2 \end{aligned}$$

when $t \neq t_i$, $|x| = \phi(t)$, $x \cdot y = |x| \phi'(t)$. Hence, condition (ii) of Corollary 3.1 is also valid. The proof is finished.

Remark 4.1. Conditions (i)–(iii) of Theorem 4.1 can be viewed as the generalized forms in the impulsive vector case of the one given in [2, Theorem V.15], which is associated with the solvability of the scalar case of (1.1) and (1.4) with f continuous.

Remark 4.2. When the impulsive effects are absent, i.e., $I_1(x) \equiv J_1(x) \equiv 0$, condition (i) of Theorem 4.1 is similar to the ones (4.1)–(4.2) and (4.6) in [1], but the latter cannot contain [2, Theorem V.15] as a special case with $n = 1$.

Remark 4.3. Even when $I_1(x) \equiv J_1(x) \equiv 0$ and $n = 1$, condition (i) of Theorem 4.1 is also an enlarged form of the one in [2, Theorem V.15], and reduces to the latter when $t_1 = \frac{a+b}{2}$.

Remark 4.4. Condition (ii) can be replaced by the condition that there exists a positive number R^* such that $I_1(x) = 0$ when $|x| \geq R^*$.

Example 4.1. Assume that

(1) $f(t, x, y) = p(t, x, y)x + q(t, x, y)y + \overline{f}(t, x, y)$, p, q : $[0, 1] \times R \times R \to R$ are continuous and

$$\begin{aligned} &p(t, x, y) \leq A_i, \, |q(t, x, y)| \leq B_i \text{ for } i = 1, 2, \ t \in (t_{i-1}, t_i), \ x, y \in R, \\ &\Gamma(A_i, B_i) > t_i - t_{i-1}, \quad i = 1, 2, \quad t_0 = 0, \quad t_2 = 1, \end{aligned}$$

where $\Gamma(A, B)$ is defined as in Theorem 4.1, $\overline{f} : [0,1] \times R^n \times R^n \to R^n$ is continuous and bounded.

(2) $I_1(x) = 0$ for $\mid x \mid \geq \overline{R} > 0$ and $J_1(\cdot)$ is bounded.

Then problem

$$\begin{aligned} x'' &= f(t, x, x'), \quad t \in (0,1), \quad t \neq t_1 \in (0,1), \\ \Delta x(t_1) &= I_1(x(t_1 - 0)), \quad \Delta x'(t_1) = J_1(x'(t_1 - 0)), \\ x(0) &= 0 = x(1) \end{aligned}$$

has at least one solution from Theorem 4.1.

Theorem 4.2. *Assume that*

(a) $-x \cdot f(t, x, y) \leq \phi^2(t) \mid x|^2 + l \mid x \mid$ *where* $l > 0$, $\phi : [a, b] \to (0, +\infty)$ *is nondecreasing on* (t_k, b) *and nonincreasing on* (a, t_k) *such that*

$$\int_a^{t_k} \phi(t)dt < \frac{\pi}{2}, \quad \int_{t_k}^b \phi(t)dt < \frac{\pi}{2}.$$

(b) *There exists* $\overline{R} > 0$ *such that* $I_i(x) = 0$ *for* $\mid x \mid \geq \overline{R}$, $i = 1, 2, \cdots, k$, *and for every* $e \in R^n$ *with* $\mid e \mid = 1$, *we have* $e \cdot J_i(x) \geq 0$ *for* $e \cdot x > \overline{R}$.

(c) f *satisfies a Nagumo's condition with respect to* $G = \{(t, x) : t \in [a, b], \mid x \mid \leq R\}$ *for every* $R > 0$.

Then problem (1.1)–(1.4) *has at least one solution.*

The proof will be given in another paper for shortening the length of this paper.

Remark 4.5. This theorem can be regarded as a generalized version in the impulsive vector case of the optimal result given by Mawhin in [5].

Example 4.2. Let $J_i(x) = \psi_i(x)x$, where $\psi_i : R^n \to R$ is continuous and satisfies $\psi_i(x) \geq 0$ for $x \in R^n$. Then condition (b) of Theorem 4.2 is valid.

When $I_i(\cdot) = J_i(\cdot) = 0$, $i = 1, 2, \cdots, k$, as a special case of Theorem 4.2, we have

Corollary 4.1. *If* $f : [a, b] \times R^n \times R^n \to R^n$ *is continuous and conditions* (a) *and* (c) *of Theorem* 4.2 *are satisfied, then problem*

$$x'' = f(t, x, x'), \quad t \in (a, b), \quad x(a) = x(b) = 0 \tag{4.1}$$

has at least one solution.

Corollary 4.2. *Assume that*

(a) $f : [a, b] \times R^n \times R^n \to R^n$ *is continuous and there exist* $t_1 \in (a, b)$, $A_1 > 0$, $A_2 > 0$ *such that*

$$-x \cdot f(t, x, y) \leq {A_i}^2 \mid x|^2 + l \mid x \mid \quad for \ \ t \in (t_i, t_{i+1}), \ \mid x \mid \geq M_1 > 0, \ \ y \in R^n,$$

where $t_0 = a$, $t_2 = b$ *and*

$$A_i(t_{i+1} - t_i) < \frac{\pi}{2} \quad for \quad i = 1, 2.$$

(b) f *satisfies a Nagumo's condition with respect to* $G = \{(t, x) : t \in [a, b], \mid x \mid \leq R\}$ *for every* $R > 0$.

Then the problem (4.1) *has at least one solution.*

Proof. This is a special case of Corollary 4.1 with $\bar{t} = t_1$ and $\phi(t) = A_i$ for $t \in (t_i, t_{i+1})$, $i = 0, 1$.

Remark 4.6. Corollaries 4.1 and 4.2 when n=1, and Theorem 4.2 when $I_i = J_i = 0$, are results on the existence of solutions of (4.1), which is similar to the recent work [6] obtained by H. Wang and Y. Li with the optimal control theory method. They have proved that for every A, B with $(k-1)^2 < A < k^2\pi^2 < B$, where k is a positive number, there exists a number $M \le k^2\pi^2$, which depends only on A and B, such that if $f, f_x : [0,1] \times R \to R$ are continuous and

$$A \le f_x(t,x) \le \beta(t) \le B, \quad \int_0^1 \beta(x)dx < M,$$

then the problem

$$x'' + f(t,x) = 0, \quad x(0) = x(1) = 0 \tag{4.2}$$

has a unique solution. If we apply Corollary 4.2 to the problem (4.1), we will derive the following result: If f, $f_x : [0,1] \times R \to R$ are continuous and

$$f_x(t,x) \le A_1^2 \quad \text{for} \quad t \in (0, t_1),$$

$$f_x(t,x) \le A_2^2 \quad \text{for} \quad t \in (t_1, 1),$$

where $t_1 \in (0,1)$ is fixed and $t_1 A_1 < \frac{\pi}{2}$, $(1-t_1)A_2 < \frac{\pi}{2}$, then (4.2) has at least a solution. But let $\beta(t) = A_i^2$ for $t \in (t_{i-1}, t_i), i = 1, 2$ with $t_0 = 0$, $t_2 = 1$, and $A_1 t_1 = \frac{\pi}{4}$; then

$$\int_0^1 \beta(t)dt = A_1^2 t_1 + A_2^2(1-t_1) \ge \frac{\pi}{4}A_1 = \frac{\pi^2}{16t_1} \to +\infty \quad \text{as} \quad t_1 \to 0^+.$$

Hence, even for the scalar case $n = 1$, Theorem 4.2 and Corollaries 4.1 and 4.2 also break some restrictions of [6].

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